Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 375–385
DOI: 10.24193/subbmath.2023.2.12

Exponential growth of solutions with L_p-norm of a nonlinear viscoelastic wave equation with strong damping and source and delay terms

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Abstract. In this work, we are concerned with a problem for a viscoelastic wave equation with strong damping, nonlinear source and delay terms. We show the exponential growth of solutions with L_p-norm. i.e. $\lim_{t\to\infty} \|u\|_p^p \to \infty$.

Mathematics Subject Classification (2010): 35L05, 35L20, 58G16, 93D20.

Keywords: Strong damping, viscoelasticity, nonlinear source, exponential growth, delay.

1. Introduction

The well known "Growth" phenomenon is one of the most important phenomena of asymptotic behavior, where many researchers omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of the equation when time arrives at infinity, it differs from the Global existence and Blow up in both mathematically and in applications point of view. Although the interest of the scientific community for the study of delayed problems is fairly recent, multiple techniques have already been explored in depth. In this direction, we are concerned with the following problem

Received 18 May 2020; Accepted 06 July 2020.

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds \\ \qquad +\mu_1 u_t + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,t) = 0, \ x \in \partial\Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \ \ (x,t) \in \Omega \times (0,\tau), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega, \end{cases} \tag{1.1}$$

where ω, b, μ_1 are positive constants, $p \geq 2$ and $\tau > 0$ is the time delay, and μ_2 is real number, and g is a differentiable function.

Viscous materials are the opposite of elastic materials that possess the ability to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of natural sciences. Many authors have given attention and attention to this problem since the beginning of the new millennium.

In the absence of the strong damping Δu_t, that is for $w = 0$, and in absence of the distributed delay term. Our problem (1.1) has been investigated by Berrimi and Messaoudi [2]. They established the local existence result by using the Galerkin method together with the contraction mapping theorem. Also, they showed that for a suitable initial data, then the local solution is global in time and in addition, they showed that the dissipation given by the viscoelastic integral term is strong enough to stabilize the oscillations of the solution with the same rate of decaying (exponential or polynomial) of the kernel g. Also their result has been obtained under weaker conditions than those used by Cavalcanti et al [4], in which a similar problem has been addressed. More precisely the authors in [5] looked into the following problem

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^{\gamma}.u = 0 \tag{1.2}$$

the authors showed a decay result of an exponential rate. This later result has been improved by Berrimi and Messaoudi [2], in which they showed that the viscoelastic dissipation alone is strong enough to stabilize the problem even with an exponential rate. In many existing works on this field, under assumptions of the kernel g. For the problem (1.1) and with μ_1 concerning Cauchy problems, Kafini and Messaoudi [14] established a blow up result for the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^\infty g(t-s)\Delta u(s)ds + u_t = |u|^{p-2}.u, \quad x \in \mathbb{R}^n, t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \end{cases} \tag{1.3}$$

where g satisfies

$$\int_0^\infty g(s)ds < (2p-4)/(2p-3),$$

and the initial data were compactly supported with negative energy such that

$$\int u_0 u_1 dx > 0.$$

In the presence of the strong damping ($w > 0$). In [23], Song and Xue considered with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) \end{cases} \tag{1.4}$$

They showed, under suitable conditions on g , that there were solutions of (1.4) with arbitrarily high initial energy that blow up in a finite time. For the same problem (1.4), in [24], Song and Zhong showed that there were solutions of (1.4) with positive initial energy that blew up in finite time. In [25], Zennir considered with the following viscoelastic equation with strong damping:

$$\begin{cases} u_{tt} - \Delta u - \omega\Delta u_t + \int_0^{t} g(t-s)\Delta u(s)ds \\ +a|u_t|^{m-2}.u_t = |u|^{p-2}.u, \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega \\ u(x,t) = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.5}$$

they proved the exponential growth result under suitable assumptions.
In [17] the authors considered the following problem for a nonlinear viscoelastic wave equation with strong damping, nonlinear damping and source terms

