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ON CERTAIN NEW CONDITIONS FOR STARLIKENESS AND STRONGLY-STARLIKENESS

GHEORGHE OROS

Abstract. In this paper we will obtain conditions for starlikeness and strongly-starlikeness starting from the differential subordination of the form:

$$\alpha zp'(z) + p^2(z) \prec h(z), \text{ where } \alpha \geq 0,$$
$$h(z) = \alpha nzq'(z) + q^2(z),$$

and q is a convex function in the unit disc U, with $q(0) = 1$ and Re $q(z) > 0$, $n \geq 1$. We will obtain our results by using the differential subordination method developed in [1], [2], [3].

1. Introduction and preliminaries

Let A_n denote the class of function of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}a^{n+2} + \dots,$$

which are analytic in the unit disc $U = \{z|\ |z| < 1\}$ and let $A = A_1$.

We let $\mathcal{H}[a, n]$ denote the class of analytic functions in U of the form:

$$f(z) = z + a_n z^n + a_{n+1}z^{n+1} + \dots, \quad z \in U$$

and let

$$S^* = \left\{f \in A,\ \text{Re}\,\frac{zf'(z)}{f(z)} > 0,\ z \in U\right\}$$

be the class of starlike functions in the unit disc U.

For $\lambda \in (0, 1]$ let

$$S^*[\lambda] = \left\{f \in A|\ \left|\arg\frac{zf'(z)}{f(z)}\right| < \lambda\frac{\pi}{2},\ z \in U\right\}$$

denote the class of strongly-starlike functions.

We will need the following notions and lemmas to prove our main results.

If f and F are analytic functions in U, then we say that f is subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there is a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, for $z \in U$ and if $f(z) = F(w(z))$, $z \in U$. If F is univalent then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Lemma A ([1], [2], [3]). *Let q be univalent in $\overline{U}$ with $q'(\zeta) \neq 0$, $|\zeta| = 1$, $q(0) = a$ and let $p(z) = a + p_n z + \dots$ be analytic in U, $p(z) \not\equiv a$ and $n \geq 1$.*

If $p \not\prec q$ then there exist $z_0 \in U$, $\zeta_0 \in \partial U$ and $m \geq n$ such that:

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0(p'(z_0) = m\zeta_0 q'(\zeta_0)$.

The function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$ is analytic and univalent in U for any $t \geq 0$ and if $L(z,t_1) \prec L(z,t_2)$ where $0 \leq t_1 \leq t_2$.

Lemma B ([7]). *The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in (0,1]$ and $M > 0$ such that:*

(i) $L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for every $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.

(ii) There is a function $p(z,t)$ analytic in U for any $t \geq 0$ measurable in $[0,\infty)$ for any $z \in U$ with $\operatorname{Re} p(z,t) > 0$ for $z \in U$, $t \geq 0$ so that

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z} p(z,t) \text{ for } |z| < r$$

and for almost any $t \geq 0$.

2. Main results

Theorem 1. *Let $\alpha \geq 0$ and let q be a convex function in the unit disc U, with $q(0) = 1$, $\operatorname{Re} q(z) > 0$ and let*

$$h(z) = \alpha n z q'(z) + q^2(z), \tag{1}$$

where n is a positive integer.

If $p \in \mathcal{H}[1,n]$, satisfies the condition:

$$\alpha z p'(z) + p^2(z) \prec h(z) \tag{2}$$

where h is given by (1) then $p(z) \prec q(z)$, and q is the best dominant.

Proof. Let

$$L(z,t) = \alpha(n+t)zq'(z) + q^2(z) = \psi(q(z),(n+t)zq'(z)). \tag{3}$$

If $t = 0$ we have

$$L(z,0) = \alpha n z q'(z) + q^2(z) = h(z).$$

We will show that condition (2) implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

From (3) we easily deduce:

$$\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} = (n+t)\left[1 + \frac{zq''(z)}{q'(z)}\right] + \frac{2}{\alpha}q(z)$$

and by using the convexity of q and condition Re $q(z) > 0$, from Theorem 1 we obtain:

$$\text{Re}\, \frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} > 0.$$

Hence by Lemma B we deduce that $L(z,t)$ is a subordination chain. In particular, the function $h(z) = L(z,0)$ is univalent and $h(z) \prec L(z,0)$, for $t \geq 0$.

