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A CHEBYSHEV SYSTEM APPROACH TO THE BOUNDARY BEHAVIOUR OF THE SUBLINEAR FUNCTIONS

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Abstract. The aim of this note is to show that the problem of the augmentation of a function system consisting of the coordinate functions of a parametrization of a convex surface in R^{n-1} with 0 in its interior, to a Chebyshev system of order $n-3$ [9] has its natural interpretation in the contex of the boundary behaviour of a strictly sublinear or a strictly superlinear function.

A strictly convex or strictly concave real function can be defined by the condition that its graph intersects every straight line in at most two distinct points. In this definition we have to do in fact with the two dimensional subspace of the affine functions which augmented by the strictly convex (or strictly concave) function in question to a three dimensional space, becomes a space having the property that each nonzero element of its vanishes in at most two points. A possible generalization of the convexity notion introduced this way is the following: Consider an arbitrary two dimensional subspace P_2 of the space $C(Q)$ of continuous real functions defined on the connected Hausdorff space Q. Then $f \in C(Q)$ is convex with respect to P_2 if every member of P_2 can agree with f in at most two distinct points. Suprisingly this generalization goes not far from the real function case: the notion is consistent for the case of Q compact if and only if this is homeomorphic with a (compact, connected) subset of the circle S^1 [5].

Starting with the above generalized convexity notion (a two dimensional underlying vector space and a convex function with respect to it) and trying to get a natural extension, we can follow two lines. To consider for instance an $n-1$ dimensional subspace P_{n-1} in $C(Q)$ and to call $f \in C(Q)$ convex with respect to it, if this f agrees with each member of P_{n-1} in at most $n-1$ distinct points. Again, this generalization is consistent for connected compact Q if and only if Q can be imbedded into S^1 and the imbedding can be surjective only when n is odd [5].

A second way is the following: to consider an $n-1$ dimensional subspace P_{n-1} of $C(Q)$ $(n \geq 3)$ and to consider $f \in C(Q)$ convex with respect to P_{n-1} if it agrees with $n-2$ linearly independent elements of P_{n-1} in at most two distinct common points.

In the case $n = 3$ the two convexity notions coincide.

The first generalization has an old history. It goes back to Popoviciu (see [8] and [11]) and is used in the constructive function theory (see also [3] and [4]). The second one is not explored explicitely, but it corresponds to a natural geometrical picture (this is emphasised also by the content of our note). We note that this second generalization can be consistent also for rather strong topological conditions on Q. This follows from some results in topological setting in [6].

Both the above two generalized convexities can be interpreted as augmentation of a given system of functions by a function to another system with prescribed properties.

The aim of this note is to show that the problem of the augmentation of a function system consisting of the coordinate functions of a parametrization of a convex surface in R^{n-1} with 0 in its interior, to a Chebyshev system of order $n-3$ [9] has its natural interpretation in the contex of the boundary behaviour of a strictly sublinear or a strictly superlinear function. Our geometric approach as well as the method used in proofs are prolific in both the convex analysis and the theory of Chebyshev systems. They emphasise the strong relation existing between these two fields.

1. **Parametrized convex surfaces in R^{n-1}**

We say that S^{n-2} is the *standard* $n-2$ *sphere* if it is the subspace of the Euclidean space R^{n-1} consisting of the set of points with the distance 1 from the origin of a Cartesian system in R^{n-1}. We say that the set C in R^{n-1} is a *topological* $n-2$ *sphere* or a *closed surface* if it is the homeomorphic image of S^{n-2}. Denote by

ϕ a homeomorphism from S^{n-2} to C. Then ϕ will be called a parametrization of C. If $\phi = (\varphi_1, \ldots, \varphi_{n-1})$, then φ_j, $j = 1, \ldots, n-1$ will be called the *coordinate functions of a parametrization of the surface C.*

We are particularly interested in the case when the topological $n-2$ sphere C in R^{n-1} is a convex (or a strictly convex) surface in the sense that it is the boundary of a convex (or respectively, of a strictly convex) body in R^{n-1}. A body B in R^{n-1} is a closed, connected and bounded set with non empty interior. The body B is convex if and only if every straight line containing an interior point of its, meets its boundary C in exactly two points. This follows from basic properties of convex sets (see e.g. [10]). Therefore a straight line can meet a convex surface C in a set having at most two connected components. If the convex surface C contains no line segment, then it is called a strictly convex surface and the set B it bounds, a strictly convex body. Thus the closed surface C is strictly convex if and only if any straight line in R^{n-1} can have an intersection with C consisting of at most two points.

