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Strongly quasilinear parabolic systems

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Abstract. Using the theory of Young measures, we prove the existence of solutions to a strongly quasilinear parabolic system

$$\frac{\partial u}{\partial t} + A(u) = f,$$

where $A(u) = -\operatorname{div}\sigma(x,t,u,Du) + \sigma_0(x,t,u,Du)$, $\sigma(x,t,u,Du)$ and $\sigma_0(x,t,u,Du)$ are satisfy some conditions and $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$.

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1. Introduction

Let $n \geq 2$ be an integer and Ω be a bounded open subset of $\mathbb{R}^n$. Let Q be $\Omega \times (0,T)$ where $T > 0$ is given. In this work we are concerned with the problem of existence of a weak solution for a class of quasilinear parabolic systems of the form

$$\frac{\partial u}{\partial t} + A(u) = f \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega; \tag{1.3}$$

where $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $u_0(x)$ is a given function in $L^2(\Omega;\mathbb{R}^m)$ and $A(u) : L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \to L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ is a Leray-Lions operator of the form $A(u) = -\operatorname{div}\sigma(x,t,u,Du) + \sigma_0(x,t,u,Du)$.

The solvability of (1.1)-(1.3) has been discussed in various papers for $m = 1$ and $m > 1$. Brezis and Browder [11] proved the existence and uniqueness of a solution of (1.1)-(1.3) when σ_0 is independent of ∇u. Landes and Mustonen [24, 25] provided

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structure conditions on a strongly nonlinear operator $A(u)$, under which (1.1) has weak solutions.

S. Demoulini [13] studied the nonlinear parabolic evolution of forward-backward type $u_t = \nabla.q(\nabla u)$ on $Q_\infty \equiv \Omega \times \mathbb{R}^+$. The author used the concept of Young measures as solutions to this kind of problems. Hungerbühler [22] considered the problem (1.1) with $\sigma_0 \equiv 0$ and proved the existence of a weak solution under classical regularity, growth, and coercivity conditions for σ, but with only very mild monotonicity assumptions for some $p \in (2n/(n+2), \infty)$. See [6, 7, 15, 17] for the utilization of Young's measure theory in elliptic case with dual or measure-valued right hand side, and [4, 16] for some kind of p-Laplacian systems.

Misawa [27] studied partial regularity results for evolutional p-Laplacian systems

$$\partial_t u^i - \sum_{\alpha,\beta=1}^{m} D_\alpha\Big(|Du|_g^{p-2} g^{\alpha\beta}(z,u) D_\beta u^i\Big) = f^i(z,u,Du),\ i = 1, ..., n,$$

with natural growth on the gradient. Dreyfuss and Hungerbühler [18] investigated a class of Navier-Stokes systems

$$\partial_t u - \operatorname{div} \sigma(x,t,u,Du) + u.\nabla u = f - \operatorname{grad} P$$

and obtained an existence result for a weak solution by the same theory as in [22]. Furthermore, the authors discussed the general case of the external force f.

In the setting of weighted Sobolev spaces, Aharouch et al. [2] studied the existence of weak solutions for (1.1) via pseudo-monotonicity, when $m = 1$. Di Nardo et al. [14] proved the existence of a renormalized solution for

$$u_t - \operatorname{div} a(x,t,u,\nabla u) + \operatorname{div} K(x,t,u) + H(x,t,\nabla u) = f - \operatorname{div} g,$$

where the data belongs to $L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega))$. For more results, the reader can see [10, 9, 12, 19].

In [5], we have investigated the problem (1.1)-(1.3) and prove the existence of weak solutions for every $f \in L^{p'}(0,T; W^{-1,p'}(\Omega;\mathbb{R}^m))$, by using the theory of Young measures and weak monotonicity assumptions. Furthermore, we have considered the following coercivity condition

$$\sigma(x,t,s,\xi) : \xi + \sigma_0(x,t,s,\xi).s \geq \beta|\xi|^p - d_2(x,t),$$

with $\beta > 0$ and $d_2 \in L^1(Q)$. The purpose of this paper, is to prove the existence of weak solutions for (1.1) by considering the coercivity condition only over σ, and the nonlinear term $\sigma_0(x,t,u,Du)$ satisfy

$$|\sigma_0(x,t,s,\xi)| \leq b(|s|)\big(d_2(x,t) + |\xi|^p\big),$$

$$\sigma_0(x,t,s,\xi).s \geq 0,$$

with $d_2 \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function. It should be noted here, in the above first condition, that there is no growth restriction on the perturbation σ_0 as a function of the unknown. This makes the resolution of (1.1) more complicate.

This paper is organized as follows: in Section 2 we recall the definition of Young measure and some its properties. Section 3 contains basic assumptions and the main result, while Section 4 is devoted to the proof of the main result.

2. Necessary facts about Young measures

In [20] it is claimed that weak convergence is a basic tool of nonlinear analysis, because it has the same compactness properties as the convergence in finite dimensional spaces. Moreover, this convergence sometimes does not behave as one desire with respect to nonlinear functionals and operators. In this situation one can use the technics of Young measures.

Consider

$$C_0(\mathbb{R}^m) = \big\{\varphi \in C(\mathbb{R}^m) : \lim_{|\lambda|\to\infty} \varphi(\lambda) = 0\big\}.$$

Its dual is the well known signed Radon measures $\mathcal{M}(\mathbb{R}^m)$ with finite mass. The duality of $\big(\mathcal{M}(\mathbb{R}^m), C_0(\mathbb{R}^m)\big)$ is given by the following integrand

$$\langle \nu, \varphi\rangle = \int_{\mathbb{R}^m} \varphi(\lambda) d\nu(\lambda), \text{ where } \nu : \Omega \to \mathcal{M}(\mathbb{R}^m).$$

Lemma 2.1 ([20]). *Let $(z_k)_k$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exist a subsequence (still denoted (z_k)) and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $\varphi \in C(\mathbb{R}^m)$ we have*

$$\varphi(z_k) \rightharpoonup^* \overline{\varphi}(x) = \langle \nu_x, \varphi\rangle \quad \textit{weakly in } L^\infty(\Omega;\mathbb{R}^m)$$

for a.e. $x \in \Omega$.

