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Necessary and sufficient conditions for oscillation of second-order differential equation with several delays

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Abstract. In this paper, necessary and sufficient conditions are establish of the solutions to second-order delay differential equations of the form

$$\Big(r(t)\big(x'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t) f_i\big(x(\sigma_i(t))\big) = 0 \text{ for } t \geq t_0,$$

We consider two cases when $f_i(u)/u^{\beta}$ is non-increasing for $\beta < \gamma$, and non-decreasing for $\beta > \gamma$ where β and γ are the quotient of two positive odd integers. Our main tool is Lebesgue's Dominated Convergence theorem. Examples illustrating the applicability of the results are also given, and state an open problem.

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1. Introduction

In this article we consider the differential equation

$$\Big(r(t)\big(x'(t)\big)^{\gamma}\Big)' + \sum_{i=1}^{m} q_i(t) f_i\big(x(\sigma_i(t))\big) = 0, \quad \text{for } t \geq t_0, \tag{1.1}$$

where γ is the quotient of two positive odd integers, and the functions f_i, p, q_i, r, σ_i are continuous that satisfy the conditions stated below;

(A1) $\sigma_i \in C([0,\infty),\mathbb{R})$, $\sigma_i(t) < t$, $\lim_{t\to\infty} \sigma_i(t) = \infty$.

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(A2) $r \in C^1([0,\infty),\mathbb{R})$, $q_i \in C([0,\infty),\mathbb{R})$; $0 < r(t)$, $0 \le q_i(t)$, for all $t \ge 0$ and $i = 1,2,\dots,m$; $\sum q_i(t)$ is not identically zero in any interval $[b,\infty)$.
(A3) $f_i \in C(\mathbb{R},\mathbb{R})$ is non-decreasing and $f_i(x)x > 0$ for $x \ne 0$, $i = 1,2,\dots,m$.
(A4) $\int_0^\infty r^{-1/\gamma}(\eta)\,d\eta = \infty$; let $R(t) = \int_0^t r^{-1/\gamma}(\eta)\,d\eta$.

The main feature of this article is having conditions that are both necessary and sufficient for the oscillation of all solutions to (1.1).

In 1978, Brands [9] has proved that for bounded delays, the solutions of

$$x''(t) + q(t)x(t-\sigma(t)) = 0$$

are oscillatory if and only if the solutions of $x''(t) + q(t)x(t) = 0$ are oscillatory. In [10, 12] Chatzarakis *et al.* have considered a more general second-order half-linear differential equation of the form

$$\big(r(x')^\alpha\big)'(t) + q(t)x^\alpha(\sigma(t)) = 0, \tag{1.2}$$

and established new oscillation criteria for (1.2) when

$$\lim_{t\to\infty} \Pi(t) = \infty \text{ and } \lim_{t\to\infty} \Pi(t) < \infty.$$

Wong [32] has obtained the necessary and sufficient conditions for oscillation of solutions of

$$\big(x(t) + px(t-\tau)\big)'' + q(t)f\big(x(t-\sigma)\big) = 0, \quad -1 < p < 0.$$

in which the neutral coefficient and delays are constants. However, we have seen in [5, 13] that the authors Baculǐkovǎ and Džurina have studied

$$\Big(r(t)\big(z'(t)\big)^\gamma\Big)' + q(t)x^\alpha(\sigma(t)) = 0, \quad z(t) = x(t) + p(t)x(\tau(t)), \quad t \ge t_0, \tag{1.3}$$