$$\begin{cases} u_{tt} - \Delta u + \int_0^{\infty} g(s)\Delta u(t-s)ds - \varepsilon_1\Delta u_t + \varepsilon_2 u_t|u_t|^{m-2} = \varepsilon_3 u|u|^{p-2}, \\ u(x,t) = 0, \quad x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.6}$$

they proved a blow up result if $p > m$, and established the global existence.
In this article, we investigated problem (1.1), in which all the damping mechanism have been considered in the same time (i.e. $w > 0$; $g \neq 0$; and $\mu_1 > 0, \mu_2 \in L^{\infty}$), these assumptions make our problem different form those studied in the literature, specially the Exponential Growth of solutions. We will prove that if the initial energy $E(0)$ of our solutions is negative (this means that our initial data are large enough), then our local solutions in bounded and

$$\|u\|_p^p \to \infty \tag{1.7}$$

as t tends to $+\infty$ used idea in [25].
Our aim in the present work is to extend the existing Exponential Growth results to strong damping for a viscoelastic problem with delay under the following assumptions.
(**A1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable and decreasing function so that

$$g(t) \geq 0, \quad 1 - \int_0^{\infty} g(s)\,ds = l > 0. \tag{1.8}$$

(**A2**) There exists a constant $\xi > 0$ such that

$$g'(t) \leq -\xi g(t) \quad , \quad t \geq 0. \tag{1.9}$$

(**A3**) μ_2 is real number so that

$$|\mu_2| \leq \mu_1. \tag{1.10}$$

2. Main results

In this section, we prove the Exponential Growth result of solution of problem (1.1). First, as in [21], we introduce the new varible

$$z(x,\rho,t) = u_t(x, t-\tau\rho)$$

then we obtain

$$\begin{cases} \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \\ z(x,0,t) = u_t(x,t) \end{cases} \tag{2.1}$$

Let us denote by

$$gou = \int_\Omega \int_0^t g(t-s)|u(t)-u(s)|^2 ds. \tag{2.2}$$

Therefore, problem (1.1) takes the form:

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ +\mu_1 u_t + \mu_2 z(x,1,t) = b|u|^{p-2}.u, \qquad x \in \Omega, t > 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \end{cases} \tag{2.3}$$

with initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \quad x \in \partial\Omega, \\ z(x,\rho,0) = f_0\left(x, -\tau\rho\right), \\ u\left(x,0\right) = u_0\left(x\right), \; u_t\left(x,0\right) = u_1\left(x\right) \end{cases} \tag{2.4}$$

where

$$(x,\rho,t) \in \Omega \times (0,1) \times (0,\infty)$$

We first state a local existence theorem that can be established by combining arguments of Georgiev and Todorova [10].

Theorem 2.1. *Assume (1.8),(1.9), and (1.10) holds. Let*

$$\begin{cases} 2 < p < \dfrac{2n-2}{n-2}, \qquad n \geq 3; \\ p \geq 2, \qquad\quad n = 1,2 \end{cases} \tag{2.5}$$

Then for any initial data

$$(u_0, u_1, f_0) \in \mathcal{H} \;/\; \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega \times (0,1))$$

the problem (2.4) has a unique solution

$$u \in C([0,T]; \mathcal{H})$$

for some $T > 0$.

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time, that is to say, the norm

$$\|u_t\|_2 + \|\nabla u\|_2 \tag{2.6}$$

in the energy space $L^2(\Omega) \times H_0^1(\Omega)$ of our solution is bounded by a constant independent of the time t. We will make use of arguments in [22].

Theorem 2.2. *Suppose that (1.8),(1.9),(1.10), and (2.5) holds. If $u_0 \in W$, $u_1 \in H_0^1(\Omega)$ and*

$$\frac{bC_*^p}{l}\left(\frac{2p}{(p-2)l}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{2.7}$$

where C_* is the best Poincaré's constant. Then the local solution $u(t,x)$ is global in time.

