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DIFFERENTIAL AND INTEGRAL OPERATORS PRESERVING FUNCTIONS WITH POSITIVE REAL PART AND HARDY CLASSES

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1. Introduction

Let $\mathcal{H}(U)$ set of denote the functions analytic in the unit disk $U = \{z : |z| < 1\}$. In [4] the authors develop differential and integral operators preserving functions with positive real part.

In [2] and [3] sharp results concerning the boundary behaviour of $I(f)$, $I_g(f)$ and $I_{\phi,\varphi}(f)$ when f belongs to the Hardy spaces H^p, $0 < p \leq \infty$, where $I(f)$, $I_g(f)$ and $I_{\phi,\varphi}(f)$ is the integral operator defined by:

$$I[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U, \text{ (Singh, 1973)} \tag{1}$$

$$I_g(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z \left[\frac{f(t)}{t}\right]^\alpha \left[\frac{g(t)}{t}\right]^\delta t^{\alpha+\delta-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U \tag{2}$$

$$I_{\phi,\varphi}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma\phi(z)}\int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U \text{ (Miller, Mocanu, 1991)} \tag{3}$$

In this paper we obtain results for the Hardy classes of these integral operators when f satisfy some differential conditions.

2. Preliminaries

For $f \in \mathcal{H}(U)$ and $z = re^{i\theta}$ we denote

$$M(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \text{ for } 0 < p < \infty$$

and

$$M_\infty(r,f) = \sup_{0\le\theta\le 2\pi} |f(re^{i\theta})| \text{ for } p=\infty.$$

A function is said to be of Hardy class H^p, $0 < p \le \infty$ if $M(r,f)$ remains bounded as $r \to 1^-$, H^∞ is the class of bounded analytic functions in the unit disk.

We shall need the following lemmas:

Lemma 1. *Let $A \ge 0$ and $B, C, D : U \to \mathbf{C}$ with*

$$\begin{aligned} &\text{Re } B(z) \ge A \\ &[\text{Im } C(z)]^2 \le [\text{Re } B(z) - A]\text{Re } [B(z) - A - 2D(z)]. \end{aligned} \tag{4}$$

If f is analytic in U with $f(0) = 1$ and

$$\text{Re } [Az^2 f''(z) + B(z)zf'(z) + C(z)f(z) + D(z)] > 0 \tag{5}$$

then Re $f(z) > 0$.

Lemma 2. *Let $\eta \ne 0$, $\eta \in \mathbf{C}$,* Re $\eta \ge 0$ *and φ, ϕ analytic functions in U, $\varphi(z)\phi(z) \ne 0$, $\varphi(0) = \phi(0)$, and*

$$\left|\text{Im } \frac{\eta\phi(z) + z\phi'(z)}{\eta\varphi(z)}\right| \le \text{Re } \frac{\phi(z)}{\eta\varphi(z)}. \tag{6}$$

Let f be analytic in U with $f(0) = 1$ and Re $f(z) > 0$, $z \in U$.

If f is defined by:

$$F(z) = \frac{\eta}{z^\eta \phi(z)} \int_0^z f(t)t^{\eta-1}\varphi(t)dt \tag{7}$$

then F is analytic in U, $F(0) = 1$ and Re $F(z) > 0$, $z \in U$.

Lemma 3. *Let β and γ be complex numbers with $\beta\gamma > 0$,* Re $\beta \ge 0$, Re $\gamma \ge 0$, *φ and ϕ be analytic in U with $\varphi(z)\phi(z) \ne 0$, $\varphi(0) = \phi(0)$ and w be analytic in U with $w(0) = 0$. Suppose that (4) holds with*

$$\begin{cases} A = \dfrac{1}{B\gamma}, \quad D(z) = -w(z) \\ B(z) = \dfrac{1}{\beta}\left[\beta + \gamma + 1 + z\dfrac{\varphi'(z)}{\varphi(z)} + \dfrac{z\phi'(z)}{\phi(z)}\right] \\ C(z) = \dfrac{1}{\beta\gamma}\left[\left(\beta + \dfrac{z\phi'(z)}{\phi(z)}\right)\left(\gamma + \dfrac{z\varphi'(z)}{\varphi(z)} + z\left(\dfrac{z\varphi'(z)}{\varphi(z)}\right)'\right)\right] \end{cases} \tag{8}$$

Let f be analytic in U with $f(0) = 1$ and Re $f(z) > 0$, $z \in U$.

