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THE CONTINUITY OF THE METRIC PROJECTION OF A FIXED POINT ONTO MOVING CLOSED-CONVEX SETS IN UNIFORMLY-CONVEX BANACH SPACES

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We will show that a result similar to Hölder continuity in Hilbert spaces of the metric projections of a fixed point onto a pseudo-Lipschitz continuous family of closed convex sets [6] holds for uniformly-convex Banach spaces. The continuity of the metric projections with respect to perturbations play an important role in the sensitivity analysis of variational inequalities in Hilbert spaces [1, 3, 4, 6, 7] and hence in a wide range of nonlinear optimization, evolution and boundary value problems. The results from this paper offer us the possibility of extending the studies involving the metric projections in a larger class of spaces.

We denote by (Λ, d) a metric space and by X a uniformly-convex Banach space. We suppose X^* locally-uniformly-convex. Let $\omega_0, x_0 \in X$, $\lambda_0 \in \Lambda$, and their neighborhoods $\Omega_0 = B(\omega_0, r)$ (the closed ball centered at ω_0 and radius r) of ω_0, Λ_0 of λ_0. Let $C : \Lambda_0 \rightsquigarrow X$ be a set-valued mapping with nonempty, closed, convex values. Let us consider the following problem:

- for $\lambda \in \Lambda_0$ and $\omega \in \Omega_0$ find $x(\omega, \lambda) = P_{C(\lambda)}(\omega) \in C(\lambda)$ such that

$$||\omega - x(\omega, \lambda)|| = \min_{x \in C(\lambda)} ||\omega - x|| \,. \tag{1}$$

In our context such an element exists for all $\omega \in \Omega_0$ and $\lambda \in \Lambda_0$ and satisfies

$$\langle J(\omega - x(\omega, \lambda)) \,, x - x(\omega, \lambda) \rangle \leq 0 \,, \quad \forall \; x \in C(\lambda) \,, \tag{2}$$

where J is the normalized duality mapping.
(2) is equivalent with

$$0 \in -J(\omega - x(\omega, \lambda)) + N_{C(\lambda)}(x(\omega, \lambda)) \,, \tag{3}$$

where

$$N_{C(\lambda)}(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq 0, \ \forall \ y \in C(\lambda)\}$$

is the normal cone to the set $C(\lambda)$ at the point x.
Hence we need to study the sensitivity with respect to λ of the following generalized equation:

$$0 \in -J(\omega - x) + N_{C(\lambda)}(x) . \tag{4}$$

For Theorem 1 it is enough to consider (Ω, d) be a metric space, $\omega_0 \in \Omega$ and Ω_0 be a neighborhood of ω_0. Let $f : X_0 \times \Omega_0 \to X^*$ be a single-valued mapping .

Definition 1. *The mappings $f(\cdot, \omega)$ are φ-monotone on X_0 for all $\omega \in \Omega_0$, if there exists an increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(r) > 0$ when $r > 0$, such that*

$$\langle f(x_1, \omega) - f(x_2, \omega), x_1 - x_2\rangle \geq \varphi(\|x_1 - x_2\|)\|x_1 - x_2\| ,$$

for all $x_1, x_2 \in X_0$ and $\omega \in \Omega_0$.

The following proposition shows that the φ-monotonicity assumption is a natural one in uniformly-convex Banach spaces.

Proposition 1. [5] *A Banach space X is uniformly-convex if and only if for each $R > 0$ there exists an increasing function $\varphi_R : \mathbf{R}_+ \to \mathbf{R}_+$, with $\varphi_R(r) > 0$ when $r > 0$, such that the normalized duality mapping $J : X \rightsquigarrow X^*$, defined by*

$$J(x) = \{ x^* \in X^* : \langle x^*, x\rangle = \|x\|^2, \|x\| = \|x^*\| \} ,$$

is φ_R-monotone in $B(0, R)$.

Definition 2. *C is pseudo-continuous around $(\lambda_0, x_0) \in Graph\, C$ if there exist neighborhoods $V \subset \Lambda_0$ of λ_0, $W \subset X_0$ of x_0 and there exists a function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0, with $\beta(0) = 0$, such that*

$$C(\lambda_1) \cap W \subset C(\lambda_2) + \beta\,(d(\lambda_1, \lambda_2))\, B(0, 1) \tag{5}$$

for all $\lambda_1, \lambda_2 \in V$.