and established sufficient conditions for oscillation of solutions of (1.3) using comparison techniques when $\gamma = \alpha = 1$, $0 \le p(t) < \infty$ and $\lim_{t\to\infty}\Pi(t) = \infty$. In same technique, Baculikova and Džurina [6] have considered (1.3) and obtained sufficient conditions for oscillation of the solutions of (1.3) by considering the assumptions $0 \le p(t) < \infty$ and $\lim_{t\to\infty}\Pi(t) = \infty$. In [31], Tripathy *et al.* have studied (1.3) and established several sufficient conditions for oscillations of the solutions of (1.3) by considering the assumptions $\lim_{t\to\infty}\Pi(t) = \infty$ and $\lim_{t\to\infty}\Pi(t) < \infty$ for different ranges of the neutral coefficient p. In [8], Bohner *et al.* have obtained sufficient conditions for oscillation of solutions of (1.3) when $\gamma = \alpha$, $\lim_{t\to\infty}\Pi(t) < \infty$ and $0 \le p(t) < 1$. Grace *et al.* [16] have established sufficient conditions for the oscillation of the solutions of (1.3) when $\gamma = \alpha$ and by considering the assumptions $\lim_{t\to\infty}\Pi(t) < \infty$, $\lim_{t\to\infty}\Pi(t) = \infty$ and $0 \le p(t) < 1$. In [19], Li *et al.* have established sufficient conditions for the oscillation of the solutions of (1.3), under the assumptions $\lim_{t\to\infty}\Pi(t) < \infty$ and $p(t) \ge 0$. Karpuz and Santra [18] have obtained several sufficient conditions for the oscillatory and asymptotic behavior of the solutions of

$$\big(r(t)(x(t) + p(t)x(\tau(t)))'\big)' + q(t)f\big(x(\sigma(t))\big) = 0,$$

by considering the assumptions $\lim_{t\to\infty}\Pi(t) < \infty$ and $\lim_{t\to\infty}\Pi(t) = \infty$ for different ranges of p.

For further work on the oscillation of the solutions to this type of equations, we refer the readers to [1, 2, 3, 4, 7, 11, 14, 16, 21, 22, 23, 20, 24, 25, 26, 27, 28, 35]. Note that the majority of publications consider only sufficient conditions, and and merely a few consider necessary and sufficient conditions. Hence, the objective in this work is to establish both necessary and sufficient conditions for the oscillatory and asymptotic behavior of solutions of (1.1) without using the comparison and the Riccati techniques.

Delay differential equations have several applications in the natural sciences and engineering. For example, they often appear in the study of distributed networks containing lossless transmission lines (see for e.g. [17]). In this paper, we restrict our attention to the study (1.1), which includes the class of functional differential equations of neutral type.

By a solution to equation (1.1), we mean a function $x \in \mathrm{C}([T_x, \infty), \mathbb{R})$, where $T_x \geq t_0$, such that $rx' \in \mathrm{C}^1([T_x, \infty), \mathbb{R})$, satisfying (1.1) on the interval $[T_x, \infty)$. A solution x of (1.1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| : \ t \geq T\} > 0$ for all $T \geq T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$; otherwise, it is said to be non-oscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

Remark 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. Main Results

Lemma 2.1. *Assume* (A1)–(A4)*, and that x is an eventually positive solution of* (1.1)*. Then there exist $t_1 \geq t_0$ and $\delta > 0$ such that*

$$0 < x(t) \leq \delta R(t), \tag{2.1}$$

$$\big(R(t) - R(t_1)\big)\Big[\int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\Big]^{1/\gamma} \leq x(t)\,, \tag{2.2}$$

for $t \geq t_1$.

Proof. Let x be an eventually positive solution. Then by (A1) there exists a t^* such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma_i(t)) > 0$ for all $t \geq t^*$ and $i = 1, 2, \dots, m$. From (1.1) it follows that

$$\Big(r(t)\big(x'(t)\big)^\gamma\Big)' = -\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big) \leq 0. \tag{2.3}$$