The following lemma shows that the associated energy of the problem is nonincreasing under the condition (1.10), there exist ξ such that

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad |\mu_2| < \mu_1 \tag{2.8}$$

We introduce the energy functional

Lemma 2.3. *Assume (1.8),(1.9),(2.8) and (2.5) hold, let $u(t)$ be a solution of (2.3), then $E(t)$ is non-increasing, that is*

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) \\ &\quad + \frac{\xi}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx - \frac{b}{p}\|u\|_p^p. \end{aligned} \tag{2.9}$$

satisfies

$$E(t) \leq -c_1(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx) \tag{2.10}$$

Proof. By multiplying the equation $(2.3)_1$ by u_t and integrating over Ω, we get

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(go\nabla u) - \frac{b}{p}\|u\|_p^p\right\} \\ = &-\mu_1\|u_t\|_2^2 - \mu_2\int_\Omega u_t z(x,1,t)dx + \frac{1}{2}(g'o\nabla u) - \frac{1}{2}g(t)\|\nabla u\|_2^2 - \omega\|\nabla u_t\|_2^2 \end{aligned} \tag{2.11}$$

and, multiplying $(2.3)_2$ by $\frac{\xi}{\tau}z$, we have

$$\begin{aligned} \frac{d}{dt}\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx &= -\frac{\xi}{\tau}\int_\Omega\int_0^1 zz_\rho d\rho dx \\ &= -\frac{\xi}{2\tau}\int_\Omega [z^2(x,1,t) - z^2(x,0,t)]dx \\ &= \frac{\xi}{2\tau}\|u_t\|_2^2 - \frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx. \end{aligned} \tag{2.12}$$

then, we get

$$\begin{aligned}\frac{d}{dt}E(t) &= -\mu_1\|u_t\|_2^2-\mu_2\int_\Omega u_t z(x,1,t)dx+\frac{1}{2}(g'o\nabla u)\\ &\quad -\frac{1}{2}g(t)\|\nabla u\|_2^2-\omega\|\nabla u_t\|_2^2+\frac{\xi}{2\tau}\|u_t\|_2^2\\ &\quad -\frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx \end{aligned}\tag{2.13}$$

By (2.11) and (2.12), we get (2.9).
And by using Young's inequality, (1.8),(1.9) in (2.13) , we get

$$\frac{d}{dt}E(t)\le -\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\|u_t\|_2^2-\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\int_\Omega z^2(x,1,t)dx \tag{2.14}$$

by (2.8) , we obtain (2.10). □

Now we are ready to state and prove our main result. For this purpose, we define

$$\begin{aligned}H(t)=-E(t) &= \frac{b}{p}\|u\|_p^p-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2\\ &\quad -\frac{1}{2}(go\nabla u)-\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx.\end{aligned}\tag{2.15}$$

Theorem 2.4. *Suppose that (1.8)-(1.10), and (2.5). Assume further that* $E(0)<0$ *holds. Then the unique local solution of problem (2.3) grows exponentially.*

Proof. From (2.9), we have

$$E(t)\le E(0)\le 0. \tag{2.16}$$

Hence

$$\begin{aligned}H'(t)=-E'(t) &\ge c_1\left(\|u_t\|_2^2+\int_\Omega z^2(x,1,t)dx\right)\\ &\ge c_1\int_\Omega z^2(x,1,t)dx\ge 0.\end{aligned}\tag{2.17}$$

and

$$0\le H(0)\le H(t)\le \frac{b}{p}\|u\|_p^p. \tag{2.18}$$

We set

$$\mathcal{K}(t)=H(t)+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2dx+\frac{\varepsilon\omega}{2}\int_\Omega(\nabla u)^2dx. \tag{2.19}$$

where $\varepsilon>0$ to be specified later.
By multiplying $(2.3)_1$ by u and taking a derivative of (2.19), we obtain

$$\begin{aligned}\mathcal{K}'(t) &= H'(t)+\varepsilon\|u_t\|_2^2+\varepsilon\int_\Omega\nabla u\int_0^t g(t-s)\nabla u(s)dsdx\\ &\quad -\varepsilon\|\nabla u\|_2^2+\varepsilon b\int_\Omega|u|^pdx-\varepsilon\mu_2\int_\Omega uz(x,1,t)dx.\end{aligned}\tag{2.20}$$

