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A NUMERICAL METHOD FOR APPROXIMATINGTHE SOLUTION OF AN INTEGRAL EQUATION FROM BIOMATHEMATICS

C. IANCU

Abstract. 'In lucrare se dă o metodă numerică unei ecuaţii integrale cu argument modificat, care modelează procesul de răspândire a unei infecţii. Rezultatul principal al lucrării este enunţat sub forma teoremei 5.1

1. Introduction

In the study of the problem which appears in the population dynamics, where certain periodical phenomena occur, the following integral equation holds:

$$x(t) = \int_{t-\tau}^{t} f(u, x(u))du, \quad t \in \mathbf{R} \tag{1.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}_+)$ fulfils the condition of periodicity with respect to t, that is

$$f(t+\omega, x) = f(t, x), \text{ for } t \in \mathbf{R},\ x \in \mathbf{R}_+,\ \omega > 0. \tag{1.2}$$

If we suppose that $\tau \in \mathbf{R}_+$ and

$$0 \leq f(t, x) \leq M, \text{ for } t \in \mathbf{R} \text{ and } x \in \mathbf{R}_+ \tag{1.3}$$

then the problem of finding periodical solutions of equation (1.1) can be considered.

The equation (1.1) can be a mathematical model for the spreading of certain infectious diseases with a contact rate that varies seasonally. In this case $x(t)$ represents the proportion of the infectives in population at the time t, τ is the time interval an individual remains infectious and $f(t, x(t))$ represents the proportion of new infectives per unit time. In the papers [1]-[4], it is tackled this important problem and are given sufficient conditions for existence of non-trivial periodic nonnegative and continuous solutions of equation (1.1).

On the basis of these results, the aim of this paper is to present a numerical method for obtaining the solutions of equation (1.1).

2. The existence and uniqueness of solution

In [5] the following mapping is attached to equation (1.1):

$$A : X_+ \to C(\mathbf{R}),$$

which is defined by the right-hand side of (1.1), where

$$X_+ = \{x \in X \mid x(t) > 0,\ (\forall)\ t \in \mathbf{R}\},$$

and

$$X = \{x \in C(\mathbf{R}) \mid x(t+\omega) = x(t),\ (\forall)\ t \in \mathbf{R}\}.$$

Because we have

$$(Ax)(t+\omega) = \int_{t+\omega-\tau}^{t+\omega} f(s, x(s))ds = \int_{t-\tau}^{t} f(s+\omega, x(s+\omega))ds =$$
$$= \int_{t-\tau}^{t} f(s, x(s))ds = (Ax)(t)$$

and

$$t - \tau < t, \quad f \geq 0,$$

it results that X_+ is a invariant subset of A.

If we suppose that

$$|f(t,x) - f(t,y)| \leq a(t)|x-y|,\ (\forall)\ t \in \mathbf{R} \text{ and } x, y \in \mathbf{R}_+ \tag{2.1}$$

$$\int_{t-\tau}^{t} a(s)ds \leq q \leq 1 \text{ for all } t \in \mathbf{R} \tag{2.2}$$

then A is a contraction mapping.

The following result is given in [5]:

Theorem 2.1. *If the conditions (1.2), (1.3), (2.1) and (2.2) are satisfied, then in $C(\mathbf{R}, \mathbf{R}_+)$ the equation (1.1) has a unique periodic continuous nonnegative solution which can be obtained by the method of successive approximations.*

Also, in [3], is proved the following theorem.

Theorem 2.2. *If the following assumptions are satisfied: (i) $f(t,x)$ is nonnegative and continuous for*

$$-\tau \leq t \leq T \text{ and } x \geq 0,\ T > 0$$

(ii) $\phi(t)$ is continuous and $0 < a \leq \phi(t)$ foe $-\tau \leq t \leq 0$ where the proportion $\phi(t)$ of infectives in population is known for $-\tau \leq t \leq 0$, i.e.

$$x(t) = \phi(t), \text{ for } -\tau \leq t \leq 0$$

and

$$\phi(0) = b = \int_{-\tau}^{0} f(s, \phi(s))ds$$

(iii) there exists an integrable function $g(t)$ such that $f(t,x) \geq g(t)$ for $-\tau \leq t \leq T$, $x \geq a$ and

$$\int_{t-\tau}^{t} g(s)ds \geq a \text{ for } 0 \leq t \leq T$$

(iv) there exists $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y|$$

for all $t \in [-\tau, T]$ and $x, y \in [a, \infty)$, then equation (1.1) has a unique continuous solution $x(t)$, $x(t) \geq a$, for $-\tau \leq t \leq T$, which satisfies the condition $x(t) = \phi(t)$, for $-\tau \leq t \leq 0$; moreover,

$$\max_{0 \leq t \leq T} |x_n(t) - x(t)| \to 0 \text{ as } n \to \infty$$

where $x_n(t) = \phi(t)$ for $-\tau \leq t \leq 0$ $(n = 0, 1, 2, \ldots)$, $x_0(t) = b$ and $x_n(t) = \int_{t-\tau}^{t} f(s, x_{n-1}(s))ds$, $0 \leq t \leq T$ $(n = 1, 2, \ldots)$.