Definition 2.2. The family $\nu = \{\nu_x\}_{x\in\Omega}$ is called Young measures associated with (generated by) the subsequence $(z_k)_k$.

In [8], it is shown that if for all $R > 0$

$$\limsup_{L\to\infty} \big|\{x \in \Omega \cap B_R(0) : |z_k(x)| \geq L\}\big| = 0,$$

then for any measurable $\Omega' \subset \Omega$, we have

$$\varphi(x, z_k) \rightharpoonup \langle \nu_x, \varphi(x, .)\rangle = \int_{\mathbb{R}^m} \varphi(x, \lambda) d\nu_x(\lambda) \quad \text{in } L^1(\Omega'),$$

for every Carathéodory function $\varphi : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that $(\varphi(x, z_k(x)))_k$ is equiintegrable.

The following lemmas are useful for us.

Lemma 2.3 ([21]). *(i) If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence (z_k), then there holds*

$$z_k \longrightarrow z \textit{ in measure } \Leftrightarrow \nu_x = \delta_{z(x)} \quad \textit{for a.e. } x \in \Omega.$$

(ii) If the sequence (v_k) generates the Young measure $\delta_{v(x)}$, then (z_k, v_k) generates the Young measure $\nu_x \otimes \delta_{v(x)}$.

It should be noted that the above properties remain true when $z_k = Dw_k$, with $w_k : \Omega \to \mathbb{R}^m$ and Ω can be repalced by the cylinder Q. We denote by $\mathbb{M}^{m\times n}$ the set of $m \times n$ matrices equipped with the inner product $\xi : \eta = \sum_{i,j} \xi_{ij}\eta_{ij}$.

Lemma 2.4 ([23]). *Let $\varphi : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and (w_k) be a sequence of measurable functions, where $w_k : Q \to \mathbb{R}^m$, such that $w_k \to w$ in measure and such that Dw_k generates the Young measure $\nu_{(x,t)}$. Then*

$$\liminf_{k\to\infty} \int_Q \varphi(x,t,w_k,Dw_k)dxdt \geq \int_Q \int_{\mathbb{M}^{m\times n}} \varphi(x,t,w,\lambda)d\nu_{(x,t)}(\lambda)dxdt$$

provided that the negative part $\varphi^-(x,t,w_k,Dw_k)$ is equiintegrable.

We conclude this section by recalling the following lemma which describes limits points of gradients sequences by means of the Young measures.

Lemma 2.5 ([5]). *The Young measure $\nu_{(x,t)}$ generated by Dw_k in $L^p(0,T;L^p(\Omega))$ satisfy the following properties:*

(i) *$\nu_{(x,t)}$ is a probability measure, i.e., $\|\nu_{(x,t)}\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for a.e. $(x,t) \in Q$.*
(ii) *The weak L^1-limit of Dw_k is given by $\langle \nu_{(x,t)}, id\rangle$.*
(iii) *For a.e. $(x,t) \in Q$, $\langle \nu_{(x,t)}, id\rangle = Dw(x,t)$.*

3. Basic assumptions and the main result

Let $Q = \Omega \times (0,T)$, where Ω is a bounded open subset of $\mathbb{R}^n$ and $T > 0$. Consider the problem (1.1)-(1.3) with $f \in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $p' = p/(p-1)$. To study this problem we assume the following hypothesis.

(H0) $\sigma : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $\sigma_0 : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ are Carathéodory functions (i.e., continuous with respect to $(t,s,\xi) \in (0,T) \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$ for a.e. $x \in \Omega$ and measurable with respect to x for all $(t,s,\xi) \in (0,T) \times \mathbb{R}^m \times \mathbb{M}^{m\times n}$). Moreover, the mapping $\xi \to \sigma_0(x,t,s,\xi)$ is linear.

(H1) There exist $\alpha > 0$, $d_1 \in L^{p'}(Q)$ and $d_2 \in L^1(Q)$ such that

$$\big|\sigma(x,t,s,\xi)\big| \leq d_1(x,t) + |s|^{p-1} + |\xi|^{p-1},$$
$$\sigma(x,t,s,\xi) : \xi \geq \alpha|\xi|^p,$$
$$\big|\sigma_0(x,t,s,\xi)\big| \leq b(|s|)\big(d_2(x,t) + |\xi|^p\big),$$
$$\sigma_0(x,t,s,\xi).s \geq 0,$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function.

(H2) σ satisfies one of the following (monotonicity) conditions:

(i) for all $(x,t) \in Q$ and all $u \in \mathbb{R}^m$, the map $\xi \mapsto \sigma(x,t,u,\xi)$ is a C^1-function and is monotone, i.e.,

$$\big(\sigma(x,t,u,\xi) - \sigma(x,t,u,\eta)\big) : (\xi - \eta) \geq 0 \quad \forall \xi,\eta \in \mathbb{M}^{m\times n}.$$

(ii) there exists a function $b : Q \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that

$$\sigma(x,t,u,\xi) = \big(\partial b/\partial \xi\big)(x,t,u,\xi) := D_\xi b(x,t,u,\xi),$$

and $\xi \mapsto b(x,t,u,\xi)$ is convex and a C^1-function for all $(x,t) \in Q$ and all $u \in \mathbb{R}^m$.

(iii) σ is strictly monotone, i.e., σ is monotone and

$$\big(\sigma(x,t,u,\xi)-\sigma(x,t,u,\eta)\big):(\xi-\eta)=0 \quad \text{implies} \quad \xi=\eta.$$

(iv) σ is strictly p-quasimonotone, i.e.,

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,\overline{\lambda})\big):(\lambda-\overline{\lambda})d\nu_x(x,t)(\lambda)dxdt>0,$$

where $\overline{\lambda}=\langle\nu_{(x,t)},id\rangle$, $\nu=\{\nu_{(x,t)}\}_{(x,t)\in Q}$ is any family of Young measures generated by a sequence in $L^p(Q)$ which are not a single Dirac mass.