If the function β *is defined as* $\beta(r) = Lr$, *with* $L \geq 0$, *([2]) then we say that* C *is pseudo-Lipschitz continuous around* (λ_0, x_0).

Theorem 1. *Let us suppose that:*

a) $0 \in f(x_0, \omega_0) + N_{C(\lambda_0)}(x_0)$;

b) f *is continous on* $X_0 \times \Omega_0$;

c) the mappings $f(\cdot, \omega)$ *are* φ*-monotone in* X_0 *for all* $\omega \in \Omega_0$;

d) C *is pseudo-continuous around* (λ_0, x_0).

Then there exist neighborhoods Λ_1 *of* λ_0, Ω_1 *of* ω_0 *and a unique continuous mapping* $x : \Omega_1 \times \Lambda_1 \to X_0$, *such that* $x(\omega_0, \lambda_0) = x_0$ *and* $x(\omega, \lambda)$ *is a solution of the variational inequality*

$$0 \in f(x, \omega) + N_{C(\lambda)}(x) ,$$

for all $(\omega, \lambda) \in \Omega_1 \times \Lambda_1$.

Proof. Let us note that assumptions *b), c)* imply that $\varphi(r) \to 0$, $\varphi(r)r \to 0$ iff $r \to 0$. We choose positive constants s, r, ε such that $B(x_0, s) \subset X_0$, $B(\lambda_0, \varepsilon) \subset \Lambda_0$, $B(\omega_0, r) \subset \Omega_0$, $\beta(d(\lambda, \lambda_0)) \leq s$ for all $\lambda \in B(\lambda_0, \varepsilon)$ and the pseudo-continuity of C to be written as:

$$C(\lambda_1) \cap B(x_0, s) \subset C(\lambda_2) + \beta(d(\lambda_1, \lambda_2))\, B(0, 1)$$

for all $\lambda_1, \lambda_2 \in B(\lambda_0, \varepsilon)$.

Let $\lambda \in B(\lambda_0, \varepsilon)$ and $\omega \in B(\omega_0, r)$ be arbitrarily choosen. Then the inclusion

$$x_0 \in C(\lambda_0) \cap B(x_0, s) \subset C(\lambda) + Ld(\lambda, \lambda_0)\, B(0, 1)$$

implies the existence of an $u_\lambda \in C(\lambda)$ such that

$$||x_0 - u_\lambda|| \leq \beta(d(\lambda, \lambda_0)) \leq s .$$

This means that $C(\lambda) \cap B(x_0, s)$ is nonempty for all $\lambda \in B(\lambda_0, \varepsilon)$. Corollary 32.35 from [8] shows that the variational inequality

$$0 \in f(x,\omega) + N_{C(\lambda)\cap B(x_0,s)}(x)$$

has a unique solution $x(\omega, \lambda) \in C(\lambda) \cap B(x_0, s)$. So

$$\langle f(x(\omega,\lambda), \omega), u - x(\omega,\lambda)\rangle \geq 0$$

for all $u \in C(\lambda) \cap B(x_0, s)$.

The pseudo-Lipschitz continuity of the set-valued mapping C implies that for $x(\omega, \lambda)$ there exists an element $u_0 \in C(\lambda_0)$ such that $||x(\omega,\lambda) - u_0|| \leq \beta(d(\lambda, \lambda_0))$.

Using the φ-monotonicity of $f(\cdot, \omega)$ we obtain

$$\begin{aligned}
&\varphi(||x(\omega,\lambda) - x_0||)\, ||x(\omega,\lambda) - x_0|| \leq \\
&\leq \langle f(x(\omega,\lambda), \omega) - f(x_0,\omega), x(\omega,\lambda) - x_0\rangle \leq \\
&\leq \langle f(x(\omega,\lambda), \omega) - f(x_0,\omega), x(\omega,\lambda) - x_0\rangle + \langle f(x_0,\omega_0), u_0 - x_0\rangle + \\
&+ \langle f(x(\omega,\lambda), \omega), u_\lambda - x(\omega,\lambda)\rangle = \\
&= \langle f(x(\omega,\lambda), \omega), u_\lambda - x_0\rangle + \langle f(x_0,\omega), u_0 - x(\omega,\lambda)\rangle + \\
&+ \langle f(x_0,\omega_0) - f(x_0,\omega), u_0 - x_0\rangle \leq \\
&\leq ||f(x(\omega,\lambda), \omega)||\, ||u_\lambda - x_0|| + ||f(x_0,\omega)||\, ||u_0 - x(\omega,\lambda)|| + \\
&+ ||f(x_0,\omega_0) - f(x_0,\omega)||\, ||u_0 - x_0|| .
\end{aligned}$$

