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ON THE STABILITY OF THE ALTERNATIVE METHOD

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Abstract. The stability of the alternative method is investigated. An optimization of the volume of computation for the numerical approximation of a solution of the equation $Lu = Nu$ is also given.

1. Introduction

The stability of the fixed point iteration procedures has been investigated by A.M. Harder and T.L. Hicks [1]:

Definition 1. *Let (X, d) be a metric space, $T : X \to X$, $x_0 \in X$ and the iteration procedure $x_{n+1} = f(T, x_n)$. If $x_n \to p$, where p is a fixed point of T, let $y_n \in X$ and $\varepsilon_n = d(y_{n+1}, f(T, y_n))$. If $\varepsilon_n \to 0$ implies $y_n \to p$ then the iteration procedure f is T-stable relating to T.*

If T is a contraction, a theorem of Ostrowski [1] shows that the iteration procedure $f(T, x_n) = Tx_n$ is T-stable:

Theorem 1. *Let $T : X \to X$ be a contraction on the complete metric space (X, d). Let p a fixed point of T, $x_0 \in X$, $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$ be. Let $y_n \in X$ and $\varepsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, \ldots$. Then*

1. $d(p, y_{n+1}) \leq (1 - k)^{-1}(\varepsilon_n + kd(y_n, y_{n+1}))$
2. $d(p, y_{n+1}) \leq d(p, x_{n+1}) + k^{n+1}d(x_0, y_0) + \sum_{i=0}^{n} k^{n-i}\varepsilon_i$
3. $y_n \to p$ *if and only if* $\varepsilon_n \to 0$.

If X is a Banach space, $E : X_E \to X$ is a linear operator and $N : X_N \to X$ is a nonlinear operator, let us consider the equation $Eu = Nu$, $u \in X_E \cap X_N$.

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If E is an invertible operator, this equation is equivalent to $u = E^{-1}Nu$, a fixed point problem for $T = E^{-1}N$. If T is a contraction, Theorem 2 applies. If T is not a contraction or E is not invertible, the equation $Eu = Nu$ is studied by the alternative (Lyapunov-Schmidt) method. Using an idea of Sanchez [2] it is easy to conclude that the alternative method is T-stable. An optimization of the volume of computation for the numerical approximation of the solution of the equation $Eu = Nu$ by the alternative method is also given.

2. The stability of the alternative method

Let X be a Banach space, $E : X_E \to X$ a linear operator, $N : X_N \to X$ a nonlinear operator and we suppose that

a): there exists a projection $P : X \to X$ such that $X = R(P) \oplus R(I - P)$ and $PE = EP$

b): there exists $H : R(I - P) \to R(I - P)$, a linear operator such that

$$H(I - P)Eu = (I - P)u \text{ for all } u \in X_E$$
$$EH(I - P)Nu = (I - P)Nu \text{ for all } u \in X_N$$

c): all the fixed points of $P + H(I - P)N$ are in X_E.

Theorem 2. *$Eu = Nu$ if and only if*

$$(I - P)u = H(I - P)Nu$$
$$P(EPu - Nu) = 0$$

Let $D : R(P) \to R(P)$ be a linear, invertible and with bounded inverse operator. For $a, b > 0$ we define

$$C = \{(v, w)\,|v \in R(P), ||v - v_0|| \leq a, w \in R(I - P), ||w|| \leq b\}$$

where $v_0 \in R(P)$ is an approximation of the solution of the equation $Eu = Nu$. On C we define $||(v, w)|| = ||v|| + ||w||$. Let $p \in \mathbb{N}, \mathrm{u} = \mathrm{v} + \mathrm{w}$ where $(v, w) \in C$, $w^0 = w, w^i = H(I - P)N(v + w^{i-1})$, for $i = 1, 2, ..., p + 1$ and $W = w^{p+1}$. Let $V = v - D^{-1}P(Ev - N(v + W))$. We define an operator on C by $T(v, w) = (V, W)$. We remark that in the paper of Sanchez [2], $p = 0$.