In what follows, $\langle .,.\rangle$ denotes the duality pairing between $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$ and $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$, $Q_\tau=\Omega\times(0,\tau)$ for $\tau\in(0,T]$.

Definition 3.1. A function $u\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is a weak solution of problem (1.1)-(1.3) if $\sigma_0(x,t,u,Du)\in L^1(Q;\mathbb{R}^m)$, $\sigma_0(x,t,u,Du)u\in L^1(Q;\mathbb{R}^m)$ and

$$-\int_Q u\frac{\partial\varphi}{\partial t}dxdt+\int_\Omega u\varphi dx\Big|_0^T+\int_Q \sigma(x,t,u,Du):D\varphi dxdt$$
$$+\int_Q \sigma_0(x,t,u,Du)\varphi dxdt=\langle f,\varphi\rangle$$

holds for all $\varphi\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap L^\infty(Q;\mathbb{R}^m)$.

Our main result is the following

Theorem 3.2. *Let $f\in L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m))$ and $u_0\in L^2(\Omega;\mathbb{R}^m)$. Assume that (H0)-(H2) are fulfilled. Then there exists a weak solution $u\in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))\cap C(0,T;L^2(\Omega;\mathbb{R}^m))$ of the problem* (1.1)-(1.3) *in the sense of Definition 3.1.*

4. Proof of the main result

We divide the proof into several steps.

Step 1 Galerkin solutions. We choose a sequence of functions

$$\{w_i\}_{i\geq 1}\subset C_0^\infty(\Omega;\mathbb{R}^m)$$

orthonormal with respect to $L^2(\Omega;\mathbb{R}^m)$ such that $\cup_{j\geq 1}V_j$, where

$$V_j=span\{w_1,...,w_j\}$$

is dense in $H_0^s(\Omega;\mathbb{R}^m)$ with s large enough such as $s>n/2+1$, so that $H_0^s(\Omega;\mathbb{R}^m)$ is continuously embedded in $C^1(\overline{\Omega})$ (see [1]). We define $W_j=C^1(0,T;V_j)$. Therefore, we have $C_0^\infty(Q;\mathbb{R}^m)\subset\overline{\cup_{j\geq 1}W_j}^{C^1(Q;\mathbb{R}^m)}$. Note that there exists $u_0^k\subset\cup_{j\geq 1}V_j$ such that $u_0^k\to u_0$ in $L^2(\Omega;\mathbb{R}^m)$.

Definition 4.1. A function $u_k\in W_k$ is called Galerkin solution of (1.1)-(1.3) if and only if

$$\int_\Omega \frac{\partial u_k}{\partial t}vdx+\int_\Omega \sigma(x,t,u_k,Du_k):Dvdx+\int_\Omega \sigma_0(x,t,u_k,Du_k).vdx=\int_\Omega f(t)vdx \tag{4.1}$$

for all $v \in V_k$ and all $t \in [0,T]$ with $u_k(x,0) = u_0^k(x)$.

Setting

$$u_k(x,t) = \sum_{i=1}^{k} d_i(t) w_i(x),$$

we then try to look for the coefficients $d_i \in C^1([0,T])$. To do this, we define a vector valued function $y_k : [0,T] \times \mathbb{R}^k \to \mathbb{R}^k$ for $d = (d_1, ..., d_k)$ by

$$\begin{aligned}\Big(y_k(t,d)\Big)_i = &\int_\Omega \sigma\Big(x,t,\sum_{j=1}^{k} d_j(t)w_j(x), \sum_{j=1}^{k} d_j(t)Dw_j(x)\Big) : Dw_i(x)dx \\ &+ \int_\Omega \sigma_0\Big(x,t,\sum_{j=1}^{k} d_j(t)w_j(x), \sum_{j=1}^{k} d_j(t)Dw_j(x)\Big).w_i(x)dx,\end{aligned}$$

for $i = 1, ..., k$. Note that $y_k(t,d)$ is continuous because σ and σ_0 are both Carathéodory functions. Therefore, we obtain the following system of ordinary differential equations

$$\begin{cases} d' + y_k(t,d) &= F, \\ d(0) &= v_k, \end{cases}$$

where

$$\big(F(t)\big)_i = \int_\Omega f(t)w_i dx \text{ and } (v_k)_i = \int_\Omega u_0^k w_i dx, \text{ for } i = 1, ..., k.$$

Multiplying the first equation by $d(t)$ and using (H1) (coercivity of σ and sign condition of σ_0) one gets $y_k(t,d)d \geq 0$. By virtue of the Young inequality, it yields

$$\frac{1}{2}\frac{d}{dt}\big|d(t)\big|^2 \leq \big|F(t)\big|\big|d(t)\big| \leq \frac{1}{2}\big(\big|F(t)\big|^2 + \big|d(t)\big|^2\big).$$

Then, Gronwall's lemma allows to deduce that

$$\big|d(t)\big| \leq C(T).$$

Thus, we get $\big|d(t) - d(0)\big| \leq 2C(T)$. Now, let us define $A_k = \max_{t\in[0,T]} \big|F - y_k(t,d(t))\big|$ and $q = \min\big\{T, \frac{2C(T)}{A_k}\big\}$. By the Cauchy-Peano theorem (cf. [3]) we obtain a local solution in $[0,q]$. Starting with the initial value q, we obtain a local solution in $[q,2q]$ and so on we get a local solution d_k in $C^1([0,T])$. Therefore, by construction, we know that the function $u_k(x,t) = \sum_{i=1}^{k} d_{ki}(t)w_i(x)$, which belongs to W_k, is a Galerkin solution for (1.1)-(1.3) satisfying

$$\begin{aligned}\int_{Q_\tau} \frac{\partial u_k}{\partial t} v dxdt + \int_{Q_\tau} \sigma(x,t,u_k,Du_k) : Dv dxdt& \\ + \int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).v dxdt = \int_{Q_\tau} f v dxdt,&\end{aligned} \tag{4.2}$$

for all $v \in W_k$ and all $\tau \in (0,T]$ with $u_k(x,0) = u_0^k(x)$.

