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AVERAGING INTEGRAL OPERATORS AND HARDY CLASSES

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1. Introduction

Let $\mathcal{H}(U)$ denote the spaces of analytic functions in the unit disk $U = \{z : |z| > 1\}$ and $H_0 = \{f \in \mathcal{H}(U) : f(0) = 0\}$. If $K \subset \mathcal{H}(U)$ then an operator $A : K \to \mathcal{H}(U)$ is said to be an averaging operator on K if $A(f(0)) = f(0)$ and $A(f)[U] \subset co\, f(U)$, for all $f \in K$, where $co\, f(U)$ is the convex hull of $f(U)$. In [4] was obtained the integral averaging operator:

$$A[f](z) = \frac{1}{z^\gamma \phi(z)} \int_0^z f(t) t^{\gamma-1} \varphi(t) dt \tag{1}$$

and in [6] was obtained the second-order averaging integral operator

$$F(z) = \frac{1}{\alpha z^\gamma \varphi(z)} \int_0^z \frac{\varphi(t)}{\phi(t)} t^{\gamma-\beta-1} \int_0^t f(s) s^{\beta-1} \phi(s) ds dt. \tag{2}$$

In this paper we obtain Hardy classes for these operators and we obtain result for a more general operator A_φ, $\varphi \in H_0$ defined by $A_\varphi(f) = A(f) + f'(0)A(\varphi)$, $\varphi \in H_0$.

In [2] and [3] were obtained Hardy classes for integral operators

$$I[f](z) = \left[\frac{\beta+\gamma}{z^\gamma} \int_0^z f^\beta(t) t^{\gamma-1} dt\right]^{\frac{1}{\beta}}, \quad z \in U \text{ (Singh, 1973)} \tag{3}$$

$$I_{\phi,\varphi}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t) \varphi(t) t^{\delta-1} dt\right]^{\frac{1}{\beta}}, \quad z \in U. \tag{4}$$

In this paper we obtain Hardy classes for these operators, using averaging operators.

2. Preliminaries

For f analytic in U and $z = re^{i\theta}$ we denote

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & \text{for } 0 < p < \infty \\ \sup\limits_{0\le\theta<2\pi} |f(re^{i\theta})|, & \text{for } p = \infty. \end{cases}$$

A function is said to be of Hardy class H^p, $0 < p \le \infty$ if $M_p(r,f)$ remains bounded as $r \to 1^-$, H^∞ is the class of bounded analytic functions in the unit disk.

If f, g analytic in U, then function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

A function h is said to be convex if h is univalent and $h(U)$ is a convex domain.

It is easy to show that an operator $A : K \to \mathcal{H}(U)$ is an averaging operator on K if and only if $[f \in K,\ h$ convex and $f \prec h] \Rightarrow A(f) \prec h$.

We shall need the following lemmas.

Lemma 1. *Let $h \in H_0$, convex and let $A \ge 0$. Suppose that $k > \frac{4}{|h'(0)|}$ and $B(z), C(z), D(z)$ are analytic in U and satisfy*

$$\text{Re } B(z) \ge A + |C(z) - 1| - \text{Re } [C(z) - 1] + k|D(z)|, \quad z \in U. \tag{5}$$

If $p \in H_0$ satisfies the differential subordination

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z) \tag{6}$$

then $p(z) \prec h(z)$.

Lemma 2. *Let $\delta \in \mathbf{C}$, $\delta \ne -1, -2, \ldots$ and let $\varphi, \phi \in \mathcal{H}(U)$ analytic functions with $\varphi(z)\phi(z) \ne 0$, $z \in U$. If*

$$\text{Re } B(z) \ge |C(z) - 1| - \text{Re } [C(z) - 1], \quad z \in U, \tag{7}$$

where $B(z) = \frac{\phi(z)}{\varphi(z)}$ and $C(z) = \frac{\gamma\phi(z) + z\phi'(z)}{\varphi(z)}$, then the integral operator A defined by (1) is an averaging operator on H_0.

