STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLIII, Number 4, December 1998

RIGIDITY OF HARMONIC MEASURE OF TOTALLY DISCONNECTED FRACTALS

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Abstract. Let $f : V \to U$ be a generalized polynomial-like map. Suppose that harmonic measure $\omega = \omega(\cdot, \infty)$ on the Julia set J_f is equal to measure of maximal entropy μ for $f : J_f \hookleftarrow$. Then the dynamics (f, V, U) is called maximal. We are going to give a necessary condition for the dynamics to be conformally equivalent to a maximal one, that is to be conformally maximal. Namely the purpose of the paper is to prove that if the Julia set is totally disconnected then $\omega \approx \mu$ implies that the system (f, U, V) is conformally maximal. This shows that maximal systems are natural substitutes for polynomials in the class of genereralized polynomial-like mappings.

0. Introduction

Let $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ be a polynomial of degree d. A result of H. Brolin (see [Br]) says that the backward orbits of f are equidistributed with respect to the measure ω : the harmonic measure on the boundary of the domain of attraction to ∞ and evaluated at ∞.

Almost twenty years later the ergodic theory of rational maps has started by the works of M. Lyubich ([Ly1], [Ly2]) and independently by A. Freire, A. Lopes and R. Mañé ([FLM], [Ma1]). It was established that for any rational map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ there is a unique f -invariant probability measure μ on the Borel σ-algebra such that:

$$\mu(f(E)) = d \cdot \mu(E) \tag{0.1}$$

for any Borel set E such that $f|_E$ is injective. The measure μ is the unique f-invariant probability measure that maximizes the entropy i.e. $h_\mu(f) = \log d$. In the light of these works Brolin's result can be interpreted as the fact that for polynomials we have $\omega = \mu$.

Conversely A. Lopes proved in [Lo] that if we have a rational map $f : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that $\omega = \mu$ (where ω is the harmonic measure on the boundary of an attracting forward invariant component of the Fatou set) then f is conjugated to a polynomial by a Möbius transformation. A simpler proof of this result was given in [MR].

The purpose of this paper is to extend the result of Lopes to the class of generelized polynomial-like mappings. Let us recall this definition from [BPV], [BV1], [BV2].

We will be considering triples (f, V, U), where U is a topological disc and $V = V_1 \cup \cdots \cup V_k$ is the union of topological discs whose closures are disjoint and are contained in U. Also $f_i := f|V_i$ is a regular or branched covering $V_i \to U$ of degree d_i (so "regular" means that $d_i = 1$). By $d = d_1 + \cdots + d_k$ we denote the degree of the map $f : V \to U$. These dynamical systems will be called generalized polynomial-like systems or GPL. The limit set ($\equiv$ *Julia set*) is $J_f = \partial K_f$, where $K_f = \bigcap_{n \geq 0} f^{-n}(U)$ is the filled Julia set.

If $k = 1$, $d \geq 2$, we come to a class of polynomial-like systems (PL) introduced in [DH] and playing an important role in classification of polynomial dynamics.

Being GPL means to be quasiconformally equivalent to a polynomial:

$$f \in \mathrm{GPL} \Rightarrow \exists\, h \in qc(U) : f = h^{-1} \circ \mathrm{poly} \circ h.$$

The starting point is to see whether the result of Lopes is true under the weaker assumption $\omega \approx \mu$. Here "$\approx$" means that ω and μ are mutually absolutely continuous and in addition to that we assume that there exists $M > 0$ such that for any $x \in J$ and $r > 0$:

$$\frac{1}{M} \leq \frac{\mu(B(x,r))}{\omega(B(x,r))} \leq M. \tag{0.2}$$

In other words we ask the following question: is it true that if for a GPL (f, U, V) we have $\omega \approx \mu$ does it follow that (f, U, V) is conformally conjugated to a polynomial?

This is a rigidity-type question aiming to rule out quasiconformal deformations.

The converse is obviously true: if our GPL is conformally conjugated to a polynomial then by Brolin's result and Harnack's inequality we obtain $\omega \approx \mu$.

It is quite a suprise to see that the answer to this question is generally negative. This was shown by M. Lyubich and A. Volberg. Namely a GPL (f, U, V) was constructed in [LyV] where $\mu = \omega$ but f is not conformally conjugated to a polynomial.

To formulate the appropriate question for the class of GPL we call a GPL system (f, U, V) *maximal* if $\mu_f = \omega_f$. Maximal GPL systems have been introduced in [BPV] as natural substitues for polynomials.

Next we call a GPL system (f, U, V) *conformally maximal* if it is conformally equivalent to a maximal system; that is:

$$f = H^{-1} \circ g \circ H,$$

where $H : U_f \to U_g$ is a conformal map and (g, U_g, V_g) is a maximal system.

In this paper we are going to prove the following rigidity result:

Theorem 4.1 *Let (f, U, V) be a GPL with totally disconnected Julia set. Then $\omega_f \approx \mu_f$ implies that (f, U, V) is conformally maximal.*

The converse of this result follows immediately by Harnack's inequality.

A weaker form of Theorem 4.1 under the condition of semihyperbolicity of f was proved in [BPV]. Also in [BPV] it was explained that this result is an analog of a theorem of Shub and Sullivan (see [SS]) on "wild" (i.e. totally disconnected) J_f and nonexpanding f.

1. Idea of proof.

Let us start with the following criterion of conformal maximality proven in [BPV]:

Theorem A *Let (f, V, U) be a GPL system. Two assertions are equivalent:*

1) *(f, V, U) is conformally maximal;*
2) *there exists a non-negative subharmonic function τ on U, which is positive and harmonic in $U \setminus K_f$, vanishes on K_f and satisfies*

$$\tau(fz) = d\tau(z). \tag{Aut}$$

If ν is an arbitrary probability measure on J_f then a general theorem [Pa], says that there exists *Jacobian $J_\nu = J_\nu(f)$ on a set of full measure ν.* It means that there exists $Y \subset J$, $\nu(J \setminus Y) = 0$, and a-ν integrable function J_ν such that for every $E \subset Y$ on which f is 1-to-1 onto $f(E)$ we have $\nu(f(E)) = \int_E J_\nu d\nu$. By (0.1) the Jacobian of μ is $J_\mu = d$. We denote the Jacobian of the harmonic measure by J_ω.