3. The statement of the problem

We consider the nonlinear integral equation (1.1) and we suppose that the hypotheses of the Theorems 2.1 and 2.2 are satisfied. Then this equation has a unique solution on the interval $[-\tau, T]$. Let φ be the solution, which, by virtue of the theorem 2.2, can be obtained by successive approximation method. So, we have

$$\begin{cases} \varphi(t) = \phi(t), \text{ for } t \in [-\tau, 0) \text{ and for } t \in [0, T] \\ \qquad \text{we have :} \\ \varphi_0(t) = \phi(0) = b = \int_{-\tau}^{0} (s, \phi(s))ds \\ \varphi_1(t) = \int_{t-\tau}^{t} f(s, \varphi_0(s))ds \\ \varphi_2(t) = \int_{t-\tau}^{t} f(s, \varphi_1(s))ds \\ \dots \\ \varphi_m(t) = \int_{t-\tau}^{t} f(s, \varphi_{m-1}(s))ds, \\ \dots \end{cases} \tag{3.1}$$

To obtain the sequence of successive approximations (3.1), it is necessary to calculate the integrals which appear in the right-hand side. In general, this problem is difficult. We shall use the trapezoidal rule.

Let an interval $[a, b] \subseteq \mathbf{R}$ be given, and the function $f \in C^2[a, b]$.

Divide the interval $[a, b]$ by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{3.2}$$

into n equal parts of length $\Delta x = \dfrac{b-a}{n}$.

Then we have the trapezoidal formula:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{i=1}^{n-1} f(x_i)\right] + r_n(f) \tag{3.3}$$

where $r_n(f)$ is the remainder of the formula.

To evaluate the approximation error of the trapezoidal formula there exists the following result.

Theorem 4.1. *For every function $f \in C^2[a,b]$, the remainder $r_n(f)$ from the trapezoidal formula (3.3), satisfies the inequality:*

$$|r_n(f)| \leq \frac{(b-a)^3}{12n^2} \max_{x\in[a,b]} |f''(x)|. \tag{3.4}$$

4. The calculation of the integrals which appear in the successive approximations methods, (3.3) [6]

Now we suppose that $f \in C^2([0,T] \times \mathbf{R}_+)$, and in order to calculate the integral φ_m from (3.1), we apply the formula (3.3). Then we divide the interval $[0,T]$ by the points:

$$0 = t_0 < t_1 < \cdots < t_n = T \tag{4.1}$$

where: $t_i = t_{i-1} + h$, $h = \frac{\tau}{2^v}$, $v = 0,1,2,\ldots$, $i = \overline{1,n}$, $n = \left[\frac{T}{h}\right]$ ($[\cdot]$ is integer part).

Thus we have

$$\varphi_m(t_k) = \int_{t_k-\tau}^{t_k} f(s,\varphi_{m-1}(s))ds = \tag{4.2}$$

$$= \frac{\tau}{2n}\left[f(t_k-\tau,\varphi_{m-1}(t_k-\tau)) + f(t_k,\varphi_{m-1}(t_k)) + 2\sum_{i=1}^{n-1} f(t_i,\varphi_{m-1}(t_i))\right] + r_{m,k}(f),$$

where, for the remainder $r_{m,k}(f)$, we have the estimation:

$$|r_{m,k}(f)| \leq \frac{\tau^3}{12n^2} \max_{s\in[0,T]} |[f(s,\varphi_{m-1}(s))]''_s|, \quad k = \overline{0,n},\ m \in \mathbf{N}.$$