Step 2 A priori estimates. In the sequel, C will denotes a positive constant which may change values from line to line and which depends on the parameters of our problem. Let u_k be a Galerkin solution of (1.1)-(1.3). Choosing u_k as test function in (4.2), we get

$$\int_{Q_\tau} \frac{\partial u_k}{\partial t} u_k dxdt + \int_{Q_\tau} \sigma(x,t,u_k,Du_k) : Du_k dxdt \\ + \int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).u_k dxdt = \int_{Q_\tau} f u_k dxdt, \tag{4.3}$$

for every $\tau \in (0,T]$. By virtue of (H1) (coercivity condition) and Hölder's inequality, we can write

$$\frac{1}{2}\|u_k(\tau)\|^2_{L^2(\Omega)} + \alpha \int_{Q_\tau} |Du_k|^p dxdt \\ \leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \|u_k\|_{L^p(0,T;W^{1,p}_0(\Omega))} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}, \tag{4.4}$$

which implies that

$$\alpha \|Du_k\|^p_p \leq \|f\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \|u_k\|_{L^p(0,T;W^{1,p}_0(\Omega))} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Therefore

$$\|u_k\|_{L^p(0,T;W^{1,p}_0(\Omega))} \leq C. \tag{4.5}$$

By virtue of (4.4), the sequence (u_k) is bounded in $L^p(0,T;W^{1,p}_0(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$. Since

$$\int_{Q_\tau} \big|\sigma(x,t,u_k,Du_k)\big|^{p'} dxdt \leq \int_{Q_\tau} \big(d_1(x,t)^{p'} + |u_k|^p + |Du_k|^p\big) dxdt \leq C,$$

then

$$\big\|\sigma(x,t,u_k,Du_k)\big\|_{L^{p'}(Q;\mathbb{M}^{m\times n})} \leq C.$$

Going back to (4.3), we obtain

$$0 \leq \int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).u_k dxdt \leq C. \tag{4.6}$$

Let $N > 0$ be fixed. By the condition (H1) and above inequality we can write

$$\begin{aligned} &\int_{Q_\tau} \big|\sigma_0(x,t,u_k,Du_k)\big| dxdt \\ &= \int_{Q_\tau \cap \{|u_k| \leq N\}} \big|\sigma_0(x,t,u_k,Du_k)\big| dxdt + \int_{Q_\tau \cap \{|u_k| > N\}} \big|\sigma_0(x,t,u_k,Du_k)\big| dxdt \\ &\leq \int_{Q_\tau \cap \{|u_k| \leq N\}} \big|\sigma_0(x,t,u_k,Du_k)\big| dxdt + \frac{1}{N}\int_{Q_\tau} \sigma_0(x,t,u_k,Du_k).u_k dxdt \\ &\leq \int_{Q_\tau \cap \{|u_k| \leq N\}} b(|u_k|)\big(d_3(x,t) + |Du_k|^p\big) dxdt + \frac{C}{N} \\ &\leq b(N)\big(\|d_2\|_{L^1(Q_\tau)} + \|Du_k\|^p_{L^p(Q_\tau)}\big) + \frac{C}{N} \leq C. \end{aligned} \tag{4.7}$$

Hence, the sequence $\sigma_0(x,t,u_k,Du_k)$ is uniformly bounded in $L^1(Q;\mathbb{R}^m)$. Therefore, for a subsequence still indexed by k and for a measurable functions $u \in L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, $\Sigma \in L^{p'}(Q;\mathbb{M}^{m\times n})$ and $\Sigma_0 \in L^1(Q;\mathbb{R}^m)$

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^m)),\\ u_k &\rightharpoonup^* u \quad \text{weakly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^m)),\\ \sigma(x,t,u_k,Du_k) &\rightharpoonup \Sigma \quad \text{weakly in } L^{p'}(Q;\mathbb{M}^{m\times n}),\\ \sigma_0(x,t,u_k,Du_k) &\rightharpoonup \Sigma_0 \quad \text{weakly in } L^1(Q;\mathbb{R}^m),\\ u_k &\longrightarrow u \quad \text{strongly in } L^1(Q;\mathbb{R}^m). \end{aligned} \tag{4.8}$$

The last property in (4.8) comes from the fact that,

$$\frac{\partial u_k}{\partial t} = f + \text{div}\sigma(x,t,u_k,Du_k) - \sigma_0(x,t,u_k,Du_k)$$

is bounded in $L^{p'}(0,T;W^{-1,p'}(\Omega;\mathbb{R}^m)) + L^1(Q;\mathbb{R}^m)$.

Lemma 4.2. *The sequence (u_k) constructed above satisfy $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$ and $u(.,0) = u_0(.)$.*

Proof. Since (u_k) is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$, up to a subsequence, we have

$$u_k(.,T) \rightharpoonup z \quad \text{in } L^2(\Omega;\mathbb{R}^m) \quad \text{as} \quad k\to\infty.$$

Let us denote $u(.,T)$ as $u(T)$ and $u(.,0)$ as $u(0)$ (for simplicity).
Let $v \in V_j \cap L^\infty(\Omega;\mathbb{R}^m)$, $j \le k$ and $\psi \in C^\infty([0,T])$, then we have (take $\tau = T$)

$$\int_Q \frac{\partial u_k}{\partial t} v\psi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : Dv\psi dxdt + \int_Q \sigma_0(x,t,u_k,Du_k).v\psi dxdt = \int_Q fv\psi dxdt.$$

The integration of the first term allows to write

$$\int_\Omega u_k(T)\psi(T)vdx - \int_\Omega u_k(0)\psi(0)vdx + \int_Q \sigma(x,t,u_k,Du_k) : Dv\psi dxdt + \int_Q \sigma_0(x,t,u_k,Du_k).v\psi dxdt = \int_Q fv\psi dxdt + \int_Q u_k v\psi' dxdt.$$