For a better exposition I would like to sketch the strategy of the approach in [BPV]:

-if the dynamics f is semihyperbolic and J is totally disconnected the function $\varphi(x) = \log J_\omega(x)$ is Hölder continuous on J,

-Hölder continuity of φ together with $\omega \approx \mu$ is used to prove that there exists a Hölder continuous function $u : J \to \mathbf{R}$ satisfying the homologous equation:

$$\varphi(x) - \log d = u(fx) - u(x) \ \ \forall x \in J, \tag{1.1}$$

-starting from (1.1) we can build an automorphic function τ as required by Theorem A to prove conformal maximality of (f, U, V).

In our more general case we do not have the Hölder continuity of $\varphi = \log J_\omega$. Therefore the above approach based on thermodynamic formalism is not applicable. Still we have a modified strategy as follows:

-Pesin theory gives a certain regularity property of φ,

-we consider the function $\phi = \varphi - \log d$ and the sequence of random variables $\{\phi \circ f^k\}_k$ on the probability space (J, μ),

-$\omega \approx \mu$ implies that $\int_J \phi \, d\mu = 0$,

-using a technique from [DPU] it follows that the sequence $\{\phi \circ f^k\}_k$ obeys the law of Central Limit Theorem or CLT

- applying CLT we obtain a function $u \in L^2(\mu)$ satisfying the homologous equation:

$$\varphi(x) - \log d = u(fx) - u(x) \text{ for } \mu \text{ a.e. } x \in J, \tag{1.2}$$

-staring from (1.2) we can construct again the automorphic function τ required by Theorem A.

2. Jacobian of the harmonic measure.

In this section we study regularity properties of the function $\varphi = \log J_\omega$. Our first ingredient is a result of F. Grishin (see [Gr]):

Lemma B *Let $\infty \notin K \subset \hat{\mathbf{C}}$ be a compact set and denote by ω the harmonic measure in $\hat{\mathbf{C}} \setminus K$ evaluated at ∞. Let $\mathcal{O}$ be an open set containing K and let u , $v \geq 0$ be two continuous subharmonic function; positive and harmonic in $\mathcal{O} \setminus K$ and vanishing on K. Let us suppose that the limit:*

$$\rho(x) = \lim_{\substack{z \to x \\ z \in \mathcal{O} \setminus K}} \frac{u(z)}{v(z)} \quad \text{exists for } \omega \text{ a.e. } x \in \partial K.$$

Then we have that $d\mu_u = \rho d\mu_v$ where μ_u and μ_v denote the Riesz measures of u and v.

In our applications we put $K = J$, $\mathcal{O} = U$, $u = G \circ f$ and $v = G$, where G is Green's function of $\hat{\mathbf{C}} \setminus J$ with pole at ∞.

Let us suppose for the moment that the limit $\lim_{\substack{z\to x \\ z\in U\setminus J}} \frac{G(fz)}{G(z)}$ exists for ω a.e. $x \in J$. Using that $\omega = \Delta G$, $\omega \circ f = \Delta(G\circ f)$ Lemma B gives :

$$J_\omega(x) = \lim_{\substack{z\to x \\ z\in U\setminus J}} \frac{G(fz)}{G(z)} \text{ for } \omega \text{ a.e. } x \in J \tag{2.1}$$

To establish the existence of the above limit we introduce:

Definition 2.1 *Let $K \subset \hat{\mathbf{C}}$ and $\mathcal{O}$ be as in Lemma B and let us fix a number $\beta > 0$. We say that a set $E \subset \mathcal{O}$ is k-nested if there exist annuli $\{A_i\}_{i=1}^k$ with the properties:*

(1) $mod A_i > \beta$,

(2) $A_i \subset \mathcal{O} \setminus K$,

(3) $E \subset inA_k \subset inA_{k-1} \subset \ldots \subset inA_1$,

where inA_i denotes the component of $\hat{\mathbf{C}} \setminus A_i$ containing E.

The existence of the limit in (2.1) will be based on the following result called Boundary Harnack Principle (see [MaV] or [BV2] for the proof):

Lemma C *Let K, $\mathcal{O}$, u, v be as in Lemma B and $\beta > 0$ be fixed. There exist $C > 0$, $0 < q < 1$ depending only on K and $\beta > 0$ such that if $E \subset \mathcal{O}$ is k-nested we have :*

$$\left|\log \frac{u(x)}{v(x)} - \log \frac{u(y)}{v(y)}\right| \leq C \cdot q^k \;\; \forall x, y \in E \setminus K \tag{2.2}$$

If we choose as before $K = J$, $\mathcal{O} = U$, $u = G\circ f$, $v = G$ (2.2) becomes:

$$\left|\log \frac{G(fx)}{G(x)} - \log \frac{G(fy)}{G(y)}\right| \leq C \cdot q^k, \tag{2.3}$$

for any $x, y \in U \setminus J$ such that $\{x, y\}$ is k-nested. We also mention that as $\beta > 0$ will be fixed we have $C > 0$, $0 < q < 1$ fixed throughout the paper.

Definition 2.2 *A point $x \in J$ is called a good point if it is ∞-tely nested.*

We will see that the limit in (2.1) exists for the good points but first we prove:

Lemma 2.3 *Suppose that $\omega \approx \mu$ then there exists $\beta > 0$ such that ω a.e. point is a good point.*

Proof. We are going to show that μ a.e. point is a good point.

We consider the natural extension $(\tilde{f}, \tilde{J}, \tilde{\mu})$ of the dynamical system (f, J, μ); that is :

$$\tilde{f} : \tilde{J} := \{(x_k)_{k \in -\mathbf{N}} : f(x_k) = x_{k+1} \ (k \leq -1\} \to \tilde{J},$$

where

$$\tilde{f}((x_k)_{k \in -\mathbf{N}}) = (f(x_k))_{k \in -\mathbf{N}}.$$

Then the Borel σ-field in J defines a σ-field M_0 in $\tilde{J}$ by $M_0 := \pi^{-1}(B)$ where π denotes the projection of $\tilde{J}$ onto the first coordinate. It is clear that $(\tilde{f})^{-1} M_0 \subset M_0$. Finally denote by $\tilde{\mu}$ the natural extension of μ to $\tilde{J}$.

A standard fact in Pesin theory (see [Pr1] pp.16) shows that for $\tilde{\mu}$ a.e. $\tilde{x} \in \tilde{J}$ there exists $r = r(\tilde{x}) > 0$ such that univalent branches f_n of f^{-n} on $B(\pi(\tilde{x}), r(\tilde{x}))$ for $n = 1, 2, \ldots$ such that $f_n(\pi(\tilde{x})) = \pi((\tilde{f})^{-n}(\tilde{x}))$ exist.