If we suppose that $p(z)$ is not subordinate to $q(z)$, then, from Lemma A, there exist $z_0 \in U$, and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ with $|\zeta_0| = 1$, and $z_0 p'(z_0) = (n+t)\zeta_0 q'(\zeta_0)$, with $t \geq 0$.

Hence

$$\psi_0 = \psi(p(z_0), z_0 p'(z_0)) = \psi(q(\zeta_0),(n+t)\zeta_0 q'(\zeta_0)) = L(\zeta_0,t),\ t \geq 0.$$

Since $h(z_0) = L(z_0,0)$, we deduce that $\psi_0 \notin h(U)$, which contradicts condition (2). Therefore, we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

If we let $p(z) = \frac{zf'(z)}{f(z)}$, (where $f \in A_n$), then Theorem 1 can be written in the following equivalent form:

Theorem 2. *Let $\alpha \geq 0$, q be a convex function in the unit disc U, with $q(0) = 1$, Re $q(z) > 0$, $n \geq 1$.*

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, $z \in U$, satisfies the condition

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z),$$

where h is given by (1) then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

In the case $\alpha = 1$ this result was obtained in [4].

3. Particular cases

1) If we let $q(z) = 1 + z$, then

$$h(z) = 1 + (\alpha n + 2)z + z^2, \quad n \geq 1$$

and Theorem 1 can be written as:

Theorem 3. *Let $\alpha \geq 0$, and let n be a positive integer.*

If $p \in \mathcal{H}[1, n]$, satisfies the condition:

$$\alpha z p'(z) + p^2(z) \prec 1 + (\alpha n + 2)z + z^2,$$

then

$$p(z) \prec 1 + z$$

and this result is sharp.

In this case Theorem 2 becomes:

Theorem 4. *Let $\alpha \geq 0$, and let n be a positive integer.*

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec 1 + (\alpha n + 2)z + z^2$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + z,$$

and this result is sharp.

2) If we let $q(z) = \frac{1+z}{1-z}$, then

$$h(z) = \frac{1 + 2(1 + \alpha n)z + z^2}{(1 - z)^2}.$$

and Theorem 1 becomes:

Theorem 5. *Let $\alpha \geq 0$ and let n be a positive integer.*

If $p \in \mathcal{H}[1, n]$, satisfies the condition

$$\alpha zp'(z) + p^2(z) \prec \frac{1 + 2(1 + \alpha n)z + z^2}{(1 - z)^2},$$

then

$$p(z) \prec \frac{1 + z}{1 - z}$$

and this result is sharp.

Theorem 2 becomes the following criterion for starlikeness:

Theorem 6. *Let $\alpha \geq 0$, and let n be a positive integer.*

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec \frac{1 + 2(1 + \alpha n)z + z^2}{(1 - z)^2}$$

then $f \in S^*$.

3) If we let $q(z) = \left(\frac{1+z}{1-z}\right)^{\lambda}$, where $0 < \lambda < 1$, then

$$h(z) = \left(\frac{1 + z}{1 - z}\right)^{\lambda} \left[\frac{2\alpha n\lambda z}{1 - z^2} + \left(\frac{1 + z}{1 - z}\right)^{\lambda}\right]$$

and Theorem 1 becomes:

Theorem 7. *Let $\alpha \geq 0$, $0 < \lambda < 1$, let n be a positive integer and let*

$$h(z) = \left(\frac{1 + z}{1 - z}\right)^{\lambda} \left[\frac{2\alpha n\lambda z}{1 - z^2} + \left(\frac{1 + z}{1 - z}\right)^{\lambda}\right] = \left(\frac{1 + z}{1 - z}\right)^{\lambda - 1} \left[\frac{2\alpha n\lambda z}{(1 - z)^2} + \left(\frac{1 + z}{1 - z}\right)^{\lambda + 1}\right]. \tag{4}$$

If $p \in \mathcal{H}[1, n]$, satisfies the condition:

$$\alpha zp'(z) + p^2(z) \prec h(z),$$

where h is given by (4), then

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\lambda}$$

and this result is sharp.

From Theorem 2 we deduce the following criterion for strongly-starlikeness.