Theorem 3. *If*

1. *there exists* $\eta \geq 0$ *such that* $(v, w) \in C$ *implies* $\|N(v+w)\| \leq \eta$
2. $H(I-P)$ *is a bounded operator and* $\|H(I-P)\| \leq b/\eta$
3. *there exists* $\sigma \geq 0$ *such that* $\|D^{-1}\| \sigma < 1$ *and if* $(v_1, w), (v_2, w) \in C$ *then* $\|D(v_1 - v_2) - P(Ev_1 - N(v_1 + w) - Ev_2 + N(v_2 + w))\| \leq \sigma \|v_1 - v_2\|$
4. *there exists* $\gamma \geq 0$ *such that* $(v_0, w) \in C$ *implies*

$$\left\|D^{-1} P(Ev_0 - N(v_0 + w))\right\| \leq \gamma \leq (1 - \left\|D^{-1}\right\| \sigma)a,$$

then T *applies* C *into* C.

Proof. From $(v, w) \in C$ we have $(v, w^k) \in C$ for all k thus

$$\|W\| = \|H(I-P)N(v + w^p)\| \leq b/\eta \cdot \eta = b$$

We have also

$$\|V - v_0\| \leq \left\|D^{-1}\right\| \|D(v - v_0) - P(Ev - N(v + W)) - P(Ev_0 - N(v_0 + W))\| \leq$$
$$\leq \left\|D^{-1}\right\| \sigma \|v - v_0\| + (1 - \left\|D^{-1}\right\| \sigma)a \leq a$$

Theorem 4. *If the conditions 1-4 of theorem 4 hold and*

5) *there exists* $L > 0$ *such that if* $u_i = v_i + w_i$, $(v_i, w_i) \in C, i = 1, 2$ *then* $\|Nu_1 - Nu_2\| \leq L \|u_1 - u_2\|$

6) $\mu = \left\|D^{-1}\right\| \sigma + (1 + \left\|D^{-1}P\right\| L)(\theta + \ldots + \theta^{p+1}) < 1$, *where* $\theta = \|H(I-P)\| L$,

then T *is a contraction.*

Proof. Let $T(v_i, w_i) = (V_i, W_i), i = 1, 2$. We have

$$\|W_1 - W_2\| \leq \|H(I-P)\| L(\|v_1 - v_2\| + \|w_1^p - w_2^p\|)$$

But

$$\|w_1^p - w_2^p\| \leq (\|v_1 - v_2\| + \left\|w_1^{p-1} - w_2^{p-1}\right\|) \|H(I-P)\| L$$

thus

$$\|W_1 - W_2\| \leq \|v_1 - v_2\| (\theta + \ldots + \theta^{p+1}) + \theta^{p+1} \|w_1 - w_2\|$$

Consequently,

$$||V_1 - V_2|| + ||W_1 - W_2|| \le \left[\left\|D^{-1}\right\| \sigma + (1 + \left\|D^{-1}P\right\| L)(\theta + ... + \theta^{p+1})\right] ||v_1 - v_2|| +$$
$$+(1 + \left\|D^{-1}P\right\| L)\theta^{p+1} ||w_1 - w_2|| \le \mu (||v_1 - v_2|| + ||w_1 - w_2||)$$

Hence T has an unique fixed point $(v, w) = (V, W) \in C$ that may be obtained by the iteration procedure $(v_{k+1}, w_{k+1}) = T(v_k, w_k)$.

Theorem 5. *If the conditions of the theorems 4,5 hold, then* $u = V + W$ *is a solution of the equation* $Eu = Nu$.

Proof. We have $w^1 = W, ..., w^p = W$, that is $W = H(I - P)N(V + W)$. Then $V = V - D^{-1}P(EV - N(V + W))$ and consequently, $P(EV - N(V + W)) = 0$ and $Eu = Nu$ from theorem 3.