Therefore, $r(t)\big(x'(t)\big)^\gamma$ is non-increasing for $t \geq t^*$. Next we show the $r(t)\big(x'(t)\big)^\gamma$ is positive. By contradiction assume that $r(t)\big(x'(t)\big)^\gamma \leq 0$ at a certain time $t \geq t^*$. Using that $\sum q_i$ is not identically zero on any interval $[b, \infty)$, and that $f(x) > 0$ for $x > 0$, by (2.3), there exist $t_2 \geq t^*$ such that

$$r(t)\big(x'(t)\big)^\gamma \leq r(t_2)\big(x'(t_2)\big)^\gamma < 0 \quad \text{for all } t \geq t_2.$$

Recall that γ is the quotient of two positive odd integers. Then

$$x'(t) \le \Big(\frac{r(t_2)}{r(t)}\Big)^{1/\gamma} x'(t_2) \quad \text{for } t \ge t_2\,.$$

Integrating from t_2 to t, we have

$$x(t) \le x(t_2) + \big(r(t_2)\big)^{1/\gamma} x'(t_2)\big(R(t) - R(t_2)\big)\,. \tag{2.4}$$

By (A4), the right-hand side approaches $-\infty$; then $\lim_{t\to\infty} x(t) = -\infty$. This is a contradiction to the fact that $x(t) > 0$. Therefore $r(t)\big(x'(t)\big)^\gamma > 0$ for all $t \ge t^*$. From $r(t)\big(x'(t)\big)^\gamma$ being non-increasing, we have

$$x'(t) \le \Big(\frac{r(t_1)}{r(t)}\Big)^{1/\gamma} x'(t_1) \quad \text{for } t \ge t_1\,.$$

Integrating this inequality from t_1 to t, and using that x is continuous,

$$x(t) \le x(t_1) + \big(r(t_1)\big)^{1/\gamma} x'(t_1)\big(R(t) - R(t_1)\big)\,.$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists a positive constant δ such that (2.1) holds. Since $r(t)\big(x'(t)\big)^\gamma$ is positive and non-increasing, $\lim_{t\to\infty} r(t)\big(x'(t)\big)^\gamma$ exists and is non-negative. Integrating (1.1) from t to a, we have

$$r(a)\big(x'(a)\big)^\gamma - r(t)\big(x'(t)\big)^\gamma + \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\, d\eta = 0$$

Computing the limit as $a \to \infty$,

$$r(t)\big(x'(t)\big)^\gamma \ge \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\, d\eta. \tag{2.5}$$

Then

$$x'(t) \ge \Big[\frac{1}{r(t)} \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\, d\eta\Big]^{1/\gamma}.$$

Since $x(t_1) > 0$, integrating the above inequality yields

$$x(t) \ge \int_{t_1}^\eta \Big[\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\Big]^{1/\gamma} d\eta$$

Since the integrand is positive, we can increase the lower limit of integration from η to t, and then use the definition of $R(t)$, to obtain

$$x(t) \ge \big(R(t) - R(t_1)\big)\Big[\int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\Big]^{1/\gamma},$$

which yields (2.2). □

For the next theorem we assume that there exists a constant β, the quotient of two positive odd integers, with $\beta < \gamma$, such that

$$\frac{f_i(u)}{u^\beta} \text{ is non-increasing for } 0 < u,\ i = 1, 2, \dots, m\,. \tag{2.6}$$

For example $f_i(u) = |u|^\alpha \operatorname{sgn}(u)$, with $0 < \alpha < \beta$ satisfies this condition.

Theorem 2.2. *Under assumptions* (A1)–(A4) *and* (2.6), *each solution of* (1.1) *is oscillatory if and only if*

$$\int_0^\infty \sum_{i=1}^m q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\, d\eta = \infty \quad \forall \delta > 0\,. \tag{2.7}$$