By virtue to (4.8), we obtain in passing to the limit as $k\to\infty$

$$\int_\Omega z\psi(T)vdx - \int_\Omega u_0\psi(0)vdx + \int_Q \Sigma : Dv\psi dxdt + \int_Q \Sigma_0.v\psi dxdt = \int_Q fv\psi dxdt + \int_Q uv\psi' dxdt. \tag{4.9}$$

Let $\psi(T) = \psi(0) = 0$, then

$$\begin{aligned}\int_Q \Sigma : Dv\psi dxdt + \int_Q \Sigma_0.v\psi dxdt &= \int_Q fv\psi dxdt + \int_Q uv\psi dxdt \\ &= \int_Q fv\psi dxdt - \int_Q u'v\psi dxdt.\end{aligned}$$

Going back to (4.9), one has

$$\begin{aligned}\int_\Omega z\psi(T)vdx - \int_\Omega u_0\psi(0)vdx &= \int_Q u'v\psi dxdt + \int_Q uv\psi' dxdt \\ &= \int_\Omega u(T)\psi(T)vdx - \int_\Omega u(0)\psi(0)vdx.\end{aligned}$$

Now, tending j to ∞, if we take $\psi(T) = 0$ and $\psi(0) = 1$, then we obtain $u(0) = u_0$, if we take $\psi(T) = 1$ and $\psi(0) = 0$, then $u(T) = z$. Therefore $u_k(.,T) \rightharpoonup u(.,T)$ in $L^2(\Omega;\mathbb{R}^m)$. □

Step 3 div-curl inequality. As stated in the introduction we will use the tool of Young measures to pass to the limit. To this purpose, since (u_k) is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, there exists a Young measure $\nu_{(x,t)}$ generated by Du_k in $L^p(0,T;L^p(\Omega))$, by Lemma 2.1. Moreover, $\nu_{(x,t)}$ satisfy the properties of Lemma 2.5.

The crucial point in the proof of this Section is the following lemma, namely div-curl inequality, which allows the passage to the limit in the approximating equations.

Lemma 4.3. *Assume that (H0)-(H2) hold. Then the Young measure $\nu_{(x,t)}$ generated by Du_k satisfies*

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du)d\nu_{(x,t)}(\lambda)dxdt \le 0.$$

Proof. Let us consider the sequence

$$I_k := \big(\sigma(x,t,u_k,Du_k) - \sigma(x,t,u,Du)\big) : (Du_k - Du),$$

and let us prove that its negative part I_k^- is equiintegrable on Q. To do this, we write I_k^- in the form

$$\begin{aligned}I_k &= \sigma(x,t,u_k,Du_k) : (Du_k - Du) - \sigma(x,t,u,Du) : (Du_k - Du) \\ &=: I_{k,1} + I_{k,2}.\end{aligned}$$

Since $d_1 \in L^{p'}(Q)$, it follows by (H1) that

$$\int_Q \big|\sigma(x,t,u,Du)\big|^{p'} dxdt \le C.$$

Thus, $\sigma(.,.,u,Du) \in L^{p'}(Q;\mathbb{M}^{m\times n})$ for arbitrary $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, and Lemma 2.5 allows to write

$$\liminf_{k\to\infty} \int_Q I_{k,2}dxdt = \int_Q \sigma(x,t,u,Du) : \Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) - Du\Big)dxdt = 0. \quad (4.10)$$

Let Q' be a measurable subset of Q, by the Hölder inequality and (H1) it follows that

$$\begin{aligned}\int_{Q'} &\big|\sigma(x,t,u_k,Du_k):Du\big|dxdt\\ &\le \Big(\int_{Q'}\big|\sigma(x,t,u_k,Du_k)\big|^{p'}dxdt\Big)^{\frac{1}{p'}}\Big(\int_{Q'}|Du|^p dxdt\Big)^{\frac{1}{p}}\\ &\le \Big(\int_{Q'}\big|d_1(x,t)\big|^{p'}+|u_k|^p+|Du_k|^p\big)dxdt\Big)^{\frac{1}{p'}}\Big(\int_{Q'}|Du|^p dxdt\Big)^{\frac{1}{p}}.\end{aligned}$$

The first integral on the right hand side of the above inequality is uniformly bounded, by the boundedness of $(u_k)_k$. The second integral is arbitrary small if the measure of Q' is chosen small enough. Hence, $\big(\sigma(x,t,u_k,Du_k):Du\big)$ is equiintegrable. A similar argument gives the equiintegrability of $\big(\sigma(x,t,u_k,Du_k):Du_k\big)$. Therefore $I_{k,1}$ is equiintegrable, and by virtue of Lemma 2.4

$$I:=\liminf_{k\to\infty}\int_Q I_k dxdt \ge \int_Q\int_{\mathbb{M}^{m\times n}}\sigma(x,t,u,\lambda):(\lambda-Du)d\nu_{(x,t)}(\lambda)dxdt.$$

To deduce the needed inequality, it is sufficient to show that $I\le 0$. We have

$$\begin{aligned}\int_Q\frac{\partial u_k}{\partial t}u_k dxdt+\int_Q\sigma(x,t,u_k,Du_k):Du_k dxdt&\\ +\int_Q\sigma_0(x,t,u_k,Du_k).u_k dxdt&=\int_Q fu_k dxdt,\end{aligned}$$

then

$$\begin{aligned}I&=\liminf_{k\to\infty}\int_Q\sigma(x,t,u_k,Du_k):(Du_k-Du)dxdt\\ &=\liminf_{k\to\infty}\Big(\int_Q\sigma(x,t,u_k,Du_k):Du_k dxdt-\int_Q\sigma(x,t,u_k,Du_k):Dudxdt\Big)\\ &=\liminf_{k\to\infty}\Big(\int_Q fu_k dxdt-\int_Q\frac{\partial u_k}{\partial t}u_k dxdt-\int_Q\sigma_0(x,t,u_k,Du_k).u_k dxdt\\ &\qquad -\int_Q\sigma(x,t,u_k,Du_k):Dudxdt\Big).\end{aligned}\tag{4.11}$$