Taking into account the fact that:

$$[f(s,\varphi_{m-1}(s))]''_s = \frac{\partial^2 f(s,\varphi_{m-1}(s))}{\partial s^2} + 2\frac{\partial^2 f(s,\varphi_{m-1}(s))}{\partial s\partial\varphi}\varphi'_{m-1}(s) +$$

$$+\frac{\partial^2 f(s,\varphi_{m-1}(s))}{\partial\varphi^2}(\varphi'_{m-1}(s))^2 + \frac{\partial f(s,\varphi_{m-1}(s))}{\partial\varphi}\varphi''_{m-1}(s)$$

and:

$$\varphi_{m-1}(t) = \int_{t-\tau}^{t} f(s,\varphi_{m-2}(s))ds$$

$$\varphi'_{m-1}(t) = \int_{t-\tau}^{t} \frac{\partial f(s,\varphi_{m-2}(s))}{\partial s}ds$$

$$\varphi''_{m-1}(t) = \int_{t-\tau}^{t} \frac{\partial^2 f(s, \varphi_{m-2}(s))}{\partial s^2} ds$$

and denoting by

$$M_1 = \max_{\substack{|\alpha|\leq 2 \\ s\in[0,T] \\ |u|\leq R}} \left| \frac{\partial^\alpha f(s,u)}{\partial s^{\alpha_1} \partial u^{\alpha_2}} \right|,$$

we obtain

$$|\varphi_{m-1}(t)| \leq \tau M_1; \quad |\varphi'_{m-1}(t)| \leq \tau M_1; \quad |\varphi''_{m-1}(t)| \leq \tau M_1$$

again from here we have:

$$|[f(s, \varphi_{m-1}(s))]''_s| \leq M_0 \tag{4.3}$$

where $M_0 = M_1 + 3\tau M_1^2 + \tau^2 M_1^3$ and M_0 does not depend on m and k.

For the remainder $r_{m,k}(f)$, from the formula (4.2) we have:

$$|r_{m,k}(f)| \leq \frac{\tau^3}{12n^2} M_0, \quad m = 0, 1, 2, \ldots, \ k = \overline{0, n}. \tag{4.4}$$

In this way we have obtained a formula for the approximate calculation of the integrals from (3.1).

5. The approximate calculation of the terms of the successive approximations sequence

Using the approximation (3.1) and the formula (4.2) with the remainder estimation (4.4), we shall present further down an algorithm for the approximate solution of equation (1.1).

So, we have:

$$\varphi_1(t_k) = \int_{t_{k-\tau}}^{t_k} f(s, \varphi_0(s)) ds =$$

$$= \frac{\tau}{2n} \left[f(t_k - \tau, \varphi_0(t_k - \tau)) + 2 \sum_{i=1}^{n-1} f(t_i, \varphi(t_i)) + f(t_k, \varphi_0(t_k)) \right] + r_{1,k}(f) =$$

$$= \tilde{\varphi}_1(t_k) + r_{1,k}(f), \quad k = \overline{0, n}$$

$$\varphi_2(t_k) = \int_{t_{k-\tau}}^{t_k} f(s, \varphi_1(s)) ds = \frac{\tau}{2n} \Big[f(t_k - \tau, \tilde{\varphi}_1(t_k - \tau) + r_{1,0}(f)) +$$

$$+2\sum_{i=1}^{n-1} f(t_i,\tilde{\varphi}_1(t_i)+r_{1,i}(f))+f(t_k,\tilde{\varphi}_1(t_k)+r_{1,n}(f))\Bigg]+r_{2,k}(f)=$$

$$=\frac{\tau}{2n}\left[f(t_k-\tau,\tilde{\varphi}_1(t_k-\tau))+2\sum_{i=1}^{n-1} f(t_i,\tilde{\varphi}_1(t_i))+f(t_k,\tilde{\varphi}_1(t_k))\right]+\tilde{r}_{2,k}(f)=$$

$$=\tilde{\varphi}_2(t_k)+\tilde{r}_{2,k}(f).$$

Observe that $\tilde{r}_{2,k}(f)=\varphi_2(t_k)-\tilde{\varphi}_2(t_k)$.

Taking into account Theorem 2.2,(iv), and the remainder estimation given by (4.4), we have:

$$|\tilde{r}_{2,k}(f)|\le\frac{\tau}{2n}L\left[|r_{1,0}(f)|+\sum_{i=1}^{n-1}|r_{1,i}(f)|+|r_{1,n}(f)|\right]+|r_{2,k}(f)|\le$$

$$\le\frac{\tau}{2n}L\left(\frac{\tau^3}{12n^2}M_0+(n-1)\frac{\tau^3}{12n^2}M_0+\frac{\tau^3}{12n^2}M_0\right)+\frac{\tau^3}{12n^2}M_0=$$

$$=\frac{\tau}{2n}L\cdot\frac{\tau^3}{12n^2}M_0(1+n-1+1)+\frac{\tau^3}{12n^2}M_0=$$

$$=\frac{\tau^3}{12n^2}M_0\left[\frac{(n+1)\tau}{2n}L+1\right]\le\frac{\tau^3}{12n^2}M_0(\tau L+1).$$