Lemma 3. *Let* $\alpha \geq 0$, $\beta, \gamma \in \mathbf{C}$, *with* $\operatorname{Re}\beta > -1$ *and* $\operatorname{Re}\gamma > -1$ *and let* φ, ϕ *analytic functions with* $\varphi(z)\phi(z) \neq 0$, $z \in U$. *Let*

$$\begin{cases} B(z) = \alpha\left[\beta + \gamma + 1 + \dfrac{z\phi'(z)}{\phi(z)} + \dfrac{z\phi'(z)}{\varphi(z)}\right] \\ C(z) = \alpha\left[\left(\beta + \dfrac{z\phi'(z)}{\phi(z)}\right)\left(\gamma + \dfrac{z\varphi'(z)}{\varphi(z)} + z\left(\dfrac{z\varphi'(z)}{\varphi(z)}\right)'\right)\right]. \end{cases} \tag{8}$$

If $\theta \in H_0$ *and* $\operatorname{Re} B(z) \geq \alpha - |C(z) - 1| - \operatorname{Re}[C(z) - 1] + 4|\theta(z)|$, $z \in U$, *then the operator*

$$F_\theta[f] = F[f] + f'(0)F[\theta], \quad f \in H_0$$

where F *is defined by (2), is an averaging operator on* H_0.

These lemmas were proced in [5].

Lemma 4. *If* $f \in H^p$, $\beta > 0$ *and* I *is the integral operator of Singh (3) then*

(i) if $\beta > p$ *then* $I[f] \in H^{\frac{\beta p}{\beta - p}}$

(ii) if $\beta \leq p$ *then* $I[f] \in H^\infty$.

Lemma 5. *If* $f \in H^p$, $\varphi \in H^q$, $\dfrac{1}{\phi} \in H^r$, $\alpha, \beta > 0$ *then*

(i) if $pq < p + \alpha q$ *then* $I_{\phi,\varphi}[f] \in H^{\frac{\beta pqr}{pq + pr + \alpha qr - pqr}}$

(ii) if $pq \geq p + \alpha q$ *then* $I_{\phi,\varphi}[f] \in H^r$.

Lemma 4 was proved in [2] and Lemma 5 in [3].

3. Main Results

Theorem 1. *Let* $h \in H_0$, *be convex and* $A \geq 0$, $k > \dfrac{4}{|h'(0)|}$, $B(z), C(z), D(z)$ *analytic in* U *satisfies (5) and* $p \in H_0$ *satisfies the differential subordination (6) then* $p(z) \in H^\lambda$, $\lambda < 1$.

Proof. From Lemma 1 we obtain $p(z) \prec h(z)$. From subordination theorem of Littlewood [1] we deduce $M_\lambda(r, p) \leq M_\lambda(r, h)$. Since h is convex is very know that $h \in H^\lambda$, $\lambda < 1$. Hence $p(z) \in H^\lambda$, $\lambda < 1$. □

Theorem 2. *Let* $\delta \in \mathbf{C}$ *with* $\delta \neq -1, -2, \ldots$ *and* $\varphi, \phi \in \mathcal{H}(U)$ *with* $\varphi(z)\phi(z) \neq 0$, $z \in U$, *satisfying conditions (7) and* A *is operator defined by (1) then* $A(f) \in H^p$, $p < 1$, *for all* f, $f \in H_0$.

Proof. From Lemma 2 the integral operator A is averaging operator on H_0. Hence $I[f](U) \subset co\, f(U)$ for all $f \in H_0$. Since $co\, f(U)$ is convex domain, from conformal mappings's theorem (Riemann) there is a function g analytic in U such that $g(U) = co\, f(U)$. Since $g(U)$ is convex domain we deduce that g is convex. Since $M_\lambda(r, I[f]) \leq M_\lambda(r, g)$ we obtain $I(f) \in H^p$, $p < 1$. □

Theorem 3. *Let $\gamma \in \mathbf{C}$ with* Re $\gamma > 0$ *and let $g \in H_0$ with* Re $\dfrac{\gamma z g'(z)}{g(z)} > 0$ *in U. If A is defined by*

$$A[f](z) = \frac{\gamma}{g(z)} \int_0^z f(t) g(t)^{\gamma-1} g'(t) dt$$

then $A[f] \in H^p$, $p < 1$ for all $f \in H_0$.

Proof. If in A $\varphi(z) = [g(z)]^{\gamma-1} g'(z) z^{1-\gamma}$ and $\phi(z) = [g(z)]^{\gamma} z^{-\gamma} \gamma^{-1}$ then is satisfying condition (7) and from Lemma 2, A is averaging operator in H_0 and from Theorem 2 we obtain the result.