3. The optimization of the numerical computation of the solutions

We approximate the 2π–periodic solutions of the equation

$$-u''(t) = f(t, u(t))$$

Let X be the Banach space of 2π–periodic, continuous functions $u : \mathbb{R} \to \mathbb{R}$, $||u|| = \sup_{t \in [0,2\pi]} |u(t)|$, f a continuous, 2π– periodic function on t, differentiable in u, with locally bounded derivative. Let $X_E = H^2(0, 2\pi), X_N = X, Eu = -u'', Nu = f(\cdot, u)$.

If $u \in X$ let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

his Fourier series. We define, for $m \in \mathbb{N}$,

$$P_m u = \frac{a_0}{2} + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt)$$

$$H(I - P_m)u = \sum_{k=m+1}^{\infty} (a_k \cos kt + b_k \sin kt)/k^2$$

¿From [3] we have $||H(I - P_m)|| \to 0$ when $m \to \infty$. For an approximation $v_0 \in P_m X$ we define the sequence $w^s = H(I - P_m)N(v + w^{s-1})$, $w^0 = 0$, for $s = 1, 2, ..., p + 1$. If m is sufficiently great then $(v, w^s) \in C$ if $||v - v_0|| \le a$, $v \in P_m X$. The second equation $P(EPu - Nu) = 0$ becomes an equation for the Fourier coefficients of v, $F(c) = 0$, where $c = (a_0/2, a_1, b_1, ...a_m, b_m)$.

If the Jacobian $J(c_0)$ of F in v_0 is invertible, let $D = J(c_0)$ and we use a theorem of Urabe [4]:

Theorem 6. *Let us consider the system* $F(c) = 0, F = (F_1, ..., F_n)$, $c = (c_1, ..., c_n)$ *for* $n \in \mathbb{N}$. *We suppose that* $F \in C^1(\Omega)$ *and that there exists* $k \in [0, 1)$ *and* $\delta > 0$ *such that*

1. $\Omega_\delta = \{c \in P_m X \, |||c - c_0|| \leq \delta\} \subset \Omega$
2. $||J(c) - J(c_0)|| \leq k/M$
3. $Mr/(1-k) \leq \delta$

where $M \geq \left\|J^{-1}(c_0)\right\|, r \geq ||F(c_0)||$.

Then the system $F(c) = 0$ *has an unique solution* $\bar{c} \in \Omega_\delta$ *and* $||\bar{c} - c_0|| \leq Mr/(1-k)$.

For a sufficiently great m the conditions of theorems 4,5 are consequences of the hypothesis of the Urabe's theorem. Hence the 2π–periodic solution u of the equation $-u'' = f(t, u)$ is $u = V + W$, where W is obtained by a fixed point iteration procedure for $P_m + H(I - P_m)N$ and V is obtained by the Newton's algorithm for the system $F(c) = 0$ (every step requires the iterations for W).

We consider the following error sources:

a) The computation of the Fourier coefficients (cf. [5]) of $w^s = H(I - P_m)N(v + w^{s-1})$.

Theorem 7. *If* $g(t)$ *is* p *times continuously differentiable,* 2π*–periodic and his Fourier series is*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

then the Fourier coefficients may be approximated by

$$a_k \approx \frac{1}{N}\sum_{i=1}^{2N} g(t_i) \cos nt_i \qquad b_k \approx \frac{1}{N}\sum_{i=1}^{2N} g(t_i) \sin nt_i$$

where $t_i = (2i-1)\pi/(2N), i = 1, ..., 2N$ *and* $k = 1, ..., N-1$ *and the approximation error is*

$$2\sigma_p(N-1)\left[\frac{1}{2\pi}\int_0^{2\pi} g^{(p)}(t)^2 dt\right]^{1/2}$$

where

$$\sigma_p(N-1) = \sqrt{2}\left[\frac{1}{N^{2p}} + \frac{1}{(N+1)^{2p}} + ...\right]^{1/2} < \sqrt{\frac{2}{2p-1}}(N-1)^{-p+1/2}$$

b) The truncation of the Fourier series at rank $N-1$ (cf. [5]). We have

$$\left|g(t) - \frac{a_0}{2} - \sum_{k=1}^{N-1}(a_k\cos kt + b_k \sin kt)\right| \le \sigma_p(N-1)\left[\frac{1}{2\pi}\int_0^{2\pi} g^{(p)}(t)^2\, dt\right]^{\frac{1}{2}}$$