Moreover for an arbitrary λ : $1/d < \lambda < 1$ (not depending on $\tilde{x}$) and a constant $C = C(\tilde{x}) > 0$:

$$\left| f_n'(\pi(\tilde{x})) \right| \leq C \cdot \lambda^n \quad \text{and} \quad \frac{|f_n'(\pi(\tilde{x}))|}{|f_n'(z)|} \leq C\,, \tag{2.4}$$

for every $z \in B(\pi(\tilde{x}, r), \ n > 0$.

Furthermore r and C are measurable functions of $\tilde{x}$.

To use this fact observe that there are $C, r > 0$ and a set $\tilde{Y} \subset \tilde{J}$ with $\tilde{\mu}(\tilde{Y}) > 0$ such that the above properties hold for $\tilde{x} \in \tilde{Y}$ and for these C and r. As $\tilde{\mu}$ is ergodic, by Birkhoff's ergodic theorem, there exists a set $\tilde{X} \subset \tilde{J}, \ \tilde{\mu}(\tilde{X}) = 1$ such that:

$$\lim_{n \to \infty} \frac{\#\{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\}}{n} = \tilde{\mu}(\tilde{Y}) > 0, \ \ \forall \tilde{x} \in \tilde{X}. \tag{2.5}$$

Let us put $X := \pi(\tilde{X}) \subset J$. Then $\mu(X) = 1$ and our goal is now to prove that there exists $\chi > 0$ such that $\forall x \in X$ there is $N = N(x)$ with the property that for any $n \geq N$ we have that x is $\chi \cdot n$-nested. Once this is proved we are done.

Let us pick $x \in X$ and consider $\tilde{x} \in \tilde{X}$ such that $\pi(\tilde{x}) = x$. By (2.5) there exists $N = N(\tilde{x})$ such that if $n \geq N$:

$$\#\{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\} \geq \frac{\mu(\tilde{Y})}{2} \cdot n. \tag{2.6}$$

Let $A(n,x) := \{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\}$. We denote by $B_k(x)$ the component of $f^{-k}(B(f^k(x),r))$ which contains x. If $k \in A(n,x)$ we have that the mapping $f^k : B_k(x) \to B(f^k(x),r)$ is univalent.

We are going to pull annuli from $B(f^k(x),r)$ to $B_k(x)$ using the univalency of f^k. First we have the following:

Claim: There exist numbers $\beta > 0,\ r' > 0$ such that for any $y \in J$ there exists an annulus $A(y,r) \subset B(y,r) \setminus J$ such that $y \in inA(y,r)$ with the properties:

(a) $mod\ A(y,r) > \beta$

(b) $dist\ (A(y,r),\ y) > r'$

To prove the claim consider an annulus $A_0 \subset U$ such that $J \subset inA_0,\ \ mod(A_0) = \beta_0$. Since J is totally disconnected and $J = \cap f^{-n}(U)$ there exists $N_0 > 0$ such that $diam\ B_{N_0} < r$ for any component B_{N_0} of $f^{-N_0}(U)$.

For $y \in J$ let us denote by $A_{N_0}(y) \subset B_{N_0}(y) \subset B(y,r) \setminus J$ the component of $f^{-N_0}(A_0)$ such that $y \in inA_{N_0}(y)$. Because N_0 is fixed properties (a) and (b) follow for $A(y,r) := A_{N_0}(y)$. This proves the claim.

- Let us put now $A_k(x) := f^{-k}(A(f^k(x),r))$ for any $k \in A(n,x)$. It is clear that $x \in inA_k(x),\ \ A_k(x) \subset B_k(x) \setminus J,$ and $mod\ A_k(x) = mod\ A(f^k(x),r) > \beta_0$ by the univalency of $f^k|_{B_k(x)}$.

Our annuli will be selected from $A_k(x),\ k \in A(n,x)$; however we need to exclude some of them to make sure that they are nested inside each other.

To do that we use (2.4) to see that there exists $L > 0$ (independent of n and x) such that if $k_1, k_2 \in A(n,x),\ \ k_2 \geq k_1 + L$ we have $B_{k_2-k_1}(f^{k_1}(x)) \subset B(f^{k_1}(x),r')$. This implies that $x \in inA_{k_2}(x) \subset inA_{k_1}(x)$.

By the above consideration we obtain for $n \geq N(x)$ at least $\frac{1}{L} \cdot \#A(n,x)$ annuli nested inside each other, containing x and with modulus greater than $\beta_0 > 0$. If we put now $\chi := \frac{\tilde{\mu}(\tilde{Y})}{2L}$ we are done by (2.6).

Lemma 2.3 has the following useful:

Corolarry 2.4 *There exists a set $X \subset J$ such that $\mu(X) = 1$ and the function $\varphi : X \to \mathbf{R}$ by:*

$$\varphi(x) = \lim_{\substack{z \to x \\ z \in U \setminus J}} \log \frac{G(fz)}{G(z)}$$

is well defined and continuous on X.

Proof Let X be the full measure set given by Lemma 2.3 and let $x \in X$. If $y, z \in U \setminus J$ are close to x it follows that $\{x, y, z\}$ is k-nested for $k = k(x, y, z)$. Furthermore $k \to \infty$ as $y, z \to x$. By Lemma C we have:

$$\left| \log \frac{G(fz)}{G(z)} - \log \frac{G(fy)}{G(y)} \right| \leq C \cdot q^k. \tag{2.7}$$

Now (2.7) implies the existence of $\varphi(x)$ for $x \in X$. Furthermore there exists $C_1 > 0$ such that if $x \in X$ and $z \in U \setminus J$ are so that $\{x, z\}$ is $k = k(x, z)$-nested we have:

$$\left| \varphi(x) - \log \frac{G(fz)}{G(z)} \right| \leq C_1 \cdot q^k. \tag{2.8}$$

To see the continuity at $x \in X$ let $y \in X$, $y \to x$. Then $\{x, y\}$ is contained in a topological disc $D(x, y)$ which is $n(x, y)$-nested with $n(x, y) \to \infty$ as $y \to x$. By (2.8) we have

$$|\varphi(x) - \varphi(y)| \leq 2C_1 \cdot q^{n(x,y)}, \tag{2.9}$$

which proves the continuity of φ.

As we do not have control on the locations and sizes of the nests we cannot extend the definition and continuity of φ to the whole J.

We would like to mention that whenever we obtain a full measure set X with a certain property we can assume that it is f-invariant. Indeed , if X is not invariant we just replace it by:

$$\hat{X} = J \setminus (\bigcup_{n\geq 0} f^{-n}(\bigcup_{m\geq 0} f^m(J\setminus X))).$$

It is clear that $\hat{X} \subset X$, $\mu(\hat{X}) = 1$ and $f^{-1}(\hat{X}) = \hat{X}$.