We continue in this manner, for $m=3,\dots$, by induction, and obtain:

$$\varphi_m(t_k)=\frac{\tau}{2n}\Bigg[f(t_k-\tau,\tilde{\varphi}_{m-1}(t_k-\tau)+\tilde{r}_{m-1,0}(f))+$$

$$+2\sum_{i=1}^{n-1} f(t_i,\tilde{\varphi}_{m-1}(t_i)+\tilde{r}_{m-1,i}(f))+$$

$$+f(t_k,\tilde{\varphi}_{m-1}(t_k)+\tilde{r}_{m-1,n}(f))\Bigg]+r_{m,k}(f)=$$

$$=\frac{\tau}{2n}\Bigg[f(t_k-\tau,\tilde{\varphi}_{m-1}(t_k-\tau))+2\sum_{i=1}^{n-1} f(t_i,\tilde{\varphi}_{m-1}(t_i))+$$

$$+f(t_k,\tilde{\varphi}_{m-1}(t_k))\Bigg]+\tilde{r}_{m,k}(f)=\tilde{\varphi}_m(t_k)+\tilde{r}_{m,k}(f),\quad k=\overline{0,n}$$

where

$$|\tilde{r}_{m,k}(f)|=|\varphi_m(t_k)-\tilde{\varphi}_m(t_k)|\le$$

$$\le\frac{\tau^3}{12n^2}M_0[\tau^{m-1}L^{m-1}+\dots+1],\quad k=\overline{0,n}$$

or

$$|\tilde{r}_{m,k}(f)| \leq \frac{\tau^3}{12n^2} M_0 \frac{1-\tau^m L^m}{1-\tau L} \leq \frac{\tau^3 M_0}{12n^2(1-\tau L)}.$$

In this way we got the sequence

$$(\tilde{\varphi}_m(t_k))_{m\in\mathbf{N}}, \quad k=\overline{0,n}$$

which approximates the sequence of successive approximation (3.1) on the knots (4.1), with the error

$$|\varphi_m(t_k) - \tilde{\varphi}_m(t_k)| \leq \frac{\tau^3 M_0}{12n^2(1-\tau L)}. \tag{5.1}$$

By Picard's theorem, [6], we have the following estimation

$$|\varphi(t_k) - \varphi_m(t_k)| \leq \frac{\tau^m L^m}{1-\tau L} ||\varphi_0 - \varphi_1||_{C[0,T]}. \tag{5.2}$$

In this way there was obtained the main result of our paper:

Theorem 5.1. *Consider the integral equation (1.1) under the conditions of Theorems 2.1 and 2.2. If the exact solution φ is approximated by the sequence $(\tilde{\varphi}_m(t_k))_{m\in\mathbf{N}}$, $k=\overline{0,n}$, on the knots (4.1), by the successive approximations method (3.1), combined with the trapezoidal rule (3.3), then the following error estimation holds:*

$$|\varphi(t_k) - \tilde{\varphi}_m(t_k)| \leq \frac{\tau^3}{1-\tau L}\left[\tau^{m-3} L^m ||\varphi_0 - \varphi_1||_{C[0,T]} + \frac{M_0}{12n^2}\right], \tag{5.3}$$

$$m = 1, 2, \ldots, \quad k = \overline{0,n}.$$

Proof. We have

$$|\varphi(t_k) - \tilde{\varphi}_m(t_k)| = |\varphi(t_k) - \varphi_m(t_k) + \varphi_m(t_k) - \tilde{\varphi}_m(t_k)| \leq$$

$$\leq |\varphi(t_k) - \varphi_m(t_k)| + |\varphi_m(t_k) - \tilde{\varphi}_m(t_k)|$$

which, by virtue of formulae (5.1) and (5.2), can also be written

$$|\varphi(t_k) - \tilde{\varphi}_m(t_k)| \leq \frac{\tau^m L^m}{1-\tau L} ||\varphi_0 - \varphi_1||_{C[0,T]} + \frac{\tau^3 M_0}{12n^2(1-\tau L)}$$

and, from here, it results immediately (5.3). The theorem is proved. □

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BABEŞ-BOLYAI UNIVERSITY, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
RO-3400 CLUJ-NAPOCA, ROMANIA