Using

$$\varepsilon\mu_2\int_\Omega uz(x,1,t)dx \le \varepsilon|\mu_2|\left\{\delta_1\|u\|_2^2+\frac{1}{4\delta_1}\int_\Omega z^2(x,1,t)dx\right\}. \tag{2.21}$$

and

$$\begin{aligned}\varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.\nabla u(s)dxds &= \varepsilon\int_0^t g(t-s)ds\int_\Omega \nabla u.(\nabla u(s)-\nabla u(t))dxds\\ &\quad +\varepsilon\int_0^t g(s)ds\|\nabla u\|_2^2\\ &\ge \frac{\varepsilon}{2}\int_0^t g(s)ds\|\nabla u\|_2^2-\frac{\varepsilon}{2}(go\nabla u).\end{aligned} \tag{2.22}$$

we obtain, from (2.20),

$$\begin{aligned}\mathcal{K}'(t) &\ge H'(t)+\varepsilon\|u_t\|_2^2-\varepsilon\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\varepsilon b\|u\|_p^p\\ &\quad -\varepsilon\delta_1|\mu_2|\|u\|_2^2-\frac{\varepsilon|\mu_2|}{4\delta_1}\int_\Omega z^2(x,1,t)dx\\ &\quad -\frac{\varepsilon}{2}(go\nabla u).\end{aligned} \tag{2.23}$$

Therefore, using (2.17) and by setting δ_1 so that

$$\frac{|\mu_2|}{4\delta_1 c_1}=\kappa,$$

substituting in (2.23), we get

$$\begin{aligned}\mathcal{K}'(t) &\ge [1-\varepsilon\kappa]H'(t)+\varepsilon\|u_t\|_2^2\\ &\quad -\varepsilon\left[\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\right]\|\nabla u\|_2^2+\varepsilon b\|u\|_p^p\\ &\quad -\frac{\varepsilon|\mu_2|^2}{4c_1\kappa}\|u\|_2^2-\frac{\varepsilon}{2}(go\nabla u).\end{aligned} \tag{2.24}$$

For $0<a<1$, from (2.15),

$$\begin{aligned}\varepsilon b\|u\|_p^p &= \varepsilon p(1-a)H(t)+\frac{\varepsilon p(1-a)}{2}\|u_t\|_2^2+\varepsilon ba\|u\|_p^p\\ &\quad +\frac{\varepsilon p(1-a)}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+\frac{\varepsilon}{2}p(1-a)(go\nabla u)\\ &\quad +\frac{\varepsilon p(1-a)\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx.\end{aligned} \tag{2.25}$$

substituting in (2.24), we get

$$\begin{aligned}\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]H'(t)+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|_2^2\\ &+\varepsilon\left[\left(\frac{p(1-a)}{2}\right)\left(1-\int_0^t g(s)ds\right)-\left(1-\frac{1}{2}\int_0^t g(s)ds\right)\right]\|\nabla u\|_2^2\\ &-\frac{\varepsilon|\mu_2|^2}{4c_1\kappa}\|u\|_2^2+\varepsilon p(1-a)H(t)+\varepsilon ba\|u\|_p^p \\ &+\frac{\varepsilon p(1-a)\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{\varepsilon}{2}[p(1-a)+1](go\nabla u)\end{aligned} \tag{2.26}$$

Using Poincaré's inequality, we obtain

$$\begin{aligned}\mathcal{K}'(t) \geq\ & [1-\varepsilon\kappa]H'(t)+\varepsilon\left[\frac{p(1-a)}{2}+1\right]\|u_t\|_2^2+\frac{\varepsilon}{2}(p(1-a)-1)(go\nabla u)\\ &+\varepsilon\left\{\left(\frac{p(1-a)}{2}-1\right)-\int_0^t g(s)ds(\frac{p(1-a)-1}{2})-\frac{c|\mu_2|^2}{4c_1\kappa}\right\}\|\nabla u\|_2^2\\ &+\varepsilon ab\|u\|_p^p+\varepsilon p(1-a)H(t)+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\end{aligned} \tag{2.27}$$

