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ON THE UNIVALENCE OF AN INTEGRAL OPERATOR

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Abstract. In this paper we investigate the conditions of univalence for the analicity and univalence in the unit disc of the integral $\int_0^z \left[\frac{g(u)}{u}\right]^\gamma du$.

1. INTRODUCTION

Let A be the class of analytic functions f in the unit disc $U = \{z \in C; |z| < 1\}$, $f(0) = f'(0) - 1 = 0$ and S be subclass of univalent functions in the class A.

Kim and Merkes [3] investigated the univalence of the integral $\int_0^z \left[\frac{g(\xi)}{\xi}\right]^\gamma d\xi$.

THEOREM A [3]. If the function f belongs to the class S then for any complex number γ, $|\gamma| \leq \frac{1}{4}$ the function

$$F_\gamma(z) = \int_0^z \left[\frac{f(\xi)}{\xi}\right]^\gamma d\xi \tag{1}$$

is in S.

2. PRELIMINARIES

We will need the following theorem and lemma for proving our main result.

THEOREM B [1]. If the function f is regular in the unit disc, $f(z) = z + a_2 z^2 + \ldots$ and

$$\left(1 - |z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{2}$$

for all $z \in U$, then the function f is univalent in U.

LEMMA SCHWARZ [2]. If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold

$$|g(z)| \leq |z| \tag{3}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (3) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3. MAIN RESULT

THEOREM 1. Let γ be a complex number and the function $h \in S$, $h(z) = z + a_2 z^2 + \cdots$. If

$$\left|\frac{zh'(z)}{h(z)} - 1\right| \leq 1 \tag{4}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{3\sqrt{3}}{2} \tag{5}$$

then the function

$$F_\gamma(z) = \int_0^z \left[\frac{h(u)}{u}\right]^\gamma du \tag{6}$$

is in S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[\frac{h(u)}{u}\right]^\gamma du. \tag{7}$$

The function

$$p(z) = \frac{1}{\gamma}\,\frac{zf''(z)}{f'(z)} \tag{8}$$

where the constant γ satisfies the inequality (5) is regular in U. From (8) and (7) it follows that

$$p(z) = \frac{\gamma}{|\gamma|}\left[\frac{zh'(z)}{h(z)} - 1\right]. \tag{9}$$

Using (9) and (4) we have

$$|p(z)| \leq 1 \tag{10}$$

for all $z \in U$. From (9) we obtain $p(0) = 0$ and applying Schwarz-Lemma we have

$$\frac{1}{|\gamma|}\left|\frac{zf''(z)}{f'(z)}\right| \leq |z| \tag{11}$$

for all $z \in U$, and hence, we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\left(1-|z|^2\right)|z|. \tag{12}$$

Because $\max\limits_{|z|\leq 1}\left(1-|z|^2\right)|z| = \frac{2}{3\sqrt{3}}$,from (12) and (5) we obtain

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1. \tag{13}$$

for all $z \in U$. From (13), (7), (6) and Theorem B it follows that F_γ is in the class S. □

References

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