Assumption *a)* implies that $||f(x_0, \omega_0)|| < \infty$, and hence using the continuity of f, we can suppose that $||f(x,\omega)|| \leq M < \infty$, for all $x \in B(x_0, s)$ and $\omega \in B(\omega_0, r)$.

We know also that

$$\begin{aligned}
||u_0 - x_0|| &\leq ||u_0 - x(\omega,\lambda)|| + ||x(\omega,\lambda) - x_0|| \leq \\
&\leq \beta(d(\lambda,\lambda_0)) + s .
\end{aligned}$$

So,

$$\begin{aligned}
&\varphi(||x(\omega,\lambda) - x_0||)\, ||x(\omega,\lambda) - x_0|| \leq \\
&\leq 2M\beta(d(\lambda,\lambda_0)) + ||f(x_0,\omega_0) - f(x_0,\omega)||(\beta(d(\lambda,\lambda_0)) + s) .
\end{aligned}$$

This means that $x(\omega,\lambda) \to x_0$, when $(\omega,\lambda) \to (\omega_0,\lambda_0)$. Thus we can choose neighborhoods $\Omega_1 \subset B(\omega_0, r)$ of ω_0 and $\Lambda_1 \subset B(\lambda_0, \varepsilon)$ of λ_0 such that $x(\omega,\lambda) \in intB(x_0, s)$, for all $(\omega,\lambda) \in \Omega_1 \times \Lambda_1$. Hence

$$0 \in f(x(\omega,\lambda), \omega) + N_{C(\lambda)}(x(\omega,\lambda)) ,$$

because

$$N_{C(\lambda)}(x(\omega,\lambda)) = N_{C(\lambda)\cap B(x_0,s)}(x(\omega,\lambda)) ,$$

for all $(\omega,\lambda) \in \Omega_1 \times \Lambda_1$.

Let us choose $\lambda_1, \lambda_2 \in \Lambda_1$ and $\omega_1, \omega_2 \in \Omega_1$.

For $x(\omega_1,\lambda_1) \in C(\lambda_1) \cap B(x_0, s)$ there exists $u_2 \in C(\lambda_2)$, such that

$$\|x(\omega_1,\lambda_1) - u_2\| \leq \beta(d(\lambda_1,\lambda_2)) .$$

For $x(\omega_1,\lambda_2) \in C(\lambda_2) \cap B(x_0, s)$ there exists $u_1 \in C(\lambda_1)$ such that

$$\|x(\omega_1,\lambda_2) - u_1\| \leq \beta(d(\lambda_1,\lambda_2)) .$$

Then

$$\begin{aligned}
&\varphi\left(\|x(\omega_1,\lambda_1) - x(\omega_1,\lambda_2)\|\right) \|x(\omega_1,\lambda_1) - x(\omega_1,\lambda_2)\| \leq \\
&\leq \langle f(x(\omega_1,\lambda_1), \omega_1) - f(x(\omega_1,\lambda_2), \omega_1), x(\omega_1,\lambda_1) - x(\omega_1,\lambda_2)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_1), \omega_1), u_1 - x(\omega_1,\lambda_1)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_2), \omega_1), u_2 - x(\omega_1,\lambda_2)\rangle = \\
&= \langle f(x(\omega_1,\lambda_1), \omega_1), u_1 - x(\omega_1,\lambda_2)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_2), \omega_1), u_2 - x(\omega_1,\lambda_1)\rangle \leq \\
&\leq 2M\beta(d(\lambda_1,\lambda_2)) .
\end{aligned}$$