Consequently, if w^s is approximated by $\widetilde{w}^s$ we have

$$\|w^s - \widetilde{w}^s\| \le 2\sqrt{N-m}\sigma_p(N-1)\left[\tfrac{1}{2\pi}\int_0^{2\pi} N\left(v + w^{s-1}\right)^{(p)}(t)^2\, dt\right]^{\frac{1}{2}}(\sigma(m) - \sigma(N)) +$$
$$+\sigma_p(N-1)\left[\tfrac{1}{2\pi}\int_0^{2\pi} N\left(v + w^{s-1}\right)^{(p)}(t)^2\, dt\right]^{\frac{1}{2}}$$

where

$$\sigma(m) = \left(\sum_{i=m+1}^{\infty}\frac{1}{i^2}\right)^{\frac{1}{2}}$$

At every step we have an error $\varepsilon_s \le \mathcal{K}(N_s - 1)^{-p+1/2}$, where

$$\mathcal{K} = \sqrt{\frac{2}{2p-1}}\left(1 + 2\sqrt{N-m}\right)\sigma(m)\max_{s\le S}\left[\frac{1}{2\pi}\int_0^{2\pi} N\left(v + w^{s-1}\right)^{(p)}(t)^2\, dt\right]^{\frac{1}{2}}$$

if $N_s \le N$ for $s = 1, 2, ..., S$.

The whole error for S iterations is

$$\|W - \widetilde{w}^{s+1}\| \le \frac{\theta^{s+1}b}{1-\theta} + \sum_{i=0}^{S}\frac{\theta^S}{\theta^i}\frac{\mathcal{K}}{(N_i-1)^{p-1/2}} \equiv \varepsilon_0$$

for a computational effort proportional to $2(N_0 + ... + N_S)$.

Our problem is now to minimize this effort for a given error ε_0. Let $S \in \mathbb{N}$. We have to minimize $N_0 + ... + N_S$ if

$$\sum_{i=0}^{S}\frac{1}{\theta^i(N_i-1)^{p-1/2}} = \frac{\frac{\varepsilon_0}{\theta^S} - \frac{b\theta}{1-\theta}}{\mathcal{K}} \equiv A_S$$

By the Lagrange multipliers rule, let

$$L = N_0 + \ldots + N_S + \lambda \left(\sum_{i=0}^{S} \frac{1}{\theta^i (N_i - 1)^{p-1/2}} - A_S \right)$$

We have the system

$$1 - \frac{\lambda \left(p - \frac{1}{2}\right)}{\theta^i (N_i - 1)^{p+1/2}} = 0 \text{ for } i = 0, 1, \ldots, S$$

$$\sum_{i=0}^{S} \frac{1}{\theta^i (N_i - 1)^{p-1/2}} = A_S$$

from where

$$N_i = 1 + \frac{\left(\theta^{-\frac{2(S+1)}{2p+1}} - 1\right)^{\frac{2}{2p-1}}}{A_S^{\frac{2}{2p-1}} \theta^{i \frac{2}{2p+1}} \left(\theta^{-\frac{2}{2p+1}} - 1\right)^{\frac{2}{2p+1}}}$$

for $i = 0, 1, \ldots, S$. Now we can choose S for which the computing effort is minimum.

As an example, let us consider the equation (cf. [5])

$$u'' = \sin t - u^3 (t)$$

For $m = 1, p = 2$, $\theta = 0.4$, $N_0 = 4$, $N_1 = 4$, $N_2 = 5$, $N_3 = 7$, $N_4 = 10$, $N_5 = 14$, $N_6 = 20$ and at every step the fixed point W was obtained by 64 evaluations of $Nu \equiv \sin u - u^3$ (instead of 120 evaluations if at every step we choose $N = 20$, for the same precision).

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