A straight line in R^{n-1} is the intersection of $n-2$ hyperplanes, i.e., it is the locus of the points $x = (x^1, \ldots, x^{n-1}) \in R^{n-1}$ satisfying a system of the form

$$c_0^j + c_1^j x^1 + \cdots + c_{n-1}^j x^{n-1} = 0, \quad j = 1, \ldots, n-2 \tag{1}$$

with

$$(c_1^j, \ldots, c_{n-1}^j), \quad j = 1, \ldots, n-2 \tag{2}$$

linearly independent vectors (the normal vectors of the mentioned hyperplanes).

According to our above observations the surface $C = \phi(S^{n-2})$ with the parametrization $\phi = (\varphi_1, \ldots, \varphi_{n-1})$ is convex (respectively, it is strictly convex) if and only if the equations

$$c_0^j + c_1^j \varphi_1(q) + \cdots + c_{n-1}^j \varphi_{n-1}(q) = 0, \quad j = 1, \ldots, n-2 \tag{3}$$

possess a set of solutions $q \in S^{n-2}$ having at most two connected components (possess at most two distinct solutions $q \in S^{n-2}$) for every set (2) of $n-2$ linearly independent vectors.

Let us consider now instead of the vectors (2) the vectors of the form

$$c_j = (c_0^j, c_1^j, \ldots, c_{n-1}^j), \quad j = 1, \ldots, n-2. \tag{4}$$

If the vectors $c_1, \ldots, c_{n-2}$ were linearly independent but the vectors (2) were not, then the system (1) were incompatible and the equations (3) could not have any solution $q \in S^{n-2}$.

By gathering the above observations we arive to the following statement:

1.1. *The surface C in R^{n-1} with the parametrization $\phi = (\varphi_1, \ldots, \varphi_{n-1})$ is convex (respectively, it is strictly convex) if and only if for each set (4) of $n-2$ linearly independent vectors the system (3) can have a set of solutions $q \in S^{n-2}$ consisting of at most two connected components (respectively, this set of solutions can have at most two distinct points).*

2. Chebyshev systems

Denote by $C(Q)$ the vector space of the real valued continuous functions defined on the connected topological space Q. The set $\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\} \subset C(Q)$ is called an (n,k) *system* (or a Chebyshev system of order $k-1$ [9]), if it is linearly independent and any k linearly independent elements in $sp\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\}$ possess at most $n-k$ common zeros in Q. An $(n,1)$ system is a so called Chebyshev or Haar system ([3], [4]). By a *weak* (n,k) *system* we mean a set of functions of the above form relaxing the last requirement in the above definition to the following one: any k linearly independent elements in $sp\{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\}$ can have a set of common solutions having at most $n-k$ connected components. A weak $(n,1)$ system is called a weak Chebyshev system. Weak Chebyshev systems have been defined in [2] for the case Q an interval in R by an oscillation condition. For this particular case the notion agrees with ours (other equivalent conditions were considered in [1]).

We are especially interested in the case when $k = n-2$. An $(n, n-2)$ system (a weak $(n, n-2)$ system) is for $n = 3$ a Chebyshev system (respectively a weak Chebyshev system).

We have the following relation of $(n, n-2)$ systems with the surface parametrizations:

2.1. *Let C be a topological $n-2$ sphere in R^{n-1} wuth the parametrization $\phi = (\varphi_1, \dots, \varphi_{n-1})$. Then C is a convex surface (respectively, it is a strictly convex surface) if and only if the set of functions $\{1, \varphi_1, \dots, \varphi_{n-1}\}$, where 1 is the constant 1 function on S^{n-2}, is a weak $(n, n-2)$ system (respectively, it is an $(n, n-2)$ system).*

To verify this statement we note that by 1.1 C is a convex surface (respectively, it is a strictly convex surface) if and only if for every set of $n-2$ linearly independent vectors (4) the system of equations (3) can have a set of solutions q in S^{n-2} with at most two connected components (respectively, this set consists of at most two points). This is nothing but the requirement that any $n-2$ linearly independent elements in $sp\{1, \varphi_1, \dots, \varphi_{n-1}\}$ have a set of common zeros possessing at most two connected components (respectively, this set consists of at most two points). That is, the convexity of C (the strict convexity of C) is equivalent with the fact that $\{1, \varphi_1, \dots, \varphi_{n-1}\}$ is a weak $(n, n-2)$ system (respectively, it is an $(n, n-2)$ system).