Remark first that $\int_Q f(u_k-u)dxdt$ tends to zero as k tends to ∞. By Lemma 4.2 we have

$$\|u_k(.,0)\|_{L^2(\Omega)}\to\|u(.,0)\|_{L^2(\Omega)}\quad\text{and}\quad\|u(.,T)\|_{L^2(\Omega)}\le\liminf_{k\to\infty}\|u_k(.,T)\|_{L^2(\Omega)},$$

which imply

$$\liminf_{k\to\infty}\Big(-\int_Q\frac{\partial u_k}{\partial t}u_k dxdt\Big)\le\frac{1}{2}\|u(.,0)\|^2_{L^2(\Omega)}-\frac{1}{2}\|u(.,T)\|^2_{L^2(\Omega)}.$$

Now, take $\psi \in C^1(0,T;V_j) \cap L^\infty(Q;\mathbb{R}^m)$, $j \leq k$, we have

$$\int_Q \frac{\partial u_k}{\partial t}\psi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : D\psi dxdt + \int_Q \sigma_0(x,t,u_k,Du_k).\psi dxdt = \int_Q f\psi dxdt.$$

The first integral (after integration) is equal to

$$\int_\Omega u_k(.,T)\psi(.,T)dx - \int_\Omega u_k(.,0)\psi(.,0)dx - \int_Q u_k \frac{\partial \psi}{\partial t} dxdt.$$

By tending k to infinity, one has

$$\int_\Omega u(.,T)\psi(T)dx - \int_\Omega u(.,0)\psi(0)dx - \int_Q u\frac{\partial \psi}{\partial t}dxdt + \int_Q \Sigma : D\psi dxdt + \int_Q \Sigma_0.\psi dxdt = \int_Q f\psi dxdt.$$

Passing j to ∞, it result for all $\psi \in C^1(0,T;C^1(\overline{\Omega}))$ that

$$\int_\Omega u(.,T)\psi(T)dx - \int_\Omega u(.,0)\psi(0)dx - \int_Q u\frac{\partial \psi}{\partial t}dxdt + \int_Q \Sigma : D\psi dxdt + \int_Q \Sigma_0.\psi dxdt = \int_Q f\psi dxdt,$$

i.e.,

$$-\int_Q u\frac{\partial \psi}{\partial t}dxdt + \int_Q \Sigma : D\psi dxdt + \int_Q \Sigma_0.\psi dxdt = \int_Q f\psi dxdt,$$

for all $\psi \in C_0^\infty(Q) \subset C^1(0,T;C_0^\infty(\Omega;\mathbb{R}^m))$. Consequently

$$\frac{\partial u}{\partial t} - \operatorname{div}\Sigma + \Sigma_0 = f.$$

Hence, for $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(Q;\mathbb{R}^m)$

$$-\int_Q \Sigma : Dudxdt - \int_Q \Sigma_0.udxdt = -\int_Q fudxdt + \int_Q u\frac{\partial u}{\partial t}dxdt.$$

Gathering the above results in the Eq. (4.11), it result that $I \leq 0$. □

Step 4 Passage to the limit. The passage to the limit will be concern the four cases listed in assumption (H2). Remark first that from Lemma 4.3 and monotonicity of the function σ, it follows that

$$\int_Q \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du)d\nu_{(x,t)}(\lambda) \otimes dx \otimes dt = 0$$

implies

$$\big(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big) : (\lambda - Du) = 0 \quad \text{on supp } \nu_{(x,t)}. \tag{4.12}$$

Now, we have all ingredients to pass to the limit in the approximating equations.

Case (i): Let ∇ denotes the derivative of σ with respect to its last variable. We prove that

$$\sigma(x,t,u,\lambda):\xi=\sigma(x,t,u,Du):\xi+\big(\nabla\sigma(x,t,u,Du)\xi\big):(Du-\lambda)$$

holds on $\operatorname{supp}\nu_{(x,t)}$, for all $\xi\in\mathbb{M}^{m\times n}$. Let $\tau\in\mathbb{R}$, from the monotonicity of σ we infer that

$$\big(\sigma(x,t,u,\lambda)-\sigma(x,t,u,Du+\tau\xi)\big):(\lambda-Du-\tau\xi)\geq 0.$$

The above inequality together with (4.12) imply

$$\begin{aligned}-\sigma(x,t,u,&\lambda):\tau\xi\\ &\geq-\sigma(x,t,u,\lambda):(\lambda-Du)+\sigma(x,t,u,Du+\tau\xi):(\lambda-Du-\tau\xi)\\ &=-\sigma(x,t,u,Du):(\lambda-Du)+\sigma(x,t,u,Du+\tau\xi):(\lambda-Du-\tau\xi).\end{aligned}$$

Since $\sigma(x,t,u,Du+\tau\xi)=\sigma(x,t,u,Du)+\nabla\sigma(x,t,u,Du)\tau\xi+o(\tau)$, we get

$$-\sigma(x,t,u,\lambda):\tau\xi\geq\tau\Big(\big(\nabla\sigma(x,t,u,Du)\big)\xi:(\lambda-Du)-\sigma(x,t,u,Du):\xi\Big).$$

The choice of τ to be arbitrary in $\mathbb{R}$ implies the needed equality

$$\sigma(x,t,u,\lambda):\xi=\sigma(x,t,u,Du):\xi+\big(\nabla\sigma(x,t,u,Du)\xi\big):(Du-\lambda).$$