We finish this section with a technical result which is based on Lemma 2.3 and will be useful in the next section:

Lemma 2.5 *There exists a set* $X_0 \subset J$, $\mu(X_0) \geq 1 - 1/20$ *and numbers* $1 > \delta > 0$, $K_1 > 0$, $K_2 > 0$, $N_0 \in \mathbf{N}$ *such that for any* $n \geq N_0$, $x \in X_0$ *the ball* $B(x, \delta^n)$ *is* $\frac{1}{K_1} \cdot n$ *nested by annuli contained in* $B(x, \delta^{\frac{n}{K_2}})$.

Proof. Let $\tilde{X}$, $\tilde{Y}$ be the sets considered in the proof of Lemma 2.3 and let us introduce:

$$\tilde{X}_N := \{\tilde{x} \in \tilde{X} : \#\{k \leq n : \tilde{f}^k(\tilde{x}) \in \tilde{Y}\} \leq \frac{\tilde{\mu}(\tilde{Y})}{2} \cdot n, \ \forall\, n \geq N\}.$$

It is clear that

$$\tilde{X}_N \subset \tilde{X}_{N+1} \ , \ \ \tilde{X} = \cup_{N\geq N_1} \tilde{X}_N \ \ \forall \ N_1 > 0$$

and hence $\lim_{N\to\infty} \tilde{\mu}(\tilde{X}_N) = 1$.

If we choose N_0 such that $\tilde{\mu}(\tilde{X}_{N_0}) \geq 1 - 1/20$ and put $X_0 := \pi(\tilde{X}_{N_0})$ then $\mu(X_0) \geq 1 - 1/20$.

For $x \in X_0$ and $n \geq N_0$ there are $\chi \cdot n$ annuli nesting x obtained in Lemma 2.3. Let us consider the ones obtained as preimages using univalent branches f_k of $f^{-k}_{|B(f^k(x),r)}$ for $\frac{\chi \cdot n}{2} \leq k \leq n$. Their number is at least $\frac{\chi \cdot n}{2}$. By (2.4) these annuli are contained in $B(x, C_1 \cdot \lambda^{\frac{\chi \cdot n}{2}})$. Furthermore it is easy to see that they are nesting the ball $B(x, c_2 \cdot \frac{1}{L^n})$ where $L := \sup|f'(z)|$ and C_1, c_2 are two fixed constants.

Without loss of generality we can assume that $C_1 = c_2 = 1$. Let us put $\delta = \frac{1}{L}$ and choose $K_2 > 0$ such that $\lambda^{\frac{\chi \cdot n}{2}} \leq \delta^{\frac{n}{K_2}}$. Finally choosing $K_1 = \frac{2}{\chi}$ we are done.

3. Homologous equation.

The purpose of this section is to prove:

Proposition 3.1 *Suppose that (f, U, V) is a GPL with totally disconnected Julia set. If $\mu \approx \omega$ there exists a function $u \in L^2(\mu)$ satisfying the homologous equation:*

$$\varphi(x) - \log d = u(fx) - u(x) \quad \textit{for } \mu \textit{ a.e. } x \in J.$$

The proof of Proposition 3.1 will be done in several steps. Let us introduce the function $\phi = \varphi - \log d$. The first step is to prove:

Lemma 3.2 *Under the assumptions of Proposition 3.1 we have $\int_J \phi d\mu = 0$.*

Proof. As in the proof of Lemma 2.3 consider the natural extension $(\tilde{f}, \tilde{J}, \tilde{\mu})$ of the system (f, J, μ). Let B be a ball in $\hat{\mathbf{C}}$. We consider the "good" branches of f^{-n} defined in B. Following [Z] we say that a branch f_ν^{-n} is "good" (or δ-good for $\delta > 0$) if:

$$f_\nu^{-n} \text{ is well defined and univalent in } 2B \tag{3.1}$$

$$diam f_\nu^{-n}(B) < K \cdot e^{-n\delta}. \tag{3.2}$$

In [Z] and [PUZ] it was proved that there exists $\delta > 0$ such that for every $\tilde{\epsilon} > 0$ there exists $M \in \mathbf{N}$ such that if there are no critical values up to order M in B then one can find a subset $\tilde{K}_B \subset \tilde{B} = \pi^{-1}(B) \subset \tilde{J}$ with $\tilde{\mu}(\tilde{K}_B) > (1 - \tilde{\epsilon})\mu(B)$ and consisting of "good" trajectories. (The trajectory $\tilde{x} = (x_0 x_{-1} \dots x_k \dots)$ is "good" if x_k is the image of some "good" branch of f^{-k} defined on B.)

We are going to apply this fact in a similar way as in [Z]: let $p_1, \dots, p_s$ be critical values up to order M. Take a small $r > 0$ and let $\epsilon > 0$. Let $B_1, \dots, B_s$ be centered at p_i-s with radius r. Let $\mathcal{B}$ be cover of $\hat{\mathbf{C}} \setminus \bigcup B_i$ by balls of radius $r/4$. If $r > 0$ is small enough then:

$$\tilde{\mu}(\cup_{B \in \mathcal{B}} \tilde{K}_B > 1 - \epsilon. \tag{3.3}$$

Introduce the function $\tilde{\phi} = \phi \circ \pi$ and suppose by contradiction that

$$\int_J \phi \, d\mu = \int_{\tilde{J}} \tilde{\phi} \, d\tilde{\mu} = \chi > 0.$$

Let us consider the partial sums:

$$\tilde{S}_n(\tilde{x}) = \sum_{i=0}^{n-1} \tilde{\phi}(\tilde{f}^i(\tilde{x})) = \sum_{i=0}^{n-1} \phi(f^i(\pi(\tilde{x}))) = S_n(\pi(\tilde{x})),$$

$$\tilde{S}^1_n(\tilde{x}) = \sum_{i=0}^{n-1} \tilde{\varphi}(\tilde{f}^i(\tilde{x})) = \sum_{i=0}^{n-1} \varphi(f^i(\pi(\tilde{x}))) = S^1_n(\pi(\tilde{x})).$$

By Birkhoff's ergodic theorem there exists $\tilde{X} \subset \tilde{J}$, $\tilde{\mu}(\tilde{X}) = 1$ such that for $\tilde{x} \in \tilde{X}$

$$\lim_{n\to\infty} \frac{\tilde{S}_n(\tilde{x})}{n} = \chi > 0.$$

Let us denote by

$$\tilde{X}_n = \{\tilde{x} \in \tilde{X} : \ \frac{\tilde{S}_k(\tilde{x})}{k} > \frac{\chi}{2} \ , \ \forall k \geq n \ \}.$$