3. Wedges and cones

A non empty subset W in R^n is called a *wedge* if $W + W \subset W$ and if $tW \subset W$ for each non negative real number t. A wedge is obviously a convex set which contains the vector 0. The wedge K is called a *cone* if $K \cap (-K) = \{0\}$. Thus the wedge K is a cone if and only if from u, $-u \in K$ it follows that $u = 0$.

The subset F of the wedge W is called a *face* of W, if it is a wedge and if the conditions $u \in F$, $v \in W$ and $u - v \in W$ imply that $v \in F$.

Any wedge contained in a cone is itself a cone, hence the face of a cone is a cone.

The face F of the wedge W is called *proper face* if $\{0\} \neq F \neq W$.

We gather next some results which we shall use later. Most of them are easy consequences of the definitions or are standard results of the theory of convex sets (see e.g. [10]).

3.1. If W is a wedge and $intW \neq \emptyset$, then $W + intW \subset intW$.

3.2. No proper face of a wedge W can contain points of $intW$.

3.3. If a subspace L of dimension 2 of R^n contains three affinely independent points of the boundary ∂W of the wedge W, then $L \cap intW = \emptyset$.

3.4. If W is a wedge with $intW \neq \emptyset$ and if L is a subspace of R^n with $L \cap intW = \emptyset$, then there exists an $n-1$ dimensional subspace H of R^n with $L \subset H$ and $H \cap intW = \emptyset$.

3.5. If W is a wedge in R^n with $intW \neq \emptyset$ and if H is a hyperplane through 0 in R^n with $H \cap intW = \emptyset$, then $F = H \cap W$ is a face of W. If $F \neq 0$, it is a proper face.

The closed cone K in R is called *strictly convex cone* if it possesses only one dimensional proper faces. The condition $\dim K \geq 2$ is here intrinsic.

3.6. The intersection of two wedges is a wedge. The intersection of two strictly convex cones is a strictly convex cone if the dimension of the intersection is ≥ 2.

4. Sublinear and superlinear functions

Consider the function $f : R^{n-1} \to R$ $(n \geq 2)$. The graph, epigraph and hypograph of f are the sets

$$grf = \{(x,t) \in R^{n-1} \times R : f(x) = t\},$$

$$epif = \{(x,t) \in R^{n-1} \times R : f(x) \leq t\},$$

$$hypof = \{(x,t) \in R^{n-1} \times R : f(x) \geq t\}$$

respectively. If f is continuous then these sets are closed and $int(epif) \neq \emptyset$, $int(hypof) \neq \emptyset$.

The function $f : R^{n-1} \to R$ is called *positively homogeneous* if $f(tx) = tf(x)$ for each $x \in R^{n-1}$ and each $t \in R^{n-1}$ and each $t \in R_+ = [0, +\infty)$. The function f is called *subadditive (superadditive)* if $f(x+y) \leq f(x) + f(y)$ $(f(x+y) \geq f(x) + f(y))$ for any $x, y \in R^{n-1}$. If f is both positively homogeneous and subadditive (positively homogeneous and superadditive) then it is called *sublinear* (respectively, *superlinear*). The function f is superlinear if and only if $-f$ is sublinear.

The sublinear (superlinear) function $f : R^{n-1} \to R$ is called *strictly sublinear (strictly superlinear)* if the equality $f(x+y) = f(x) + f(y)$ for non zero x and y implies that x and y are positive multiple of each other.

The property of a function of being sublinear, superlinear, strictly sublinear or strictly superlinear can be expressed geometrically using the notions of wedges and cones:

4.1. *The positively homogeneous continuous function $f : R^{n-1} \to R$ is*

(a) sublinear (superlinear) if and only if epif (hypof) is a wedge;

(b) strictly sublinear (strictly superlinear) if and only if epig (hypof) is a strictly convex one.