If f is defined by:

$$F(z) = \frac{\beta}{z^{\gamma}\varphi(z)} \int_0^z \frac{\varphi(t)}{\phi(t)} t^{\gamma-\beta-1} \int_0^t [f(s) + w(s)]\phi(s)s^{\beta-1} ds dt \tag{9}$$

then F is analytic in U, $F(0) = 1$ and Re $F(z) > 0$, $z \in U$.

Lemma 1-3 was proved in [4].

Lemma 4. [1] *If $f \in \mathcal{H}(U)$ and* Re $f(z) > 0$, $z \in U$ *then* $f \in H^p$, $p < 1$.

Lemma 5. *If $f \in H^p$, $b > 0$ and I is the integral operator of Singh (3) then*

(i) if $\beta > p$ then $I[f] \in H^{\frac{\beta p}{\beta - p}}$;

(ii) if $\beta \leq p$ then $I[f] \in H^{\infty}$.

Lemma 6. *If $f \in H^p$, $g \in H^q$, $p, q, \alpha, \beta, \delta \in \mathbf{R}_+^*$ then*

(i) if $\dfrac{pq}{\delta p + \alpha q} < 1$ then $I_g(f) \in H^{\frac{\beta pq}{\delta p + \alpha q - pq}}$;

(ii) if $\dfrac{pq}{\delta p + \alpha q} \geq 1$ then $I_g(f) \in H^{\infty}$.

Lemma 7. *If $f \in H^p$, $\varphi \in H^q$, $\dfrac{1}{\phi} \in H^r$, $\alpha, \beta > 0$ then*

(i) if $pq < p + \alpha q$ then $I_{\phi,\varphi}[f] \in H^{\frac{\beta pqr}{pq + pr + \alpha qr - pqr}}$;

(ii) if $pq > p + \alpha q$ then $I_{\phi,\varphi}[f] \in H^r$.

Lemma 5 was proved in [2] and Lemma 6 and Lemma 7 was proved in [3].

3. Main results

Theorem 1. *Let be $A \geq 0$ and $B, C, D : U \to \mathbf{C}$ satisfying condition (4). If f analytic in U, $f(0) = 1$ and*

$$\text{Re } [Az^2 f''(z) + B(z)zf'(z) + C(z)f(z) + D(z)] > 0, \quad A_i \in \mathbf{C}$$

then

$$F(z) = A_0 + A_1 f(z) + A_2 f^2(z) + \cdots + A_n f^n(z) \in H^{\frac{\lambda}{n}}, \quad \lambda < 1.$$

Proof. From Lemma 1 and Lemma 4 we have $f(z) \in H^{\lambda}$, $\lambda < 1$. Hence we deduce $f^n(z) \in H^{\frac{\lambda}{n}}$. By applying Minkowski's inequality we obtain the result. □

Theorem 2. *Let $A \geq 0$ and $B, C, D : U \to \mathbf{C}$ satisfying conditions (4) and f analytic in U, $f(0) = 1$ and (5). If $\beta > 0$, $\gamma \in \mathbf{C}$ and I is integral operator of Singh (1) then* *(i) if* $\beta > 1$ *then* $I[f] \in H^{\frac{\beta}{\beta-\lambda}}$, $\lambda < 1$;

(ii) if $\beta \leq 1$ *then* $I[f] \in H^{\infty}$.

Proof. From Lemma 1 and Lemma 4 we obtain $f(z) \in H^{\lambda}$, $\lambda < 1$. From Lemma 5 we obtain the result. □

Theorem 3. *Let be* $\delta \neq 0$, $\mathrm{Re}\ \delta \geq 0$ *and* φ, ϕ *analytic functions in* U, *with* $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ *satisfying condition (6),* f *analytic in* U *with* $f(0) = 1$ *and* $\mathrm{Re}\ f(z) > 0$, $z \in U$ *and* F *defined by (7). If* $\beta > 0$, $\gamma \in \mathbf{C}$ *and* I *is defined by (1) then:*

(i) if $\beta > 1$ *then* $I[F] \in H^{\frac{\beta}{\beta-\lambda}}$, $\lambda < 1$;

(ii) if $\beta \leq 1$ *then* $I[F] \in H^{\infty}$.