At this point, we choose $a>0$ so small that

$$\alpha_1=\frac{p(1-a)}{2}-1>0$$

and assume

$$\int_0^\infty g(s)ds<\frac{\frac{p(1-a)}{2}-1}{\left(\frac{p(1-a)}{2}-\frac{1}{2}\right)}=\frac{2\alpha_1}{2\alpha_1+1} \tag{2.28}$$

then we choose κ so large that

$$\alpha_2=(\frac{p(1-a)}{2}-1)-\int_0^t g(s)ds\left(\frac{p(1-a)-1}{2}\right)-\frac{c|\mu_2|^2}{4c_1\kappa}>0$$

Once κ and a are fixed, we pick ε so small enough so that

$$\alpha_4=1-\varepsilon\kappa>0$$

and

$$\mathcal{K}(t)\leq\frac{b}{p}\|u\|_p^p, \tag{2.29}$$

Thus, for some $\beta>0$, estimate (2.27) becomes

$$\begin{aligned}\mathcal{K}'(t) \geq\ & \beta\{H(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+(go\nabla u)+\|u\|_p^p\\ &+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx\}.\end{aligned} \tag{2.30}$$

and

$$\mathcal{K}(t)\geq\mathcal{K}(0)>0,\quad t>0. \tag{2.31}$$

Next, using Young's and Poincaré's inequalities, from (2.19) we have

$$\begin{aligned}\mathcal{K}(t) &= \left(H+\varepsilon\int_\Omega uu_t dx+\frac{\varepsilon\mu_1}{2}\int_\Omega u^2dx+\frac{\varepsilon\omega}{2}\int_\Omega \nabla u^2dx\right)\\ &\le c\left[H(t)+\left|\int_\Omega uu_t dx\right|+\|u\|_2^2+\|\nabla u\|_2^2\right]\\ &\le c[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2]. \qquad (2.32)\end{aligned}$$

for some $c>0$: Since, $H(t)>0$, we have from (2.3)

$$\begin{aligned}&-\frac{1}{2}\|u_t\|_2^2-\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2-\frac{1}{2}(go\nabla u)\\ &-\frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx+\frac{b}{p}\|u\|_p^p>0. \qquad (2.33)\end{aligned}$$

then

$$\begin{aligned}\frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p<\frac{b}{p}\|u\|_p^p+(go\nabla u)\\ &\quad+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (2.34)\end{aligned}$$

In the other hand, using (1.8), to get

$$\begin{aligned}\frac{1}{2}(1-l)\|\nabla u\|_2^2 &< \frac{b}{p}\|u\|_p^p<\frac{b}{p}\|u\|_p^p+(go\nabla u)\\ &\quad+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (2.35)\end{aligned}$$

Consequently,

$$\begin{aligned}\|\nabla u\|_2^2 &< \frac{2b}{p}\|u\|_p^p+2(go\nabla u)+l\|\nabla u\|_2^2\\ &\quad+2\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (2.36)\end{aligned}$$

Inserting (2.36) into (2.32), to see that there exists a positive constant k_1 such that

$$\begin{aligned}\mathcal{K}(t) &\le k_1[H(t)+\|\nabla u\|_2^2+\|u_t\|_2^2+\frac{b}{p}\|u\|_p^p+(go\nabla u)(t)\\ &\quad+\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx],\forall t>0. \qquad (2.37)\end{aligned}$$

From inequalities (2.30) and (2.37) we obtain the differential inequality

$$\mathcal{K}'(t)\ge\lambda\mathcal{K}(t), \qquad (2.38)$$

where $\lambda>0$, depending only on β and k_1.
a simple integration of (2.38), we obtain

$$\mathcal{K}(t)\ge\mathcal{K}(0)e^{(\lambda t)},\forall t>0 \qquad (2.39)$$

From (2.19) and (2.29), we have

$$\mathcal{K}(t) \leq \frac{b}{p}\|u\|_p^p. \tag{2.40}$$

By (2.39) and (2.40), we have

$$\|u\|_p^p \geq Ce^{(\lambda t)}, \forall t > 0$$

Therefore, we conclude that the solution in the L_p-norm growths exponentially. This completes the proof. □

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