We prove the statement (b) for the sublinear case. The other cases can be similarly handled.

Suppose that f is strictly sublinear and denote $K = epif$. If $(x,s) \in K$ (that is, if $f(x) \leq s$) and $t \in R_+$ then $tf(x) \leq ts$ and by the positive homogeneity of f we get $f(tx)ts$, that is, $(tx,ts) = t(x,s) \in K$ which shows that $tK \subset K$, $\forall\, t \in R_+$.

Let be $(x,s),(y,t) \in K$. Then $f(x+y) \leq f(x) + f(y) \leq s+t$ and hence $(x+y, s+t) \in epif = K$. That is, $K + K \subset K$.

We have proved that K is a wedge. Suppose that F is a proper face of K. Then $F \subset \partial K$ since by 3.2, $F \cap intK = \emptyset$. By the continuity of f, $grf = \partial K$. Hence $F \subset grf$. Suppose now that $(x,s),(y,t) \in F$. Then $(x+y, s+t) \in F$ since F is a wedge. But then $F(x+y) = s+t = f(x)+f(y)$. According to the strict sublinearity of f, if x and y are non zero vectors, it follows that $y = rx$ for some $r > 0$. But then $f(y) = f(rx) = rf(x) = rs$. That is, $(y,t) = r(x,s)$. In conclusion, $\dim F = 1$.

If $(x,s), -(x,s) \in K$, then $f(x) \leq s$ and $f(-x) \leq -s$. By the sublinearity of f we have $-f(x) \leq f(-x)$. These relations give $f(x) = s$ and $f(-x) = -s$. Thus $0 = f(x-x) = f(x) + f(-x)$. From the strict sublinearity of f it follows then that $x = 0$, Hence $s = 0$ and we conclude that K is a cone.

Suppose now that f is positively homogeneous and $K = epif$ is a strictly convex cone. Assume that there exist some linearly independent vectors $x, y \in R^{n-1}$ such that $f(x+y) = f(x) + f(y)$. Put $s = f(x)$, $t = f(y)$ and consider the space L in

$R^{n-1} \times R$ engendered by the vectors $(x,s),(y,t)$. From the definition of K we have $intK \neq \emptyset$. The vectors $(x,s),(y,t)$ and $(x+y,s+t)$ are affinely independent and are contained in $grf = \partial K$. Hence $L \cap intK = \emptyset$ by 3.3. According 3.4 there exists a hyperplane H with $L \subset H$ and $H \cap intK = \emptyset$. Then $F = K \cap H$ is a face of K by 3.5. But $\dim K \geq 2$ and we get a contradiction with the hypothesis of strict convexity of K. Thus f must be strictly sublinear.

5. Traces of sublinear and superlinear functions on convex surface

Let $f : R^{n-1} \to R$ $(n \geq 3)$ be a sublinear function and let $D \subset R^{n-1}$ be a convex body with $0 \in intD$. Denote $C = \partial D$. We shall in this case say that C is a convex surface with 0 in its interior. It is standard question in the global optimization to search the maximum of f on D. Obviously, it suffices to get its maximum on the boundary C of D. It is also immediate from the position of C that the values of f on C determine this function. This motivates the investigation of the sublinear function f on convex surfaces like C.

Let $\phi : S^{n-2} \to C$, $\phi = (\varphi_1, \ldots, \varphi_{n-1})$ be a parametrization of the closed surface C. We can describe the behaviour of a function $g : R^{n-1} \to R$ on C by the function $g \circ \phi$ called the *trace of g on C*.

The geometrical approach we have outlined enables us to answer (using the notions introduced in section 2) the question whether or not a real function φ : $S^{n-2} \to R$ can be the trace on a convex surface with 0 in its interior of a sublinear (or strictly sublinear) function. We have in this contex the following result:

5.1. *Let C be a convex surface in R^{n-1} with 0 in its interior having the parametrization $\phi : S^{n-2} \to C$, $\phi = (\varphi_1, \ldots, \varphi_{n-1})$. Then the continuous function*

$$\varphi : S^{n-2} \to R$$

(a) is the trace on C of a continuous sublinear or superlinear function if and only if the set of functions

$$\{\varphi, \varphi_1, \ldots, \varphi_{n-1}\} \tag{5}$$

is a weak $(n, n-2)$ system;

(b) is the trace on C of a continuous strictly sublinear or strictly superlinear function if and only if (5) is an $(n, n-2)$ system.