Proof. We prove sufficiency by contradiction. Initially we assume that a solution x is eventually positive. So, Lemma 2.1 holds, and then there exists $t_1 \geq t_0$ such that

$$x(t) \geq \big(R(t) - R(t_1)\big) w^{1/\gamma}(t) \geq 0 \quad \forall t \geq t_1\,,$$

where

$$w(t) = \int_t^\infty \sum_{i=1}^m q_i(\zeta) f_i\big(x(\sigma_i(\zeta))\big)\, d\zeta\,.$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists $t_2 \geq t_1$, such that $R(t) - R(t_1) \geq \frac{1}{2} R(t)$ for $t \geq t_2$. Then

$$x(t) \geq \frac{1}{2} R(t) w^{1/\gamma}(t)\,. \tag{2.8}$$

Computing the derivative of w, we have

$$w'(t) = -\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big).$$

Thus w is non-negative and non-increasing. Since $x > 0$, by (A3), $f_i\big(x(\sigma_i(t))\big) > 0$, and by (A2), it follows that $\sum_{i=1}^m q_i(t) f_i\big(x(\sigma_i(t))\big)$ cannot be identically zero in any interval $[b,\infty)$; thus w' cannot be identically zero, and w can not be constant on any interval $[b,\infty)$. Therefore $w(t) > 0$ for $t \geq t_1$. Computing the derivative,

$$\big(w^{1-\beta/\gamma}(t)\big)' = \Big(1 - \frac{\beta}{\gamma}\Big) w^{-\beta/\gamma}(t) w'(t)\,. \tag{2.9}$$

Integrating (2.9) from t_2 to t, and using that $w > 0$, we have

$$\begin{aligned} w^{1-\beta/\gamma}(t_2) &\geq \Big(1 - \frac{\beta}{\gamma}\Big)\Big[-\int_{t_2}^t w^{-\beta/\gamma}(\eta) w'(\eta)\, d\eta\Big] \\ &= \Big(1 - \frac{\beta}{\gamma}\Big)\Big[\int_{t_2}^t w^{-\beta/\gamma}(\eta)\Big(\sum_{i=1}^m q_i(\eta) f_i\big(x(\sigma_i(\eta))\big)\Big)\, d\eta\Big]\,. \end{aligned} \tag{2.10}$$

Next we find a lower bound for the right-hand side of (2.10), independent of the solution x. By (A3), (2.1), (2.6), and (2.8), we have

$$\begin{aligned} f_i(x(t)) &= f_i\big(x(t)\big)\frac{x^\beta(t)}{x^\beta(t)} \geq \frac{f_i\big(\delta R(t)\big)}{(\delta R(t))^\beta} x^\beta(t) \\ &\geq \frac{f_i\big(\delta R(t)\big)}{(\delta R(t))^\beta}\Big(\frac{R(t) w^{1/\gamma}(t)}{2}\Big)^\beta = \frac{f_i\big(\delta R(t)\big)}{(2\delta)^\beta} w^{\beta/\gamma}(t) \quad \text{for } t \geq t_2\,. \end{aligned}$$

Since w is non-increasing, $\beta/\gamma > 0$, and $\sigma_i(\eta) < \eta$, it follows that

$$f_i\big(x(\sigma_i(\eta))\big) \geq \frac{f_i\big(\delta R(\sigma_i(\eta))\big)}{(2\delta)^\beta} w^{\beta/\gamma}(\sigma_i(\eta)) \geq \frac{f_i\big(\delta R(\sigma_i(\eta))\big)}{(2\delta)^\beta} w^{\beta/\gamma}(\eta)\,. \tag{2.11}$$

Going back to (2.10), we have

$$w^{1-\beta/\gamma}(t_2) \geq \frac{\left(1-\frac{\beta}{\gamma}\right)}{(2\delta)^{\beta}}\Big[\int_{t_2}^{t}\sum_{i=1}^{m} q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\,d\eta\Big]\,. \tag{2.12}$$

Since $(1-\beta/\gamma)>0$, by (2.7) the right-hand side approaches ∞ as $t\to\infty$. This contradicts (2.12) and completes the proof of sufficiency for eventually positive solutions.