Theorem 8. *Let $\alpha \geq 0$, $0 < \lambda < 1$, and let n be a positive integer.*

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

where h is given by (4), then

$$f \in S^*[\lambda].$$

By choosing certain subdomains of $h(U)$ we can deduce the following particular criteria for strongly-starlikeness.

Corollary 1. *Let $0 < \lambda < 1$, $n \geq 1$, $\alpha \geq 0$.*

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, satisfies the condition

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} J(\alpha, f; z)\right\}\right| < \phi_0(n, \alpha, \lambda), \tag{5}$$

where

$$\phi_0(n, \alpha, \lambda) = \frac{\lambda\pi}{2} + \arctan \frac{\frac{n\alpha\lambda}{1-\lambda} + \left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{1+\lambda}{2}} \sin\frac{\lambda\pi}{2}}{\left(\frac{1+\lambda}{1-\lambda}\right)^{\frac{1+\lambda}{2}} \cos\frac{\lambda\pi}{2}} \tag{6}$$

then $f \in S^*[\lambda]$.

Proof. The domain $h(U)$, where h is given by (4) is symmetric with respect to the real axis. Therefore, if $z = e^{i\theta}$, then in order to obtain the boundary of $h(U)$ it is sufficient to suppose $0 \leq \theta \leq \pi$.

Letting $\cot\frac{\theta}{2} = t$ and $h(e^{i\theta}) = u + iv$, we find:

$$\begin{cases} u(t) = t^{\lambda}\left[-\frac{\alpha n\lambda a}{2t}(1+t^2) + (b^2 - a^2)t^{\lambda}\right] \\ v(t) = t^{\lambda}\left[\frac{\alpha n\lambda b}{2t}(1+t^2) + 2abt^2\right] \end{cases} \tag{7}$$

where $a = \sin\frac{\lambda\pi}{2}$ and $b = \cos\frac{\lambda\pi}{2}$.

We also have:

$$\phi = \phi(t) = \arg h(e^{i\theta}) = \frac{\lambda\pi}{2} + \arctan \frac{\frac{\alpha n\lambda}{2}(1+t^2) + t^{\lambda+1}\sin\frac{\lambda\pi}{2}}{t^{\lambda+1}\cos\frac{\lambda\pi}{2}}. \tag{8}$$

From (8) it is easy to show that the equation $\phi'(t) = 0$, has the root:

$$t_0 = \sqrt{\frac{1+\lambda}{1-\lambda}}$$

and

$$\min_{t\geq 0} \phi(t) = \phi(t_0) = \phi_0(n, \lambda)$$

where $\phi_0(n, \lambda)$ is given by (6).

We deduce that the sector $\{w : \ |\arg w| < \phi_0(n, \alpha, \lambda\}$ is the largest sector which lies in $h(U)$. Hence (5) implies

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

where h is given by (4) and Corollary 1 follows from Theorem 2. □

Corollary 2. *Let* $0 < \lambda < 1$, $n \geq 1$, $\alpha \geq 0$.

If $f \in A_n$, *with* $\frac{f(z)}{z} \neq 0$, *satisfies the condition*

$$\left|\operatorname{Im} \frac{zf'(z)}{f(z)} J(\alpha, f; z)\right| < V(n, \alpha, \lambda), \tag{9}$$

where $V(n, \alpha, \lambda) = v(t_0)$, *with* v *given by (7) and* t_0 *is the root of the equation:*

$$4t^{\lambda+1}\sin\lambda\pi + \alpha n(\lambda+1)t^2\cos\frac{\lambda\pi}{2} + \alpha n(\lambda-1)\cos\frac{\lambda\pi}{2} = 0 \tag{10}$$

then $f \in S^*[\lambda]$.

Proof. From (7) we deduce:

$$v' = \lambda t^{\lambda-2}\left[\frac{\alpha n(\lambda-1)b}{2} + \frac{\alpha n(\lambda+1)b}{2}t^2 + 4abt^{\lambda+1}\right]$$

and the equation $v'(t) = 0$ becomes (10).

Hence

$$V(n, \alpha, \lambda) = v(t_0) = \min_{t\geq 0} v(t)$$

and we deduce that the strip $|v| < V(n, \alpha, \lambda)$ lies in $h(U)$. Therefore (9) implies

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

and Corollary 2 follows from Thereom 2. □

Corollary 3. *Let* $0 < \lambda < 1$, $n \geq 1$, $\alpha \geq 0$.