Hence we obtain some particular results. For $\gamma = 1$, $\alpha = 1$ and $g(z) = z$ we have the operator

$$\frac{1}{2} \int_0^z f(t) dt \in H^p, \quad p < 1.$$

Since, the Libera's operator is

$$\frac{2}{z} \int_0^z f(t) dt$$

we obtain Hardy classes for that operator:

$$\frac{2}{z} \int_0^z f(t) dt \in H^p, \quad p < 1.$$

For $\gamma = 0$, $\phi(z) = \dfrac{1}{2}$ and $\varphi(z) = 1$ we obtain

$$A[f](z) = \frac{1}{2} \int_0^z \frac{f(t)}{t} dt \in H^p, \quad p < 1.$$

Hence, we have Alexander's operator

$$\int_0^z f(t) t^{-1} dt \in H^p, \quad p < 1, \text{ for all } f,\ f \in H_0.$$

□

Theorem 4. *Let be $\gamma \in \mathbf{C}$, $\gamma \neq -1, -2, \ldots$ and let φ, ϕ analytic functions with $\varphi(z)\phi(z) \neq 0$, $z \in U$ and satisfying conditions (7). If $I_{\phi,\varphi}$ is defined by (4) and $\alpha = 1$, $\delta = \gamma$ then $I_{\phi,\varphi}[f] \in H^{\beta\lambda}$, $\beta > 0$, $0 < \lambda < 1$, for all $f \in H_0$.*

Proof. The operator $I_{\phi,\varphi}$ can be written as: $I_{\phi,\varphi} = B \circ A$ where $B(f) = (\beta+\gamma)[f(z)]^{\frac{1}{\beta}}$ and A is defined by (1). Since $f \in H^p$ applying Hölder's inequality we obtain $B(f) \in H^{\beta p}$. From Theorem 2 we have $A(f) \in H^\lambda$, $\lambda < 1$ for all $f \in H_0$. Hence $B(A(f)) \in H^{\beta\lambda}$ and $I_{\phi,\varphi}(f) \in H^{\beta\lambda}$ for all $f \in H_0$. □

Theorem 5. *If $\delta \in \mathbf{C}$, $\delta \neq -1, -2, \ldots$, φ, ϕ analytic functions and $\varphi(z)\phi(z) \neq 0$, $z \in U$ satisfies conditions (7), and A is the integral operator defined by (1) substituting γ with δ and I is the integral operator of Singh (3) then:*

(i) if $\beta > 1$ then $I(A) \in H^{\frac{\beta\lambda}{\beta-1}}$, $\lambda < 1$

(ii) if $0 < \beta \leq 1$ then $I(A) \in H^\infty$, for all $f \in H_0$.

Proof. From Theorem 2 we deduce $A \in H^\lambda$, $\lambda < 1$. From Lemma 4 we obtain the result. □

Theorem 6. *Let $\alpha \geq 0$, $\beta, \gamma \in \mathbf{C}$, with $\operatorname{Re} \beta > -1$ and $\operatorname{Re} \gamma > -1$ and let φ, ϕ analytic function with $\varphi(z)\phi(z) \neq 0$, $z \in U$. Let $F : H_0 \to H_0$ defined by (2) and suppose that are satisfying (8).*

If $\theta \in H_0$ and $\operatorname{Re} B(z) \geq \alpha - |C(z) - 1| - \operatorname{Re}[C(z) - 1] + 4|\theta(z)|$, $z \in U$, then the operator $J_\theta[f] = F[f] + f'(0)F[\theta]$ we have $J_\theta(f) \in H^\lambda$, $\lambda < 1$, for all $f \in H_0$.

Proof. From Lemma 3 we obtain that J_θ is averaging integral operator. Hence $J_\theta[f](U) \subset co\, f(U)$, where $co\, f(U)$ is convex domain. From Riemann's theorem exists a convex function g such that $g(U) = co\, f(U)$. Hence we deduce $M_\lambda(r, J_\theta) \leq M_\lambda(r, g)$, and we obtain the results. □

Remark 1. Analog with Theorem 4 we can obtain results for Hardy classes for $I[J_\theta]$ where I is the integral operator of Singh.

Remark 2. Analog with Theorem 7 [2] we can obtain results for Hardy classes for the n-order integral operator of Singh.

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