For an eventually negative solution x, we introduce the variables $y=-x$ and $g_i(y)=-f_i(y)$. Then y is an eventually positive solution of (1.1) with g_i instead of f_i. Note that g_i satisfies (A3) and (2.6) so can apply the above process for the solution y.

Next we show the necessity part by a contrapositive argument. When (2.7) does not hold we find a eventually positive solution that does not converge to zero. If (2.7) does not hold for some $\delta>0$, then for each $\epsilon>0$ there exists $t_1\geq t_0$ such that

$$\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(\delta R(\sigma_i(\zeta)))\,d\zeta \leq \epsilon/2 \tag{2.13}$$

for all $\eta\geq t_1$. Note that t_1 depends on δ. We define the set of continuous functions

$$M=\{x\in C([0,\infty)) : (\epsilon/2)^{1/\gamma}\big(R(t)-R(t_1)\big)\leq x(t)\leq \epsilon^{1/\gamma}\big(R(t)-R(t_1)\big),\ t\geq t_1\}\,.$$

We define an operator Φ on M by

$$(\Phi x)(t)=\begin{cases}0 & \text{if } t\leq t_1\\ \int_{t_1}^{t}\Big[\frac{1}{r(\eta)}\Big[\epsilon/2+\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(x(\sigma_i(\zeta)))\,d\zeta\Big]\Big]^{1/\gamma} d\eta & \text{if } t>t_1\,.\end{cases}$$

Note that when x is continuous, Φx is also continuous on $[0,\infty)$. If x is a fixed point of Φ, i.e. $\Phi x=x$, then x is a solution of (1.1).
First we estimate $(\Phi x)(t)$ from below. For $x\in M$, we have

$$0\leq \epsilon^{1/\gamma}\big(R(t)-R(t_1)\big)\leq x(t).$$

By (A3), we have $0\leq f_i(x(\sigma_i(\eta)))$ and by (A2) we have

$$(\Phi x)(t)\geq 0+\int_{t_1}^{t}\Big[\frac{1}{r(\eta)}[\epsilon/2+0+0]\Big]^{1/\gamma} d\eta=(\epsilon/2)^{1/\gamma}\big(R(t)-R(t_1)\big)\,.$$

Now we estimate $(\Phi x)(t)$ from above. For x in M, by (A2) and (A3), we have

$$f_i(x(\sigma_i(\zeta)))\leq f_i\big(\delta R(\sigma_i(\zeta))\big).$$

Then by (2.13),

$$\begin{aligned}(\Phi x)(t)&\leq \int_{t_1}^{t}\Big[\frac{1}{r(\eta)}\Big[\epsilon/2+\int_{\eta}^{\infty}\sum_{i=1}^{m} q_i(\zeta) f_i(\delta R(\sigma_i(\zeta)))\,d\zeta\Big]\Big]^{1/\gamma} d\eta\\ &\leq \epsilon^{1/\gamma}\big(R(t)-R(t_1)\big)\,.\end{aligned}$$

Therefore, Φ maps M to M.

Next we find a fixed point for Φ in M. Let us define a sequence of functions in M by the recurrence relation

$$u_0(t) = 0 \quad \text{for } t \geq t_0,$$

$$u_1(t) = (\Phi u_0)(t) = \begin{cases} 0 & \text{if } t < t_1 \\ \epsilon^{1/\gamma}\big(R(t) - R(t_1)\big) & \text{if } t \geq t_1 \end{cases},$$

$$u_{n+1}(t) = (\Phi u_n)(t) \quad \text{for } n \geq 1,\ t \geq t_1\,.$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using that f is non-decreasing and mathematical induction, we can show that $u_{n+1}(t) \geq u_n(t)$. Therefore, the sequence $\{u_n\}$ converges pointwise to a function u. Using the Lebesgue Dominated Convergence Theorem, we can show that u is a fixed point of Φ in M. This shows under assumption (2.13), there a non-oscillatory solution that does not converge to zero. This completes the proof. □

