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ON BROWDER'S FIXED POINT THEOREM

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1. Introduction

In 1968 F.E. Browder stated the following fixed point principle in topological vector spaces:

Theorem 1.1. ([3]) *Let X be a Hausdorff topological vector space and K be a nonempty, compact, convex subset of X. Let F be a multivalued operator such that $F : K \to P_{cv}(K)$ and for each $y \in K$ the set $F^{-1}(y) := \{x \in K \mid y \in F(x)\}$ is open. Then there exists x_0 in K such that $x_0 \in F(x_0)$.*

The key tool in his proof is the compactness of the set K, which is used to construct a continuous selection for T and, in the same time, permett to apply Schauder's fixed point theorem.

The first purpose of this note is to give another proof for this theorem, using the notion of locally selectionable multivalued operator. The virtue of this proof is that one use the compactness of K only for the application of Schauder's theorem.

On the other side, using the property of decomposability as substitute for convexity in Theorem 1.1, we get the second main result of the paper: a selection principle for multivalued operators with decomposable values.

We follow the notations and symbols from [7].

2. Main results

The concept of locally selectionable multivalued operator has been introduced because these set-valued maps do posses continuous selection on paracompact topological spaces.

Definition 2.1. Let X, Y be two nonempty sets and $F : X \to P(Y)$ a multivalued operator. Then a singlevalued operator $f : X \to Y$ is a selection for F iff $f(x) \in F(x)$, for each $x \in X$.

Definition 2.2. Let X be a nonempty set and $F : X \to P(X)$ a multivalued operator. Then $x_0 \in X$ is called a fixed point for F iff $x_0 \in F(x_0)$.

The fixed points set for F will be denoted by $FixF$.

Definition 2.3. ([1]) Let X, Y be two topological spaces. We say that $F : X \to P(Y)$ is locally selectionable at $x_0 \in X$ iff for all $y_0 \in F(x_0)$ there exist an open neighborhood $N(x_0)$ of x_0 and a continuous map $f : N(x_0) \to Y$ such that $f(x_0) = y_0$ and $f(x) \in F(x)$, for all $x \in N(x_0)$. F is said to be locally selectionable if it is locally selectionable at every $x_0 \in X$.

Remark 2.4. ([1]) Any locally selectionable multivalued operator is lower semicontinuous.

The main tools in our proof of the Browder fixed point theorem are:

Lemma 2.5. ([1]) *Let X, Y be two topological spaces and $F : X \to P(Y)$ a multivalued operator. If $F^{-1}(y)$ is open for each $y \in Y$ then F is locally selectionable.*

Lemma 2.6 ([1]) *Let X be a paracompact space and F be a locally selectionable operator with nonempty, convex values from X to a Hausdorff topological vector space Y. Then F has a continuous selection.*

The first result of this note is the following:

Theorem 2.7. *Let X be a paracompact vector space, K a nonempty, compact, convex subset of X and $F : K \to P_{cv}(K)$ a multivalued operator such that for each $y \in K$, $F^{-1}(y)$ is open. Then $FixF \neq \emptyset$.*

Proof. From Lemma 2.5, F is locally selectionable. Lemma 2.6 implies the existence of a continuous selection $f : K \to K$ of F. A simple application of the Schauder's fixed point theorem concludes the proof. □

For the second part of the paper, consider $(T, \mathcal{A}, \mu)$ a complete σ-finite and nonatomic measure space and E a Banach space. Let $L^1(T, E)$ be the Banach space of all measurable functions $u : T \to E$ which are Bochner μ-integrable. We call a set $K \subset L^1(T, E)$ decomposable iff for all $u, v \in K$ and each $t \in \mathcal{A}$ we have that

$u\chi_A + v\chi_{T\setminus A} \in K$, where χ_A stands for the characteristic function of the set A (see also [6]).

An userful result is:

Theorem 2.8. ([4]) *Let K be a bounded, decomposable set of $L^1(T, E)$. Then the Kuratowski's index of the set K is the diameter of K.*

The second main result of this paper is:

Theorem 2.9. *Let E be a Banach space such that $L^1(T, E)$ is separable. Let K be a nonempty, paracompact, decomposable subset of $L^1(T, E)$ and $F : K \to P_{dec}(K)$ be a multivalued operator such that $F^{-1}(y)$ is open, for each $y \in K$. Then F has a continuous selection.*

Proof. For each $y \in K$, $F^{-1}(y)$ is an open subset of K. Since K is compact, the open covering $(F^{-1}(y))_{y \in K}$ admits a locally finite, open refinement, so $K = \bigcup_{j \in J} F^{-1}(y_j)$, $y_i \in K$ for $J \subset \mathbf{N}$. Let $\{\psi_j\}_{j \in J}$ be a continuous partition of unity subordinate to $(F^{-1}(y_j))_{j \in J}$. Using the same construction as in the proof of Lemma 3.1 from [7] (see also Proposition 1.1 - Proposition 1.3 in [5]), we get a continuous function $f : K \to K$, $f(x) = \sum_{j \in J} f_j(x)\chi_j(x)$, where $f_j(x) \in F(x)$ for each $x \in K$. This function is a continuous selection for F. □

Remark 2.10. A "decomposable" version of the Browder's fixed point theorem is an open problem. It is well known that a compact, decomposable subset of $L^1(T, E)$ consists of only one point (see Theorem 2.8). On the other hand, each closed, decomposable subset of $L^1(T, E)$ has the compact fixed point property (see [2]). The problem is if there exists a continuous, compact selection for F.

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