Proof. From Lemma 2 and Lemma 4 we deduce $F \in H^{\lambda}$, $\lambda < 1$, and from Lemma 5 we obtain the result. □

Theorem 4. *Let be* $\eta \neq 0$, $\eta \in \mathbf{C}$, $\mathrm{Re}\ \eta \geq 0$ *and* φ, ϕ *analytic functions in* U, *with* $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ *and satisfying (6).*

If f *analytic in* U, $f(0) = 1$, $\mathrm{Re}\ f(z) > 0$, $z \in U$, F *defined by (7) and* I_g *defined by (2) then:*

(i) if $\dfrac{\lambda\mu}{\delta\lambda + \alpha\mu} < 1$ *then* $I_F[f] \in H^{\frac{\beta\lambda\mu}{\delta\lambda+\alpha\mu-\lambda\mu}}$, $0 < \lambda < 1$, $0 < \mu < 1$;

(ii) if $\dfrac{\lambda\mu}{\delta\lambda + \alpha\mu} < 1$ *then* $I_F[f] \in H^{\infty}$, $0 < \lambda < 1$, $0 < \mu < 1$.

Proof. From Lemma 2 we obtain $\mathrm{Re}\ F(z) > 0$ and from Lemma 4 we deduce $F(z) \in H^{\mu}$, $\mu < 1$. Since $\mathrm{Re}\ f(z) > 0$ we have $f \in H^{\lambda}$, $\lambda < 1$. Applying again Lemma 6 we obtain the result. □

Remark 1. An analog result we can obtain for F defined by (9).

Theorem 5. *Let* $\alpha = 1$, $\beta > 0$, $\gamma \in \mathbf{C}$, $\mathrm{Re}\ \gamma \geq 0$, $\delta = \gamma$, ϕ *and* γ *analytic functions in* U, *with* $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ *satisfying (6) and* f *analytic in* U, $f(0) = 1$, $\mathrm{Re}\ f(z) \geq 0$, $z \in U$, *then*

$$I_{\phi,\varphi}[f] \in H^{p\beta}, \quad p < 1.$$

Proof. From Lemma 2, for $\eta \in \mathbf{C}^*$, Re $\eta \geq 0$,

$$F(z) = \frac{\eta}{z^\eta \phi(z)} \int_0^z f(t)\varphi(t)t^{\eta-1}\,dt,$$

we have Re $F(z) > 0$. Hence $F \in H^p$, $p < 1$.

For $\eta = \gamma$ we obtain

$$\frac{\gamma}{\beta+\gamma} \cdot \frac{\beta+\gamma}{z^\gamma \phi(z)} \int_0^z f(t)\varphi(t)t^{\gamma-1}\,dt \in H^p$$

and

$$\frac{\gamma}{\beta+\gamma} I_{\phi,\varphi} \in H^{p\beta}$$

and

$$I_{\phi,\varphi} \in H^{p\beta}.$$

□

Theorem 6. *Let $\eta \neq 0$, $\eta \in \mathbf{C}$,* Re $\eta \geq 0$ *and φ, ϕ analytic functions in U, with $\varphi(z)\phi(z) \neq 0$, $\varphi(0) = \phi(0)$ satisfying (6). Let be f analytic in U, $f(0) = 1$,* Re $f(z) >$ *0, $z \in U$ and F defined by (7). If $i_{\phi,\varphi}$ is defined by (3), $\alpha, \beta, \gamma, \delta > 0$ and $g \in H^p$, $0 < \lambda < 1$, $0 < \mu < 1$ then*

(i) if $p\lambda < p + \alpha\lambda$ then $U_{F,f}(g) \in H^{\frac{\beta p\lambda\mu}{p\lambda+p\mu+\alpha\lambda\mu-p\lambda\mu}}$;

(ii) if $p\lambda \geq p + \alpha\lambda$ then $I_{F,f}(g) \in H^\infty$.

Proof. From Lemma 4 we have $f \in H^\lambda$, $\lambda < 1$. From Lemma 2 we obtain Re $F(z) > 0$ and Re $\frac{1}{F(z)} > 0$. From Lemma 4 we have $\frac{1}{F(z)} \in H^\mu$, $\mu < 1$.

Applying again Lemma 7 replacing ϕ with F, φ with f and f with g we obtain the result. □

Remark 2. An analog result we can obtain for F defined by (9) or g is defined by (9).

References

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