We shall prove the assertion (b). The proof of (a) is similar.

Suppose that f is strictly sublinear and $\varphi = f \circ \phi$. We have to show that (5) is an $(n, n-2)$ system. To this end, let us take $n-2$ linearly independent elements in $sp\{\varphi, \varphi_1, \dots, \varphi_{n-1}\}$:

$$c_0^j \varphi + c_1^j \varphi_1 + \dots + c_{n-1}^j \varphi_{n-1}, \quad j = 1, \dots, n-2. \tag{6}$$

The vectors $c_j = (c_0^j, c_1^j, \dots, c_{n-1}^j)$, $j = 1, \dots, n-2$ are linearly independent. Hence the set of solutions of the system

$$c_1^j u^1 + \dots + c_{n-1}^j u^{n-1} + c_0^j u^n = 0, \quad j = 1, \dots, n-2 \tag{7}$$

(in $u = (u^1, \dots, u^n)$) is a two dimensional subspace L of R^n. Let us identify the domain of f with the subspace $u^n = 0$ in R^n. Then $epif$ is by 4.1(b) a strictly convex cone K with grf being its boundary. There exist three possibilities: $L \cap \partial K = \{0\}$, $L \cap \partial K$ consists of a ray from 0 on grf or $L \cap \partial K$ consists of two distinct rays on grf from 0.

Consider the surface $C_1 = \Psi(S^{n-2}) \subset R^n$ having the parametrization $\Psi = (\varphi_1, \dots, \varphi_{n-1}, \varphi)$. Then geometrically C_1 is the intersection of grf with the cylinder with the generator parallel with the axis Ou^n and the base C in R^{n-1}. From the configuration of C each ray from 0 on grf intersects C_1 once. Hence the plane L of dimension 2 consisting of the set of solutions of the system (7) has an intersection with C_1 which is the empty set if $L \cap \partial K = \{0\}$, it contains a single point if $L \cap \partial K$ is a single ray and it consists of two points if $L \cap \partial K$ consists of two rays. The intersections of C_1 with L are given by the solutions in $q \in S^{n-2}$ of the system

$$c_0^j \varphi(q) + c_1^j \varphi_1(q) + \dots + c_{n-1}^j \varphi_{n-1}(q) = 0, \quad j = 1, \dots, n-2. \tag{8}$$

Since $\Psi = (\varphi_1, \dots, \varphi_{n-1}, \varphi)$ is one to one, to each intersection point of L with C_1 corresponds exactly one solution $q \in S^{n-2}$ of the system (8). By the above observations on the intersection of L with C_1 we conclude that (8) can have at most two distinct solutions $q \in S^{n-2}$ which is nothing but the condition for (5) to form an $(n, n-2)$ system.

Conversely, let us suppose that $\varphi \in C(S^{n-2})$ is a function with the property that (5) is an $(n, n-2)$ system. Consider C_1 as being the set $\Psi(S^{n-2})$ with $\Psi = (\varphi_1, \dots, \varphi_{n-1}, \varphi)$. Then Ψ is a parametrization of the surface C_1. the set $C = \phi(S^{n-2}) \subset R^{n-1}$ with R^{n-1} the subspace of vectors in R^n with the last component 0, is a closed convex surface containing 0 in its interior. Hence every ray from 0 in R^{n-1} intersects C in exactly one point.

Let ψ be the Minkowski functional with respect to 0 of $D = coC$. We define the function $f : R^{n-1} \to R$ by putting

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \psi(x)\varphi(\phi^{-1}(x/\psi(x))) & \text{if } x \neq 0 \end{cases}.$$

The trace of f on C is φ. Indeed, if $x \in C$, then $\psi(x) = 1$ and $x/\psi(x) = x$. Hence $f(x) = \varphi(\phi^{-1}(x))$ and denoting $q = \phi^{-1}(x)$, we have that

$$(f \circ \phi)(q) = \varphi(q).$$

The function f is positively homogeneous since ψ is so. Hence grf is engendered by a moving ray with center 0, running on C_1.