It is clear that $\tilde{X}_n \subset \tilde{X}_{n+1}$, $\forall n > 0$ and $\tilde{X} \subset \cup_{n\geq N} \tilde{X}_n$, $\forall N > 0$. It follows that for $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\tilde{\mu}(\tilde{X}_n) > 1 - \epsilon \ , \ \forall n \geq N.$$

By (3.3) it follows that

$$\tilde{\mu}(\tilde{X}_n \cap \tilde{f}^{-n}(\bigcup_{B\in\mathcal{B}} \tilde{K}_B)) > 1 - 2\epsilon \ , \ \forall n \geq N.$$

Consequently there exists $B \in \mathcal{B}$ and $\beta > 0$ such that

$$\tilde{\mu}(\tilde{X}_n \cap \tilde{f}^{-n}(\tilde{K}_B)) > \beta \ \text{ for infinitely many } n \in \mathbf{N}$$

Let us denote by $X^n := \pi(\tilde{X}_n \cap \tilde{f}^{-n}(\tilde{K}_B))$; then $\mu(X^n) > \beta$.

If $x \in X^n$ then $x = \pi(\tilde{x})$ for some $\tilde{x} \in \tilde{X}_n \cap \tilde{f}^{-n}(\tilde{K}_B)$, and thus x is a preimage of $f^n(x) \in B$ under some univalent branch $f_\nu^{-n}|_{2B}$. Let us denote the set of univalent branches of $f^{-n}|_{2B}$ by $\mathcal{G}_n$. By the above consideration we have

$$X^n \subset \bigcup_{\nu\in\mathcal{G}_n} f_\nu^{-n}(2B). \tag{3.4}$$

Our goal is to show that while $\mu(X^n) > \beta$ we have that $\omega(X^n) \to 0$ as $n \to \infty$. By (3.4) we write

$$\omega(X^n) \leq \sum_{\nu\in\mathcal{G}_n} \omega(f_\nu^{-n}(2B) \cap X^n). \tag{3.5}$$

On the other hand $x \in X^n$ implies that

$$\frac{S_n(x)}{n} > \frac{\chi}{2} \text{ for } n \geq N,$$

which means that

$$S_n^1(x) > \log d^n + n \cdot \frac{\chi}{2} \text{ for } n \geq N,\ x \in X^n. \tag{3.6}$$

Using (3.6) , for any $\nu \in \mathcal{G}_n$ we can estimate:

$$\begin{aligned} 1 \geq \omega(2B \cap f^n(X^n)) = \int_{f_\nu^{-n}(2B)\cap X^n} e^{S_n^1(x)}\, d\omega(x) \geq \\ \geq d^n \cdot e^{\frac{\chi}{2}\cdot n} \cdot \omega(f_\nu^{-n}(2B) \cap X^n). \end{aligned}$$

Consequently we obtain:

$$\omega(f_\nu^{-n}(2B) \cap X^n) \leq e^{-\frac{\chi}{2}\cdot n} \cdot d^{-n}. \tag{3.7}$$

Relations (3.5) and (3.7) now give:

$$\begin{aligned} \omega(X^n) \leq \sum_{\nu\in\mathcal{G}_n} \omega(f_\nu^{-n}(2B) \cap X^n) \leq e^{-\frac{\chi}{2}\cdot n} \cdot \sum_{\nu\in\mathcal{G}_n} d^{-n} = \\ = e^{-\frac{\chi}{2}\cdot n} \cdot \frac{1}{\mu(2B)} \cdot \sum_{\nu\in\mathcal{G}_n} \mu(f_\nu^{-n}(2B)) \leq e^{-\frac{\chi}{2}\cdot n}. \end{aligned}$$

This shows that $\omega(X^n) \leq e^{-\frac{\chi}{2}\cdot n}$ and as n can be chosen arbitrarily large we obtain that μ and ω are singular. This contradiction shows that $\int_J \phi\, d\mu = 0$.

Let us consider now the sequence $\{\phi \circ f^k\}_k$ of random variables. Our next step is

Lemma 3.3 *There exists a finite asymptotic variance:*

$$\begin{aligned} \sigma^2 := \sigma_\mu^2(\phi) := \lim_{n\to\infty} \frac{\int_J (\sum_{i=0}^{n-1} \phi \circ f^i)^2 d\mu}{n} = \\ = \int_J \phi^2 d\mu + 2 \cdot \sum_{i=0}^{\infty} \int_J \phi \cdot \phi \circ f^i d\mu. \end{aligned}$$

Moreover, the sequence $\{\phi \circ f^k\}_k$ obeys the law of Central Limit Theorem (CLT).

Proof We need to investigate the behaviour of the Perron-Frobenius-Ruelle operator $L : L^2(\mu) \to L^2(\mu)$ defined by:

$$Lu(x) = \frac{1}{d} \sum_{y \in f^{-1}(x)} u(y). \tag{3.8}$$

If x is a critical value we count the preimages in the above sum together with their multiplicities.

The formula (3.8) is correct if $u \in \mathcal{C}(J)$ and in this way $L : \mathcal{C}(J) \to \mathcal{C}(J)$ is a well defined linear operator with $\| L \|_{\mathcal{C}(J)} \leq 1$. We can extend L to $L^2(\mu)$ by continuity or by formula (3.8) on an invariant set $X \subset J$ with $\mu(X) = 1$ on which $u \in L^2(\mu)$ is defined. We have by Jensen's inequality that $\| L \|_{L^2(\mu)} \leq 1$. It is well known ([Ly1], [FLM]) that $L^*\mu = \mu$ and thus $\int_J u \cdot v \circ f \, d\mu = \int_J v \cdot Lu \, d\mu$. In other words $L : L^2(\mu) \to L^2(\mu)$ is the adjoint operator of

$$A : L^2(\mu) \to L^2(\mu), \quad Au = u \circ f$$

Our goal is to prove the following decay property of $L^k\phi$: for any $p > 0$ there exist $C = C(p)$ and $K = K(p)$ such that for $k \geq K$ we have:

$$\| L^k \phi \|_\infty < C \cdot \frac{1}{k^p}. \tag{3.9}$$

Estimate (3.9) gives the first statement immediately. For the second statement we apply a theorem of Gordin (see [Go] or [D] Theorem 1.1.2). Following exactly the same arguments as in [DPU] - Theorem 5.3 we obtain that the estimate (3.9) together with Gordin's theorem imply the second statement.

To prove (3.9) we are going to use a similar idea as in Section 4 from [DPU]. The lack of the uniform Hölder continuity of $\{L^k\phi\}_k$ is compensated by Lemma 2.5 and a result of F. Przytycki ([Pr2]).