In the next theorem, we assume the existence of a differentiable function σ_0 such that

$$0 < \sigma_0(t) \leq \sigma_i(t), \quad \exists \alpha > 0 : \alpha \leq \sigma_0'(t), \quad \text{for } t \geq t_0,\ i = 1, 2, \dots, m. \tag{2.14}$$

Also we assume that there exists a constant β, the quotient of two positive odd integers, with $\gamma < \beta$, such that

$$\frac{f_i(u)}{u^\beta} \text{ is non-decreasing for } 0 < u,\ i = 1, 2, \dots, m\,. \tag{2.15}$$

For example $f_i(u) = |u|^\alpha \operatorname{sgn}(u)$, with $\beta < \alpha$ satisfies this condition.

Theorem 2.3. *Under assumptions* (A1)–(A4), (2.14), (2.15), *and* $r(t)$ *is non-decreasing, every solution of* (1.1) *is oscillatory if and only if*

$$\int_{t_1}^\infty \Big[\frac{1}{r(\eta)} \int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\, d\zeta\Big]^{1/\gamma} d\eta = \infty\,. \tag{2.16}$$

Proof. We prove sufficiency by contradiction. Initially assume that x is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 2.1, there exists $t_1 \geq t_0$ such that: $x(\sigma_i(t)) > 0$, $x(\tau(t)) > 0$, and $r(t)\big(x'(t)\big)^\gamma$ is positive and non-increasing. Since $r(t) > 0$ so $x(t)$ is increasing for $t \geq t_1$. From (A3), $x(t) \geq x(t_1)$ and (2.15), we have

$$f_i(x(t)) \geq \frac{f_i(x(t))}{x^\beta(t)} x^\beta(t) \geq \frac{f_i(x(t_1))}{x^\beta(t_1)} x^\beta(t)\,.$$

By (A1) there exists a $t_2 \geq t_1$ such that $\sigma_i(t) \geq t_1$ for $t \geq t_2$. Then

$$f_i\big(x(\sigma_i(t))\big) \geq \frac{f_i(x(t_1))}{x^\beta(t_1)} x^\beta(\sigma_i(t)) \quad \forall t \geq t_2\,. \tag{2.17}$$

Using this inequality, (2.5), that $\sigma_i(t) \geq \sigma_0(t)$ which is an increasing function, and that z is increasing, we have

$$r(t)\big(x'(t)\big)^\gamma \geq \frac{x^\beta(\sigma_0(t))}{x^\beta(t_1)} \int_t^\infty \sum_{i=1}^m q_i(\eta) f_i(x(t_1))\, d\eta\,,$$

for $t \geq t_2$. From $r(t)\big(z'(t)\big)^\gamma$ being non-increasing and $\sigma_0(t) \leq t$, we have

$$r(\sigma_0(t))\big(x'(\sigma_0(t))\big)^\gamma \geq r(t)\big(x'(t)\big)^\gamma.$$

We use this in the left-hand side of the above inequality. Then dividing by $r(\sigma_0(t)) > 0$, raising both sides to the $1/\gamma$ power, and dividing by $z^{\beta/\gamma}(\sigma_0(t)) > 0$, we have

$$\frac{x'(\sigma_0((t))}{x^{\beta/\gamma}(\sigma_0(t))} \geq \Big[\frac{1}{r(\sigma_0(t))x^\beta(t_1)}\int_t^\infty \sum_{i=1}^m q_i(\eta)f_i(x(t_1))\,d\eta\Big]^{1/\gamma}, \text{ for } t \geq t_2.$$