Using the equiintegrability of $\sigma(x,t,u_k,Du_k)$ and above equality to deduce that its weak L^1-limit is

$$\begin{aligned}\overline{\sigma}&:=\int_{\mathbb{M}^{m\times n}}\sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda)\\ &=\int_{\operatorname{supp}\nu_{(x,t)}}\sigma(x,t,u,\lambda)d\nu_{(x,t)}(\lambda)\\ &=\int_{\operatorname{supp}\nu_{(x,t)}}\Big(\sigma(x,t,u,Du)+\big(\nabla\sigma(x,t,u,Du)\big):(Du-\lambda)\Big)d\nu_{(x,t)}(\lambda)\\ &=\sigma(x,t,u,Du)\underbrace{\int_{\operatorname{supp}\nu_{(x,t)}}d\nu_{(x,t)}(\lambda)}_{=:1}\\ &\quad+\big(\nabla\sigma(x,t,u,Du)\big)^t\underbrace{\int_{\operatorname{supp}\nu_{(x,t)}}(Du-\lambda)d\nu_{(x,t)}(\lambda)}_{=0}\\ &=\sigma(x,t,u,Du).\end{aligned}$$

We have $\sigma(x,t,u_k,Du_k)$ is bounded in $L^{p'}(Q;\mathbb{M}^{m\times n})$ reflexive, then $\sigma(x,t,u_k,Du_k)$ is weakly convergent in $L^{p'}(Q;\mathbb{M}^{m\times n})$ and its weak $L^{p'}$-limit is also $\sigma(x,t,u,Du)$.

Case (ii): In this case we prove that, if $\lambda\in\operatorname{supp}\nu_{(x,t)}$ then

$$b(x,t,u,\lambda)=b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du).$$

Suppose that $\lambda\in\operatorname{supp}\nu_{(x,t)}$, from (4.12) it follows for $\tau\in[0,1]$

$$(1-\tau)\big(\sigma(x,t,u,Du)-\sigma(x,t,u,\lambda)\big):(Du-\lambda)=0.$$

The above expression together with monotonicity of σ allow to write

$$\begin{aligned} 0 &\leq (1-\tau)\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,\lambda)\Big):(Du-\lambda) \\ &= (1-\tau)\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):(Du-\lambda). \end{aligned} \tag{4.13}$$

Since σ is monotone, we have

$$\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):\tau(\lambda-Du)\geq 0$$

which implies since $\tau\in[0,1]$

$$\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):(1-\tau)(\lambda-Du)\geq 0.$$

From this inequality and (4.13) we can infer that

$$\Big(\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big)-\sigma(x,t,u,Du)\Big):(\lambda-Du)=0 \quad \forall\tau\in[0,1],$$

i.e.,

$$\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)=\sigma(x,t,u,Du):(\lambda-Du).$$

We know that (by hypothesis)

$$\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)=\frac{\partial b}{\partial\tau}\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)$$

for $\tau\in[0,1]$. By integration of the above equation over $[0,1]$, it follows that

$$\begin{aligned} b(x,t,u,\lambda) &= b(x,t,u,Du)+\int_0^1\sigma\big(x,t,u,Du+\tau(\lambda-Du)\big):(\lambda-Du)d\tau \\ &= b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du) \end{aligned}$$

as we desired. Let us denotes

$$K_{(x,t)}=\big\{\lambda\in\mathbb{M}^{m\times n}:\ b(x,t,u,\lambda)=b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du)\big\}.$$

From the above results, $\lambda\in K_{(x,t)}$. Since b is convex, we can write

$$\underbrace{b(x,t,u,\lambda)}_{=:B_1(\lambda)}\geq\underbrace{b(x,t,u,Du)+\sigma(x,t,u,Du):(\lambda-Du)}_{=:B_2(\lambda)}.$$

Since $\lambda\mapsto B_1(\lambda)$ is C^1-function, then for $\xi\in\mathbb{M}^{m\times n}$ and $\tau\in\mathbb{R}$

$$\frac{B_1(\lambda+\tau\xi)-B_1(\lambda)}{\tau}\geq\frac{B_2(\lambda+\tau\xi)-B_2(\lambda)}{\tau}\quad\text{for }\tau>0,$$

$$\frac{B_1(\lambda+\tau\xi)-B_1(\lambda)}{\tau}\leq\frac{B_2(\lambda+\tau\xi)-B_2(\lambda)}{\tau}\quad\text{for }\tau<0.$$

Consequently $D_\lambda B_1=D_\lambda B_2$, i.e.,

$$\sigma(x,t,u,\lambda)=\sigma(x,t,u,Du)\quad\text{on }\operatorname{supp}\nu_{(x,t)}\subset K_{(x,t)}. \tag{4.14}$$

Consdier the function $g(x,t,s,\lambda)=|\sigma(x,t,s,\lambda)-\overline{\sigma}(x,t)|$. Then g is a Carathéodoroy function by that of σ. Moreover, since $\sigma(x,t,u_k,Du_k)$ is equiintegrable, thus

$g_k(x,t) := g(x,t,u_k,Du_k)$ is also equiintegrable, hence $g_k \rightharpoonup \overline{g}$ in $L^1(Q)$ (in fact, this convergence is strong since $g_k \geq 0$), where

$$\begin{aligned}\overline{g}(x,t) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} \big|\sigma(x,t,s,\lambda) - \overline{\sigma}(x,t)\big| d\delta_{u(x,t)}(s)\otimes d\nu_{(x,t)}(\lambda)\\ &= \int_{\mathbb{M}^{m\times n}} \big|\sigma(x,t,u,\lambda) - \overline{\sigma}(x,t)\big| d\nu_{(x,t)}(\lambda)\\ &= \int_{\operatorname{supp}\nu_{(x,t)}} \big|\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)\big| d\nu_{(x,t)}(\lambda) = 0\end{aligned}$$

by (4.14).

Case (iii): On the one hand, by Eq. (4.12) we deduce that $\nu_{(x,t)} = \delta_{Du(x,t)}$ for a.e. $(x,t)\in Q$. By virtue of the first property in Lemma 2.3, one gets

$$Du_k \to Du \quad \text{in measure as } k\to\infty.$$

On the other hand, since (u_k) is bounded in $L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m))$, up to a subsequence, $u_k\to u$ in measure. Therefore (for a subsequence) $u_k\to u$ and $Du_k\to Du$ almost everywhere for $k\to\infty$. The continuity of the function σ implies

$$\sigma(x,t,u_k,Du_k)\to\sigma(x,t,u,Du) \quad \text{almost everywhere as } k\to\infty.$$

The Vitali convergence theorem implies $\sigma(x,t,u_k,Du_k)\to\sigma(x,t,u,Du)$ in $L^1(Q)$, by the boundedness and equiintegrability of $\sigma(x,t,u_k,Du_k)$.