Let us start by reminding the following fact proven in [DU]: there exists a measurable Markov partition α of J and numbers $0 < \lambda < 1$, $C > 0$ such that for

$A \in \alpha,\ f(A) = J\ \mu$ a.e. and for all $n \geq 1$:

$$\mu(\cup\{A \in \vee_{j=0}^{n-1} f^{-j}(\alpha) : \operatorname{diam} f^k(A) > C_1 \cdot \lambda^{n-k} \text{ for some } k = 0, 1, \dots, n\}) < 1/20. \tag{3.10}$$

For $n \geq 0$ let α_b^n be the collection of the elements of the partition $\alpha^n = \vee_{j=0}^{n-1} f^{-j}(\alpha)$ defined in (3.10) and let $\alpha_g^n = \alpha^n \setminus \alpha_b^n$.

We also are going to use the fact (see [DPU] Lemma 4.3) that if $\psi \in L^2(\mu)$, $\int_J \psi\, d\mu = 0$ and $\Delta \geq \| \psi \|_\infty$ then

$$\mu(\{x : \psi(x) \ \leq\ \Delta/4\ \}) \geq 1/5. \tag{3.11}$$

The crucial estimate (3.9) follows immediately from the following:

Claim *For any $b > 1$ there exists an integer $j(b)$ such that if $j \geq j(b)$ and $[b^j] \leq k \leq [b^{j+1}]$ we have the estimate:*

$$\| L^k \phi \|_\infty \ \leq\ (39/40)^j. \tag{3.12}$$

In fact the closer we choose b to 1 the greater value of p can be obtained in (3.9).

To prove prove (3.12) we introduce the sequence $\{n_j\}_j$, $n_j = [b^j]$ and observe that since $\| L\psi \|_\infty \leq \| \psi \|_\infty$ it is enough to show that

$$\| L^{n_j} \phi \|_\infty \ \leq\ (39/40)^j. \tag{3.13}$$

We are going to use induction over j: let us assume (3.13) for j. By (3.11) we obtain

$$\mu(\{x :\ L^{n_j}\phi(x) \ \leq\ 1/4 \cdot (39/40)^j\ \}) \ \geq\ 1/5.$$

Let us introduce the set

$$G_j := \{x :\ L^{n_j}\phi(x) \ \leq\ 1/4 \cdot (39/40)^j\ \} \ \bigcap\ X_0,$$

where X_0 is the set from Lemma 2.5. Then it is clear that $\mu(G_j) \geq 1/10$.

Let us denote by $k_j = n_{j+1} - n_j$ and define:

$$\alpha_G^j := \{A \in \alpha_g^{k_j} \ : \ A \cap G_j \neq \emptyset \}$$

Relation (3.10) then yields: $\mu(\alpha_G^j) \geq 1/20$.

First of all we are going to show that for $y \in \cup\alpha_G^j$

$$L^{n_j}\phi(y) \ \leq \ 1/2 \cdot (39/40)^j. \tag{3.14}$$

Observe that if $y \in \cup\alpha_G^j$ there exists $x \in G_j$ such that $y \in B(x, C \cdot \lambda^{k_j})$. Consequently there is a constant $K_3 > 0$ such that $y \in B(x, \delta^{\frac{n_j}{K_3}})$.

On the other hand by Lemma 2.5 there is a number of $\frac{1}{K_4} \cdot n_j$ annuli nesting $\{x, y\}$ and contained in $B(x, \delta^{\frac{n_j}{K_5}})$.

Since $x \in G_j$ gives

$$L^{n_j}\phi(x) \ \leq \ 1/4 \cdot (39/40)^j,$$

we intend to estimate the difference

$$| \ L^{n_j}\phi(x) - L^{n_j}\phi(y) \ | \ = \ \left| \sum_{z_x \in f^{-n_j}x} 1/d^{n_j}\phi(z_x) - \sum_{z_y \in f^{-n_j}y} 1/d^{n_j}\phi(z_y) \right| .$$

Let us denote by $\{C_i^j\}_{i\in I}$ the collection of the components of $f^{-n_j}(B(x, \delta^{\frac{n_j}{K_5}}))$.

¿From Przytycki's finiteness lemma (see [Pr2] Lemma 2) it follows that there exists an integer $M = M(K_5, \delta)$ such that the degree of the maps: $f^{n_j} : C_i^j \to B(x, \delta^{\frac{n_j}{K_5}})$ is at most M.

Using Lemma 2.5 we obtain in C_i^j a number of at least $\frac{1}{K_4} \cdot n_j$ annuli with moduli bounded below by a fixed constant $\beta_1 = \beta_1(\beta, M)$ nesting $\{z_x, z_y\}$. Here we have used that the modulus of preimages under bounded degree mappings is distorted by a bounded amount.

Consequently we can use (2.9) to obtain:

$$|\phi(z_x) - \phi(z_y)| \ \leq \ C_2 \cdot q_1^{n_j},$$

where $0 < q_1 < 1$, $q_1 = q_1(\beta_1, K_4)$.

This implies that for j large enough:

$$|L^{n_j}\phi(x) - L^{n_j}\phi(y)| \leq C_2 \cdot q_1^{n_j} \leq 1/4 \cdot (39/40)^j,$$

and (3.14) follows.

For $x \in J$ we define $G_j(x) := f^{-k_j}(x) \cap \cup\alpha_G^j$, and $B_j(x) := f^{-k_j}(x) \setminus G_j(x)$. We are now ready to estimate:

$$L^{n_{j+1}}\phi(x) = L^{k_j}(L^{n_j}\phi(x) = \sum_{y\in G_j(x)} 1/d^{k_j} \cdot L^{n_j}\phi(y) + \sum_{y\in B_j(x)} 1/d^{k_j} \cdot L^{n_j}\phi(y) \leq$$

$$\leq (39/40)^j \cdot 1/d^{k_j} \cdot (1/2 \cdot \#G_j(x) + \#B_j(x)) = (39/40)^j \cdot 1/d^{k_j} \cdot (d^{k_j} - 1/2 \cdot \#G_j(x)).$$

Finally we use that

$$\#G_j(x) \cdot 1/d^{k_j} = \mu(\cup\alpha_G^j) \geq 1/20$$

and obtain

$$L^{n_{j+1}}\phi(x) \leq (39/40)^{j+1}. \tag{3.15}$$

Changing ϕ to $-\phi$ we obtain the counterpart of (3.15):

$$L^{n_{j+1}}\phi(x) \geq -(39/40)^{j+1}.$$

The above estimates yield (3.13) for $j+1$. This finishes the proof of the Claim and we are done.