Multiplying the left-hand side by $\sigma_0'(t)/\alpha \geq 1$, and integrating from t_1 to t,

$$\frac{1}{\alpha}\int_{t_1}^t \frac{z'(\sigma_0(\eta))\sigma_0'(\eta)}{z^{\beta/\gamma}(\sigma_0(\eta))}\,d\eta \geq \frac{1}{z^{\beta/\gamma}(t_1)}\int_{t_1}^t \Big[\frac{1}{r(\sigma_0(\eta))}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)f_i(x(t_1))\,d\zeta\Big]^{1/\gamma}\,d\eta. \tag{2.18}$$

On the left-hand side, since $\gamma < \beta$, integrating, we have

$$\frac{1}{\alpha(1-\beta/\gamma)}\Big[z^{1-\beta/\gamma}(\sigma_0(\eta))\Big]_{s=t_2}^t \leq \frac{1}{\alpha(\beta/\gamma-1)}z^{1-\beta/\gamma}(\sigma_0(t_2))\,.$$

On the right-hand side of (2.18), we use that $\min_{1\leq i\leq m} f_i(z(t_1)) > 0$ and that $r(\sigma_0(s)) \leq r(s)$, to conclude that (2.16) implies the right-hand side approaching ∞, as $t \to \infty$. This contradiction implies that the solution x cannot be eventually positive.

For eventually negative solutions, we use the same change of variables as in Theorem 2.2, and proceed as above.

To prove the necessity part we assume that (2.16) does not hold, and obtain an eventually positive solution that does not converge to zero. If (2.16) does not hold, then for each $\epsilon > 0$ there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^\infty \Big[\frac{1}{r(\eta)}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\,d\zeta\Big]^{1/\gamma}\,d\eta < \epsilon/2(f_i(\epsilon))^{1/\gamma} \quad \forall t \geq t_1\,. \tag{2.19}$$

Let us consider the set of continuous functions

$$M = \{x \in C([0,\infty)) : \epsilon/2 \leq x(t) \leq \epsilon \text{ for } t \geq t_1\}$$

Then we define the operator

$$(\Phi x)(t) = \begin{cases} 0 & \text{if } t \leq t_1, \\ \epsilon/2 + \int_{t_1}^t \frac{1}{r(\eta)}\Big[\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)f_i(x(\sigma_i(\zeta)))\,d\zeta\Big]^{1/\gamma}\,d\eta & \text{if } t > t_1\,. \end{cases}$$

Note that if x is continuous, Φx is also continuous at $t = t_1$. Also note that if $\Phi x = x$, then x is solution of (1.1).

First we estimate $(\Phi x)(t)$ from below. Let $x \in M$. By $0 < \epsilon/2 \leq x$, we have $(\Phi x)(t) \geq \epsilon/2 + 0 + 0$, on $[t_1, \infty)$.

Now we estimate $(\Phi x)(t)$ from above. Let $x \in M$. Then $x \leq \epsilon$ and by (2.19), we have

$$(\Phi x)(t) \leq \epsilon/2 + \big(f_i(\epsilon)\big)^{1/\gamma}\int_{t_1}^t \Big[\frac{1}{r(\eta)}\int_\eta^\infty \sum_{i=1}^m q_i(\zeta)\,d\zeta\Big]^{1/\gamma}\,d\eta \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore Φ maps M to M. To find a fixed point for Φ in M, we define a sequence of functions by the recurrence relation

$$\begin{gathered} u_0(t) = 0 \quad \text{for } t \geq t_0, \\ u_1(t) = (\Phi u_0)(t) = 1 \quad \text{for } t \geq t_1, \\ u_{n+1}(t) = (\Phi u_n)(t) \quad \text{for } n \geq 1,\ t \geq t_1 . \end{gathered}$$

Note that for each fixed t, we have $u_1(t) \geq u_0(t)$. Using that f is non-decreasing and mathematical induction, we can prove that $u_{n+1}(t) \geq u_n(t)$. Therefore $\{u_n\}$ converges pointwise to a function u in M. Then u is a fixed point of Φ and a positive solution to (1.1). This completes the proof. $\square$