Case (iv): Assume that $\nu_{(x,t)}$ is not a Dirac measure on a set $(x,t)\in Q'$ of positive measure. We have $\overline{\lambda} = \langle\nu_{(x,t)}, id\rangle = Du(x,t)$, thus

$$\begin{aligned}&\int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : (\lambda-\overline{\lambda}) d\nu_{(x,t)}(\lambda)dxdt\\ &= \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : \lambda d\nu_{(x,t)}(\lambda)dxdt\\ &- \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\overline{\lambda}) : \overline{\lambda} d\nu_{(x,t)}(\lambda)dxdt\\ &= \int_Q \sigma(x,t,u,\overline{\lambda}) : \Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda)\Big)dxdt\\ &- \int_Q \sigma(x,t,u,\overline{\lambda}) : \overline{\lambda}\Big(\int_{\mathbb{M}^{m\times n}} d\nu_{(x,t)}(\lambda)\Big)dxdt\\ &= 0.\end{aligned}$$

It follows by the strict p-quasimonotonicity of σ that

$$\int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda)dxdt > \int_Q\int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\nu_{(x,t)}(\lambda)dxdt.$$

By virtue of Lemma 4.3 (i.e., $I \leq 0$), it result that

$$\begin{aligned}\int_Q \int_{\mathbb{M}^{m\times n}} & \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dxdt \\ & > \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \overline{\lambda} d\nu_{(x,t)}(\lambda) dxdt \\ & \geq \int_Q \int_{\mathbb{M}^{m\times n}} \sigma(x,t,u,\lambda) : \lambda d\nu_{(x,t)}(\lambda) dxdt,\end{aligned}$$

and this is a contradiction. Hence $\nu_{(x,t)}$ is a Dirac measure, i.e., $\nu_{(x,t)} = \delta_{h(x,t)}$ for a.e. $(x,t) \in Q$, thus

$$h(x,t) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x,t)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) = Du(x,t).$$

Thus $\nu_{(x,t)} = \delta_{Du(x,t)}$. Owing to Lemma 2.3, we get $Du_k \to Du$ in measure. The remainder of the proof of this case is similar to that in Case (iii).
To complete the proof of the main result, it remains to pass to the limit on the nonlinearity term $\sigma_0(x,t,u_k,Du_k)$. From the convergence in measure of u_k to u and of Du_k to Du, it then follows by the continuity of σ_0, that

$$\sigma_0(x,t,u_k,Du_k) \longrightarrow \sigma_0(x,t,u,Du) \quad \text{almost everywhere in } Q,$$

(for a subsequence). Let Q' be a subset of Q and let $N > 0$. We can write

$$\begin{aligned}&\int_{Q'} |\sigma_0(x,t,u_k,Du_k)| dx \\ &= \int_{Q'\cap\{|u_k|\leq N\}} |\sigma_0(x,t,u_k,Du_k)| dxdt + \int_{Q'\cap\{|u_k|>N\}} |\sigma_0(x,t,u_k,Du_k)| dxdt.\end{aligned}$$

By the third condition in (H1) together with (4.6), we obtain

$$\begin{aligned}\int_{Q'} & |\sigma_0(x,t,u_k,Du_k)| dxdt \\ & \leq b(N) \int_{Q'} d_2(x,t) dxdt + b(N) \int_{Q'} |Du_k|^p dxdt + \frac{C}{N} \leq \epsilon,\end{aligned}$$

for some $\epsilon > 0$. Applying Vitali's theorem, we obtain

$$\sigma_0(x,t,u_k,Du_k) \longrightarrow \sigma_0(x,t,u,Du) \quad \text{strongly in } L^1(Q).$$

In addition, by Fatou's Lemma, we get $\sigma_0(x,t,u,Du)u \in L^1(Q)$.
Now, since σ_0 is linear with respect to its last variable, then

$$\begin{aligned}\sigma_0(x,t,u_k,Du_k) & \rightharpoonup \langle \nu_{(x,t)}, \sigma_0(x,t,u,.)\rangle \\ & = \int_{\mathbb{M}^{m\times n}} \sigma_0(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) \\ & = \sigma_0(x,t,u,.) o \int_{\mathbb{M}^{m\times n}} \lambda d\nu_{(x,t)}(\lambda) \\ & = \sigma_0(x,t,u,Du),\end{aligned}$$

in $L^1(Q)$, by the equiintegrability of σ_0.

Taking $\varphi \in C^1(0,T;V_j) \cap L^\infty(Q;\mathbb{R}^m)$, $j \leq k$

$$\int_Q \frac{\partial u_k}{\partial t}\varphi dxdt + \int_Q \sigma(x,t,u_k,Du_k) : D\varphi dxdt + \int_Q \sigma_0(x,t,u_k,Du_k).\varphi dxdt = \int_Q f\varphi dxdt.$$

By integrating the first term and letting $j \to \infty$, it follows from the above results, that for $\varphi \in C^1(0,T;C_0^\infty(\Omega)) \cap L^\infty(Q;\mathbb{R}^m)$

$$-\int_Q u\frac{\partial \varphi}{\partial t}dxdt + \int_\Omega u\varphi dx\Big|_0^T + \int_Q \sigma(x,t,u,Du) : D\varphi dxdt + \int_Q \sigma_0(x,t,u,Du).\varphi dxdt = \int_Q f\varphi dxdt$$

as $k \to \infty$. Hence $u \in L^p(0,T;W_0^{1,p}(\Omega;\mathbb{R}^m)) \cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^m))$ is in fact a weak solution for (1.1)-(1.3).

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