The last step toward the proof of Proposition 3.1 is:

Lemma 3.4 *Under the conditions of Proposition 3.1 we have* $\sigma^2 = 0$.

Proof Let us suppose by contradiction that $\sigma^2 > 0$. Let us consider the function $\phi_1 = -\phi = \log d - \varphi$ and apply CLT for the sequence of random variables $\{\phi_1 \circ f^k\}_k$. As in the proof of Lemma 3.2 we consider the corresponding partial sums but now for the function ϕ_1. Instead of Birkhoff's ergodic theorem we apply now CLT: for any $A > 0$ we have:

$$\tilde{\mu}(\{\tilde{x} \in \tilde{J} : \tilde{S}_n(\tilde{x}) < -A \cdot \sigma \cdot n^{1/2}\}) \to \psi(-A),$$

where $\psi(-A) = \int_{-\infty}^{-A} e^{-\frac{t^2}{2}} dt$.

Exactly as in the proof of Lemma 3.2 we consider the cover $\mathcal{B}$ satisfying the relation (3.3). Choosing $\epsilon > 0$ in (3.3) to be small we can find $\beta > 0$ and a ball $B \in \mathcal{B}$ such that:

$$\tilde{\mu}(\{\tilde{x} \in \tilde{J} \;:\; \tilde{S}_n(\tilde{x}) \;<\; -A \cdot \sigma \cdot n^{1/2}\} \;\cap\; \tilde{f}^{-n}(\tilde{K}_B)) > \beta > 0\,, \tag{3.16}$$

for infinitely many n-s.

Let us denote by $X^n := \pi(\{\tilde{x} \in \tilde{J} \;:\; \tilde{S}_n(\tilde{x}) \;<\; -A \cdot \sigma \cdot n^{1/2}\} \;\cap\; \tilde{f}^{-n}(\tilde{K}_B))$. Then $\mu(X^n) > \beta$ and

$$X^n \subset \bigcup_{\nu \in \mathcal{G}_n} f_\nu^{-n}(2B), \tag{3.17}$$

where $\mathcal{G}_n$ denotes the set of univalent branches $f_\nu^{-n}|_{2B}$.

Our goal again is to show that $\omega(X^n) \to 0$ as $n \to \infty$. As before we have

$$\omega(X^n) \leq \sum_{\nu \in \mathcal{G}_n} \omega(f_\nu^{-n}(2B) \cap X^n). \tag{3.18}$$

If $x \in X^n$ we have

$$S_n(x) < -A \cdot \sigma \cdot n^{1/2},$$

or equivalently

$$S_n^1(x) > \log d^n + A \cdot \sigma \cdot n^{1/2}. \tag{3.19}$$

Using (3.19) we can estimate for any $\nu \in \mathcal{G}_n$:

$$\begin{aligned} 1 \;\geq\; \omega(2B \cap f^n(X^n)) \;=\; \int_{f_\nu^{-n}(2B) \cap X^n} e^{S_n^1(x)}\, d\omega(x) \;\geq \\ \geq d^n \cdot e^{A\sigma n^{1/2}} \omega(f_\nu^{-n}(2B) \cap X^n). \end{aligned}$$

As a consequence we obtain

$$\omega(f_\nu^{-n}(2B) \cap X^n) \leq e^{-A\sigma n^{1/2}} \cdot d^{-n}. \tag{3.20}$$

Finally (3.18) and (3.20) give:

$$\omega(X^n) \leq e^{-A\sigma n^{1/2}}.$$

As n can be chosen to be arbitrarily large we obtain that the two measures are singular which is a contradiction proving the lemma.

Based on the above three Lemmas the proof of Proposition 3.1 follows immediately as seen e.g. in [PUZ] Lemma 1.1.

4. **Conformal maximality.**

In this final section we are going to give the proof of:

Theorem 4.1 *Let (f, U, V) be a GPL with totally disconnected Julia set. Then $\omega \approx \mu$ implies that (f, U, V) is conformally maximal.*

Proof. The proof is based on the homologous equation given by Proposition 3.1

$$\varphi(x) - \log d = u(fx) - u(x) \ , \ \mu \text{ a.e. } x \in J. \tag{4.1}$$

Starting from (4.1) we are going to construct an automorphic function τ that is required by Theorem A for conformal maximality.

Different kind of homologous equations appear naturally when investigating the relations between two measures on the Julia set as seen e.g. in [Z], [Vo], [LyV], [BPV], [BV2]. Also the techniques to handle these equations are different accordingly. Our approach is based on the main idea in [BPV] and [BV2]; however we have here the difficulty due to lack of regularity of φ and u.

Let us notice first that we can assume that the invariant set X, $\mu(X) = 1$ on which (4.1) holds consists of "good" points (in the sense of Definition 2.2).

¿From the proof it will be clear that there is no loss of generality to assume that there exists a repelling fixed point $p \in J$ of f which is not a critical value (i.e. $p \neq f^n(c)$ for all $n \geq 0$ and all critical points of f). Let us consider such $p \in J$. Notice that $p \in J$ is a good point and thus a point of continuity of $\phi = \varphi - \log d$. Our first step is to show that

$$\varphi(p) \ - \ \log d \ = \ 0. \tag{4.2}$$

(Warning: (4.2) does not follow from (4.1) since we do not know *apriori* that $p \in X$.)

Let us denote by B a small disc centered at p such that all components B_n of $f^{-n}B$ containing p are included in B and B is free from critical points of f.

It is clear that $\mu(B_n) = \frac{1}{d^n} \cdot \mu(B)$ and

$$\omega(B) = \int_{B_n} e^{S_n^1(x)}\, d\omega(x) \text{ where } S_n^1(x) = \sum_{i=0}^{n-1} \varphi(f^i(x)).$$

Furthermore observe that for any $x \in B_n \bigcap J$ and $i = 0, \dots, n-1$ we have that $\{f^i(x), p\}$ is $(n-i)$-nested. Consequently by (2.9) the inequality

$$|\varphi(f^i(x)) - \varphi(p)| \leq 2C_1 \cdot q^{n-i}$$

holds. This implies that

$$|S_n^1(x) - S_n^1(p)| \leq C_2 \; \forall x \in B_n$$

and therefore there exists $K > 0$ such that:

$$\frac{1}{K} \cdot e^{-n\cdot\varphi(p)} \cdot \omega(B) \leq \omega(B_n) \leq K \cdot e^{-n\cdot\varphi(p)} \cdot \omega(B).$$

Consequently $\frac{\omega(B_n)}{\mu(B_n)} \sim e^{n\cdot(\log d - \varphi(p))}$.