Example 2.4. Consider the delay differential equations

$$\Big(e^{-t}\big(x'(t)\big)^{11/3}\Big)' + \frac{1}{t+1}(x(t-2))^{1/3} + \frac{1}{t+2}(x(t-1))^{5/3} = 0 . \tag{2.20}$$

Here

$$\begin{gathered} \gamma = 11/3,\ r(t) = e^{-t},\ \sigma_1(t) = t-2,\ \sigma_2(t) = t-1, \\ R(t) = \int_0^t e^{5s/3}\, ds = \frac{3}{5}\big(e^{5t/3} - 1\big), \\ f_1(x) = x^{1/3} \text{ and } f_2(x) = x^{5/3}. \end{gathered}$$

For $\beta = 7/3$, we have $0 < \max\{\alpha_1, \alpha_2\} < \beta < \gamma$, and

$$f_1(x)/x^\beta = x^{-2} \text{ and } f_2(x)/x^\beta = x^{-2/3}$$

which both are decreasing functions. To check (2.7) we have

$$\begin{aligned} \int_0^\infty \sum_{i=1}^m q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\, d\eta &\geq \int_0^\infty \sum_{i=1}^m q_i(\eta) f_i(\delta R(\sigma_i(\eta)))\, d\eta \\ &\geq \int_0^\infty q_1(\eta) f_1(\delta R(\sigma_1(\eta)))\, d\eta \\ &= \int_0^\infty \frac{1}{\eta+1}\Big(\delta \frac{3}{5}\big(e^{5(\eta-2)/3} - 1\big)\Big)^{1/3} d\eta = \infty \quad \forall \delta > 0, \end{aligned}$$

since the integral approaches $+\infty$ as $\eta \to +\infty$. So, all the conditions of Theorem 2.2 hold, and therefore, each solution of (2.20) is oscillatory or converges to zero.

Example 2.5. Consider the neutral differential equations

$$\Big(\big(x'(t)\big)^{1/3}\Big)' + t(x(t-2))^{7/3} + (t+1)(x(t-1))^{11/3} = 0 . \tag{2.21}$$

Here

$$\begin{gathered} \gamma = 1/3,\ r(t) = 1,\ \sigma_1(t) = t-2,\ \sigma_2(t) = t-1, \\ f_1(v) = v^{7/3} \text{ and } f_2(v) = v^{11/3}. \end{gathered}$$

For $\beta = 5/3$, we have $\min\{\alpha_1, \alpha_2\} > \beta > \gamma$, and

$$f_1(x)/x^\beta = x^{2/3} \text{ and } f_2(x)/x^\beta = x^2$$

which both are increasing functions. To check (2.16) we have

$$\begin{aligned}\int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta &\geq \int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\int_{\eta}^{\infty}\sum_{i=1}^{m}q_i(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta\\ &\geq \int_{t_0}^{\infty}\Big[\frac{1}{r(\eta)}\int_{\eta}^{\infty}q_1(\zeta)\,d\zeta\Big]^{1/\gamma}d\eta\\ &\geq \int_{2}^{\infty}\Big[\int_{\eta}^{\infty}\zeta\,d\zeta\Big]^{3}d\eta=\infty.\end{aligned}$$

So, all the conditions of of Theorem 2.3 hold. Thus, each solution of (2.21) is oscillatory or converges to zero.

Open Problem

Based on this work and [5, 6, 8, 13, 16, 18, 20, 19, 26, 31] an open problem that arises is to establish necessary and sufficient conditions for the oscillation of the solutions of the second-order nonlinear neutral differential equation

$$\Big(r(t)\big(z'(t)\big)^{\gamma}\Big)'+\sum_{i=1}^{m}q_i(t)f_i\big(x(\sigma_i(t))\big)=0,\ \text{for } t\geq t_0,$$

where $z(t)=x(t)+p(t)x(\tau(t))$ for $p\in C(\mathbf{R},\mathbf{R})$.

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