If $f \in A_n$, *with* $\frac{f(z)}{z} \neq 0$, $z \in U$, *satisfies the condition:*

$$\text{Re}\ \left[\frac{zf'(z)}{f(z)} J(\alpha, f; z)\right] > U(n, \alpha, \lambda) \tag{11}$$

where $U(n, \alpha, \lambda) = u(t_0)$, *with* u *given by (7) and* t_0 *is the root of the equation:*

$$4t^{\lambda+1} \cos \lambda\pi - n\alpha(\lambda+1)t^2 \cos \frac{\lambda\pi}{2} - n\alpha(\lambda-1) \sin \frac{\lambda\pi}{2} = 0 \tag{12}$$

then $f \in S^*[\lambda]$.

Proof. From (7) we deduce:

$$u' = \lambda t^{\lambda-2} \left[-\frac{\alpha na(\lambda-1)}{2} - \frac{\alpha na(\lambda+1)}{2} t^2 + 2(b^2 - a^2)t^{\lambda+1}\right]$$

and the equation $u'(t) = 0$ becomes (10).

Hence

$$U(n, \lambda) = u(t_0) = \max_{t \geq 0} u(t)$$

and we deduce that the half-plane $\{w : \text{Re}\ w > U(n, \alpha, \lambda)\}$ lies in $h(U)$. Therefore (11) implies

$$\frac{zf'(z)}{f(z)} J(\alpha, f; z) \prec h(z)$$

and Corollary 3 follows from Theorem 2. □

4. Examples

1) If we let $n = 1$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, then from (6) we deduce

$$\phi_0\left(1, \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{4} + \text{arctg}\ \left(1 + \frac{1}{3^{\frac{3}{4}}\sqrt{2}}\right) = 1.7027\ldots$$

and by Corollary 1 we have the following result:

If $f \in A$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.7027\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$, i.e.

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}.$$

2) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, then from (6) we deduce

$$\phi_0\left(2, \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi}{4} + \operatorname{arctg}\left(1 + \frac{\sqrt{2}}{3^{\frac{3}{4}}}\right) = 1.863\ldots$$

and by Corollary 1 we have the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.863\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$, i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4}.$$

3) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{2}{3}$, then from (6) we deduce

$$\phi_0\left(2, \frac{1}{2}, \frac{2}{3}\right) = \frac{\pi}{3} + \operatorname{arctg}\frac{2 + 5^{\frac{5}{6}} \cdot \frac{\sqrt{3}}{2}}{5^{\frac{5}{6}} \cdot \frac{1}{2}} = 2.2725\ldots$$

and by Corollary 1 we have the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 2.2725\ldots$$

then $f \in S^*\left[\frac{2}{3}\right]$ i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{3}.$$

4) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$, then from (6) we deduce

$$\phi_0\left(2, \frac{1}{2}, \frac{1}{3}\right) = \frac{\pi}{6} + \operatorname{arctg}\frac{1 + 2^{\frac{2}{3}}}{2^{\frac{2}{3}} \cdot \sqrt{3}} = 1.2792\ldots$$

and by Corollary 1 we have the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and

$$\left|\arg\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.2792\ldots$$

then $f \in S^*\left[\frac{1}{3}\right]$ i.e.

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{6}.$$

5) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$, then equation (10) becomes:

$$16t^{\frac{3}{2}} + 3\sqrt{2}t^2 - \sqrt{2} = 0$$

which has the root $t_0 = 0.1846\ldots$ Hence by Corollary 2 we deduce the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$ and:

$$\left|\text{Im}\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| < 1.220\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$.

6) If we let $n = 2$, $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{2}$, then equation (12) becomes: $3t^2 - 1 = 0$ and from Corollary 3 we deduce the following result:

If $f \in A_2$, with $\frac{f(z)}{z} \neq 0$, $z \in U$, and:

$$\left|\text{Re}\left[\frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} + 1\right)\right]\right| > -0.610\ldots$$

then $f \in S^*\left[\frac{1}{2}\right]$.

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UNIVERSITY OF ORADEA, DEPARTMENT OF MATHEMATICS, 3700 ORADEA, ROMANIA