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SOME ANALYTIC INTEGRAL OPERATORS AND HARDY CLASSES

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1. Introduction

Let A denote the set of functions $f(x) = z + a_2z^2 + \dots$ that are analytic in the unit disk U and S denote the subset of A consisting of univalent functions. In [4] the authors show that the integral operator

$$I_{\phi,\varphi}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma\phi(z)}\int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}} \tag{1}$$

maps certain subsets of A into S.

In [2] and [3] were obtained Hardy classes for integral operator (1) and

$$I[f](z= \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}, \quad z \in U. \tag{2}$$

In this paper we obtain Hardy classes for these operators using the "open door" function [4], a special mapping from U onto a slit domain.

2. Preliminaries

Definition 1. Let c be a complex number such that Re $c > 0$ and let

$$N = N(c) = \frac{1}{\text{Re } c}\left[|c|(1+2\text{Re } c)^{\frac{1}{2}} + \text{Im } c\right].$$

If h is the univalent function $h(z) = \dfrac{2Nz}{1-z^2}$ and $b = h^{-1}(c)$ then we define the "open door" function Q_c as

$$Q_c(z) = h\left(\frac{z+b}{1+\bar{b}z}\right), \quad z \in U.$$

From its definition we see that Q_c is univalent, $Q_c(0) = c$ and $Q_o(U) = h(U)$ is the complex plane slit along the half-lines Re $w = 0$, Im $w \geq N$ and Re $w = 0$, Im $w \leq -N$.

Definition 2. Let f and g be analytic in U. The function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$ if g is univalent, $f(0) = g(0)$ and $f(U) \subset g(U)$.

For f analytic in U and $z = re^{i\theta}$ we denote

$$M_p(r,f) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, & \text{for } 0 < p < \infty \\ \sup\limits_{0\leq\theta<2\pi} |f(re^{i\theta})|, & \text{for } p = \infty. \end{cases}$$

A function is said to be of Hardy class H^p, $0 < p \leq \infty$ if $M_p(r,f)$ remains bounded as $r \to 1^-$, H^∞ is the class of bounded analitic functions in the unit disk.

We shall need the following lemmas:

Lemma 1. [5] *Let Q_c be the function given by Definition 1 and let $B(z)$ be analytic function in U satisfying $B(z) \prec Q_c(z)$.*

If p is analytic in U, $p(0) = \frac{1}{c}$ and p satisfies the differential equation $zp'(z) + B(z)p(z) = 1$ then Re $p(z) > 0$, $z \in U$.

Lemma 2. [4] *Let $\alpha, \delta \in \mathbf{C}$,* Re $(\alpha + \delta) > 0$ *and let φ be analytic function in U with $\varphi(0) = 1$, $\varphi(z) \neq 0$ in U. If $f \in A$ satisfies*

$$\alpha\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec Q_{\alpha+\delta}(z) \tag{3}$$

and F is defined by

$$F(z) = I[f](z) = \left[(\alpha+\delta)\int_0^z f^\alpha(t)t^{\delta-1}\varphi(t)dt\right]^{\frac{1}{\alpha+\delta}} \tag{4}$$

then

$$\text{Re } \left[(\alpha+\delta)\frac{zF'(z)}{F(z)}\right] > 0 \circ i \ F \in S.$$

Moreover, if $\alpha + \delta > 0$ then $F \in S^*$ (starlike functions).

Lemma 3. (Prawitz, 1927) *If $f \in S$, then $f \in H^p$, $p < \frac{1}{2}$.*

3. Main results

Theorem 1. *If $\beta > 0$, $\gamma > 0$, $f \in A$ and* $\dfrac{\beta z f'(z)}{f(z)} \prec Q_c(z)$ *then*

$$I(f) \in H^{\lambda}, \quad \lambda = \frac{q\beta}{\beta+\gamma}, \quad q < \frac{1}{2}$$

where I is defined by (2).