The function f is strictly sublinear or strictly superlinear. If none, then no $epif$, no $hypof$ can be a strictly convex cone by 4.1. This means that there exists a straight line d in R^n, not passing through 0, which meets grf, the boundary of $epif$ (and of $hypof$) in at least three distinct points u_1, u_2, u_3.

Consider the two dimensional plane L through 0 engendered by d. This plane can be represented as the set of solutions of a system of the form (7) with the vectors $(c_1^j, \dots, c_{n-1}^j)$, $j = 1, \dots, n-2$ being linearly independent.

The plane L will meet C in the points u_1, u_2, u_3. Let be $q_j = \Psi^{-1}(u_j)$, $j = 1, 2, 3$. Then q_1, q_2, q_3 will be distinct solutions of the system (8), where the vectors

$$c_j = (c_0^j, c_1^j, \dots, c_{n-1}^j), \quad j = 1, \dots, n-2$$

are linearly independent. This means that (5) cannot be an $(n, n-2)$ system, contradiction which completes the proof.

6. **Augmentation of a parametrization to $(n, n-2)$ systems**

Let the set of functions

$$\{\varphi_1, \dots, \varphi_{n-1}\} \subset C(S^{n-2}) \tag{9}$$

be the coordinate function of a parametrization ϕ of a closed surface C in R^{n-1}. We say that a continuous function $\varphi : S^{n-2} \to R$ is an *augmentation to a weak* $(n, n-2)$ *system (to an* $(n, n-2)$ *system)* of (9) if

$$\{\varphi, \varphi_1, \dots, \varphi_{n-1}\} \tag{10}$$

is a weak $(n, n-2)$ system (is an $(n, n-2)$ system).

We have seen (section 2) that if (9) are the coordinate functions of a parametrization of a convex (strictly convex) surface in R^{n-1}, then the function $\varphi = 1$ is an augmentation to a weak $(n, n-2)$ system (to an $(n, n-2)$ system) of (9). Conversely, if any constant nonzero function augmentation (9) to a weak $(n, n-2)$ system (to an $(n, n-2)$ system), then the functions (9) must be the coordinate functions of the parametrization of a closed convex (a closed strictly convex) surface.

We are prepared to consider the augmentation to a weak $(n, n-2)$ system (to an $(n, n-2)$ system) of the set of coordinate functions of the parametrization of a convex surface in R^{n-1} with 0 in its interior.

Let (9) be the set of coordinate functions of the parametrization of a convex surface C in R^{n-1} $(n \geq 3)$ with 0 in its interior. The augmentation $\varphi \in C(S^{n-2})$ to a weak $(n, n-2)$ system (to an $(n, n-2)$ system) (10) will be called *sublinear* (**strictly sublinear**) if φ is the trace C of a sublinear (of a strictly sublinear) function (see 5.1). The superlinear (strictly superlinear) augmentation is defined similarly.

Using this terminology we have the following result:

6.1. *Let (9) be the set of the coordinate functions of a parametrization of a convex surface.*

(a) The set of the sublinear (superlinear) augmentations φ *of (9) to (10)*

(b) The set of strictly sublinear (strictly superlinear) augmentations φ of (9) to (10)

is invariant with respect to the multiplication with positive scalars and its invariant with respect to taking the pointwise maximum (the pointwise minimum) of two elements.

We prove (a) for the sublinear case. The invariance of the set of augmentations with respect to the multiplication with positive scalars is obvious.

Let φ and ψ be two sublinear augmentations. Then φ and ψ are traces of the sublinear functions f and g respectively. The function $\max\{f, g\}$ is sublinear and possesses as trace on C the function $\max\{\varphi, \psi\}$. Thus by 5.1 $\max\{\varphi, \psi\}$ will be a sublinear augmentation.

(b) Suppose that φ and ψ are strictly sublinear augmentations of (9). If $f, g : R^{n-1} \to R$ are the strictly sublinear functions with traces φ and ψ respectively, then $epif$ and $epig$ are strictly convex cones. Since the relation

$$epi(\max\{f, g\}) = (epif) \cap (epig),$$

and since the set on the right hand side is a strictly convex cone (see 3.6), $\max\{f, g\}$ is a strictly sublinear function. The trace of this function on C is $\max\{\varphi, \psi\}$. Hence

$$\{\max\{\varphi, \psi\}, \varphi_1, \dots, \varphi_{n-1}\}$$

is an $(n, n-2)$ system.

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