STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLIII, Number 4, December 1998

ON AN INTEGRAL OPERATOR

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Abstract. In this paper we investigate the conditions of univalence for the analicity and univalence in the unit disc of the integral $\int_0^z \left[\frac{g(u)}{u}\right]^\gamma du$.

1. INTRODUCTION

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$

We denote by S the class of the function $f \in A$ which are univalent in U.

The integral operators which transform the class S into S are prezented in the theoremes A and B, which follow.

THEOREM A [2]. If the function f belongs to the class S then for any complex number γ, $|\gamma| \le \frac{1}{4}$ the function

$$F_\gamma(z) = \int_0^z \left[\frac{f(\xi)}{\xi}\right]^\gamma d\xi \tag{1}$$

is in S.

THEOREM B[4]. Let α, β, γ be complex numbers and $h(z) = z + a_2z^2 + \dots$ a regular and univalent function in U. If

(i) $\quad \mathrm{Re}\beta \ge \mathrm{Re}\alpha > 0$

and

(ii) $\quad |\gamma| \le \frac{Re\alpha}{2}$ for $\mathrm{Re}\alpha \in \left(0, \frac{1}{2}\right)$

or

$|\gamma| \le \frac{1}{4}$ for $Re\alpha \in \left[\frac{1}{2}, \infty\right)$.

then the function

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{h(u)}{u}\right)^\gamma du\right]^{\frac{1}{\beta}} \tag{2}$$

belongs to the class S.

2. PRELIMINARIES

We will need the following theorem and lemma for proving our main result.

THEOREM C [3]. Let α be a complex number, $Re\alpha > 0$ and $f(z) = z + a_2 z^2 + \cdots$ be a regular function in U. If

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{3}$$

for all $z \in U$, then for any complex number $\beta, Re\beta \geq Re\alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}} \tag{4}$$

is in the class S.

LEMMA SCHWARZ [1]. If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold

$$|g(z)| \leq |z| \tag{5}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (5) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. MAIN RESULT

THEOREM 1. Let α, γ be complex numbers, Re$\alpha = a > 0$ and the function $h \in S$, $h(z) = z + a_2 z^2 + \ldots$. If

$$\left| \frac{zh'(z)}{h(z)} - 1 \right| \leq 1 \tag{6}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{(2a+1)^{\frac{(2a+1)}{2a}}}{2}, \tag{7}$$

then for any complex number β, Re $\beta \geq a$ the function

$$G_{\beta,\gamma}(z) = \left[\beta \int_0^z \left(\frac{h(u)}{u} \right)^\gamma du \right]^{\frac{1}{\beta}} \tag{8}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[\frac{h(u)}{u}\right]^\gamma du. \tag{9}$$

The function

$$p(z) = \frac{1}{\gamma}\frac{zf''(z)}{f'(z)} \tag{10}$$

where the constant γ satisfies the inequality (7) is regular in U. From (10) and (9) it follows that

$$p(z) = \frac{\gamma}{|\gamma|}\left[\frac{zh'(z)}{h(z)} - 1\right]. \tag{11}$$

Using (11) and (6) we obtain

$$|p(z)| \leq 1 \tag{12}$$

for all $z \in U$. From (11) we have $p(0) = 0$ and applying Schwarz-Lemma we obtain

$$\frac{1}{|\gamma|}\left|\frac{zf''(z)}{f'(z)}\right| \leq |z| \tag{13}$$

for all $z \in U$, and hence, we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\frac{1-|z|^{2a}}{a}|z|. \tag{14}$$

Because $\max\limits_{|z|\leq 1}\frac{1-|z|^{2a}}{a}|z| = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}}$,from (14) and (7) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{15}$$

for all $z \in U$. From (15), (9) and Theorem C it follows that $G_{\beta,\gamma}$ is in the class S. □

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