Proof. The operator I as given by (2), can be written as $I : I = G_1 \circ G_2$ where

$$G_1[f](z) = \left(\frac{f^{\beta+\gamma}(z)}{z^{\gamma}}\right)^{\frac{1}{\beta}}$$

$$G_2[f](z) = \left((\beta+\gamma)\int_0^z f^{\beta}(t)t^{\gamma-1}dt\right)^{\frac{1}{\beta+\gamma}}.$$

If in Lemma 1 $\varphi(z) \equiv 1$ then condition (3) to maps $\dfrac{\beta z f'(z)}{f(z)} \prec Q_c(z)$ and the operator F defined by (4) to maps in G_2. Hence $G_2(z) \in S^*$ and $G_2 \in H^q$, $q < \dfrac{1}{2}$. On the other hand if $f \in H^p$ then $f^{\beta+\gamma} \in H^{\frac{p}{\beta+\gamma}}$ and $\dfrac{f^{\beta+\gamma}(z)}{z^{\gamma}} \in H^{\frac{p}{\beta+\gamma}}$. Hence $\left(\dfrac{f^{\beta+\gamma}(z)}{z^{\gamma}}\right)^{\frac{1}{\beta}} \in H^{\frac{\beta p}{\beta+\gamma}}$ and we obtain $G_1 \in H^{\frac{\beta p}{\beta+\gamma}}$ if $f \in H^p$. Since $G_2 \in H^q$, $q < \dfrac{1}{2}$ we obtain $G_1 \circ G_2 \in H^{\frac{q\beta}{\beta+\gamma}}$, $q < \dfrac{1}{2}$. □

Corollary 2. *Let α, δ complex numbers with* Re $(\alpha+\delta) > 0$ *and φ analytic in U, $\varphi(0) = 1$, $\varphi(z) \neq 0$, $z \in U$. If $f \in A$ and satisfying (3) then $F \in H^{\lambda}$, $\lambda < \dfrac{1}{2}$ (F defined by (4)). If $\alpha + \delta > 0$ then $F \in H^{\frac{1}{2}}$ and $F' \in H^{\frac{1}{3}}$.*

Theorem 3. *Let Q_c "open door" function and $B(z)$ analytic in U satisfying $B(z) \prec Q_c(z)$. Let ϕ an analytic function in U, $\phi(0) = \dfrac{1}{c}$, and $z\phi'(z) + B(z)\phi(z) = 1$. Let α, β, δ be real numbers $\beta > 0$, $\alpha\delta > 0$ and φ analytic in U with $\varphi(0) = 1$, $\varphi(z) \neq 0$, $z \in U$. If $f \in A$ satisfies (3) then*

$$I_{\phi,\varphi}[f] \in H^p, \quad p = \frac{q\lambda\beta}{\lambda(\alpha+\delta)+q}, \quad \lambda < 1, \quad q < \frac{1}{2}.$$

Proof. From Lemma 1 we have Re $\phi(z) > 0$. Hence Re $\frac{1}{\phi(z)} > 0$ and $\frac{1}{\phi(z)} \in H^\lambda$, $\lambda < 1$. The integral operator $I_{\phi,\varphi}$ as given by (1) can be written as: $I_{\phi,\varphi} = G \circ F$ where $G(x) = \left(\frac{f^{\alpha+\delta}(z)}{z^\gamma \phi(z)}\right)^{\frac{1}{\beta}}$ and F is defined by (4).

From Lemma 2, $F \in S^*$ and from Lemma 3, $F \in H^q$, $q < \frac{1}{2}$. Since $\frac{1}{\phi(z)} \in H^\lambda$ applying Hölder's inequality we obtain $\frac{1}{z^\gamma \phi(z)} F^{\alpha+\delta}(z) \in H^\mu$ where

$$\mu = \frac{\lambda \frac{q}{\alpha+\delta}}{\lambda + \frac{q}{\alpha+\delta}} = \frac{q\lambda}{\lambda(\alpha+\delta)+q}, \quad q < \frac{1}{2}.$$

Hence

$$G(F) \in H^p, \quad p = \frac{q\lambda\beta}{\lambda(\alpha+\delta)+q}, \quad q < \frac{1}{2}, \quad \lambda < 1.$$

□

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