Hence we obtain that $x(\omega_1,\lambda_1) \to x(\omega_1,\lambda_2)$, when $\lambda_1 \to \lambda_2$, uniformly for all $\omega_1 \in \Omega_1$. We have also that

$$\begin{aligned}
&\varphi\left(\|x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\|\right) \|x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\| \leq \\
&\leq \langle f(x(\omega_1,\lambda_2), \omega_1) - f(x(\omega_2,\lambda_2), \omega_1), x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\rangle + \\
&+ \langle f(x(\omega_1,\lambda_2), \omega_1), x(\omega_2,\lambda_2) - x(\omega_1,\lambda_2)\rangle +
\end{aligned}$$

$$+ \langle f(x(\omega_2,\lambda_2),\omega_2)\,,\, x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\rangle =$$

$$= \langle f(x(\omega_2,\lambda_2),\omega_2) - f(x(\omega_2,\lambda_2),\omega_1)\,,\, x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\rangle \leq$$

$$\leq \|f(x(\omega_2,\lambda_2),\omega_2) - f(x(\omega_2,\lambda_2),\omega_1)\| \|x(\omega_1,\lambda_2) - x(\omega_2,\lambda_2)\| \,.$$

Thus $x(\omega_1,\lambda_2) \to x(\omega_2,\lambda_2)$, when $\omega_1 \to \omega_2$.
The two convergence imply the continuity of $x(\cdot,\cdot)$ at (ω_2,λ_2). This point being choosed arbitrarily the continuity hold in $\Omega_1 \times \Lambda_1$.

As a corollary of the previous theorem we can prove the continuity of the metric projection with respect to perturbations.
Let $\Omega = X$ and $\omega_0 \in X$.

Corollary 1. *Let us suppose that:*
i) $x_0 = P_{C(\lambda)}(\omega_0)$*;*
ii) C *is pseudo-continuous around* (λ_0, x_0).
Then there exists neighborhoods Ω_0' *of* ω_0, Λ_0' *of* λ_0, *such that* $x(\cdot,\cdot) = P_{C(\cdot)}(\cdot)$ *is continuous on* $\Omega_0' \times \Lambda_0'$ *and hence* $x(\omega,\cdot) = P_{C(\cdot)}(\omega)$ *is continuous on* Λ_0' *for all* $\omega \in \Omega_0'$.

Proof. In the case of a uniformly-convex Banach space with locally-uniformly convex dual the normalized duality mapping is single-valued, φ-monotone on each closed-ball and continuous from the strong topology of X to the strong topology of X^*.

So, we can define the mapping $f(x,\omega) = -J(\omega - x)$ and we can use Theorem 1 to prove the continuity of $x(\cdot,\cdot)$ on $\Omega_0' \times \Lambda_0'$.
Hence for all $\omega \in \Omega_0'$ the metric projections $P_{C(\lambda)}(\omega)$ vary continuously with respect to λ on Λ_0'.

As we have seen, even when C is pseudo-Lipschitz continuous, this continuity is not the same $\frac{1}{2}$-Hölder type as in [6], because the normalized duality mapping is not strongly-monotone in a general uniformly-convex Banach spaces.
In the case of a Hilbert space, the $\frac{1}{2}$-Hölder-continuity with respect to λ is a consequence of Theorem 1 and Corollary 1.

References

[1] W. Alt and I. Kolumbán, Implicit function theorems for monotone mappings, *Kybernetika* **29** (1993), 210-221.

[2] J. P. Aubin and H. Frankowska, "Set-valued analysis", Birkhäuser, 1990.

[3] S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res.* **13** (1988), 421-434.

[4] R. N. Mukherjee and H. L. Verma, Sensitivity analysis of generalized variational inequalities, *J. Math. Anal. Appl.* **167** (1992), 299-304.

[5] J. Prüß, A characterization of uniform-convexity and application to accretive operators, *Hiroshima Math. J.* **11** (1981), 229-234.

[6] N. D. Yen, Hölder continuity of solutions to a parametric variational inequality, *Appl. Math. Optim.* **31** (1995), 245-255.

[7] N. D. Yen and G. M. Lee, Solution sensitivity of a class of variational inequalities, *J. Math. Anal. Appl.* **215** (1997), 48-55.

[8] E. Zeidler, "Nonlinear Functional Analysis and its Applications", II/b, Springer-Verlag, 1990.

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