On the other hand $\omega \approx \mu$ and thus (0.2) together with the above relation imply that $\varphi(p) - \log d = 0$.

Now we can start the construction of τ. This will be done in three steps:

Step I : construction of τ on B.

Let us denote by g the inverse branch of $f^{-1} : B \to B_1 \subset B$. Notice that $\{p, g^n z, g^{n-1} z\}$ is $n-1$-nested for any $z \in B$, $n \in \mathbf{N}$. By (2.8) it follows that

$$\left|\log \frac{G(g^{n-1}z)}{G(g^n z)} - \varphi(p)\right| \leq C_1 \cdot q^{n-1}.$$

Using (4.2) this gives:

$$\left|\frac{G(g^{n-1}z)}{d \cdot G(g^n z)} - 1\right| \leq C_1 \cdot q^{n-1}. \tag{4.3}$$

Now (4.3) implies that the following limit:

$$\tau^1(z) := \lim_{n\to\infty} d^n \cdot G(g^n z)\,,\ z \in B \tag{4.4}$$

represents a subharmonic function which is harmonic in $B \setminus J$ and vanishing on J. Notice also that τ^1 is automorphic on B_1: $\tau^1(fz) = d \cdot \tau^1(z)$.

Our next objective is to show that there exists a function $u^1 \in L^2_{|B}(\mu)$ such that for any $x \in X \bigcap B_1$:

$$\lim_{\substack{z\to x\\ z\in B\setminus J}} \frac{G(z)}{\tau^1(z)} = e^{u(x)-u^1(x)}. \tag{4.5}$$

To define $u^1 \in L^2_{|B}(\mu)$ observe that for any $x \in B \bigcap X$ the sequence $\{u(g^n x)\}_n$ is a Cauchy sequence.

To see this we use (4.1),(4.2) and the inequality:

$$|\varphi(p) - \varphi(g^n x)| \le 2C_1 \cdot q^n.$$

It follows that $|u(g^n x) - u(g^{n-1}x)| \le 2C_1 \cdot q^n$.

Now for $x \in B \bigcap X$ we denote by $u^1(x) := \lim_{n\to\infty} u(g^n x)$. It is clear that $u^1 \in L^2_{|B}(\mu)$.

In order to prove (4.5) notice that as $x \in X$ is a good point and $z \to x$, there exists $N(x,z)$ such that $\{x, z\}$ is $N(x,z)$ -nested and $N(x,z) \to \infty$ as $z \to x$.

Let us suppose that $x, z \in B_1$; z is close to x, thus $\{x, z\}$ is $N(x,z)$-nested for some large $N(x,z)$. By the definition of τ^1:

$$\frac{G(z)}{\tau^1(z)} = \lim_{n\to\infty} \frac{G(z)}{d^n \cdot G(g^n z)} = \prod_{n=0}^{\infty} \frac{G(g^n z)}{d \cdot G(g^{n+1} z)}. \tag{4.6}$$

Notice also that $\{g^n x, g^n z\}$ is $(N(x,z) - N_0)$-nested for some fixed N_0. Without loss of generality we can assume that $N_0 = 0$ and hence $\{g^n x, g^n z\}$ is $N(x,z)$-nested.

Let us put $N := N(x,z)$ and consider $i \le 2N$. Because $\{g^{i-1}x, g^{i-1}z\}$ is N-nested (4.1) and (2.8) give:

$$\left|\log \frac{G(g^{i-1}z)}{d \cdot G(g^i z)} - (u(g^{i-1}x) - u(g^i x))\right| \le C_1 \cdot q^N,$$

which means

$$e^{u(g^{i-1}x)-u(g^ix)-C_1\cdot q^N} \leq \frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(g^{i-1}x)-u(g^ix)+C_1\cdot q^N}.$$

This implies:

$$e^{u(x)-u(g^{2N}x)-2C_1N\cdot q^N} \leq \prod_{i=0}^{2N}\frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(x)-u(g^{2N}x)+2C_1N\cdot q^N}. \tag{4.7}$$

On the other hand for $i > 2N$ we are going to use that $\{g^{i-1}x, g^{i-1}z\}$ is i-nested and so

$$e^{u(g^{i-1}x)-u(g^ix)-C_1\cdot q^i} \leq \frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(g^{i-1}x)-u(g^ix)+C_1\cdot q^i}.$$

For $n > 2N$ this implies:

$$e^{u(g^{2N}x)-u(g^nx)-C_2\cdot q^N} \leq \prod_{i=2N}^{n}\frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(g^{2N}x)-u(g^nx)+C_2\cdot q^N}. \tag{4.8}$$

Now (4.7) and (4.8) imply:

$$e^{u(x)-u(g^nx)-C_3\cdot q^{N/2}} \leq \prod_{i=0}^{n}\frac{G(g^{i-1}z)}{d\cdot G(g^iz)} \leq e^{u(x)-u(g^nx)+C_3\cdot q^{N/2}}. \tag{4.9}$$

Consequently if $\{x, z\}$ is N-nested (4.6) and (4.9) give:

$$e^{u(x)-u^1(x)-C_3\cdot q^{N/2}} \leq \frac{G(z)}{\tau^1(z)} \leq e^{u(x)-u^1(x)+C_3\cdot q^{N/2}}. \tag{4.10}$$

Recalling that $N = N(x, z) \to \infty$ as $z \to x$ the estimate (4.10) gives (4.5).

Let us consider now the union of backward orbits of B : $\mathcal{O} := \bigcup_{n\geq 0} f^{-n}B$.

Step II: extension of τ to $\mathcal{O}$

Let B_θ be a component of $f^{-n}B$ for some $n > 0$. We define a function τ_θ^2 on B_θ by:

$$\tau_\theta^2(z) = \frac{1}{d^n}\tau_1(f^nz), \quad z \in B_\theta.$$

We would like to prove that τ_θ^2 (or a symmetrized version of it) does not depend on θ (and n).

We are going to calculate first the limit:

$$\lim_{\substack{z \to x \\ z \in B_\theta \setminus J}} \frac{G(z)}{\tau_\theta^2(z)} \text{ for } x \in f^{-n}(X \cap B) \cap B_\theta.$$

To do that we write:

$$\frac{G(z)}{\tau_\theta^2(z)} = \frac{G(z)}{\frac{1}{d^n} \cdot \tau^1(f^n z)} = \frac{G(f^n z)}{\tau^1(f^n z)} \cdot \frac{d \cdot G(z)}{G(fz)} \cdots \frac{d \cdot G(f^{n-1} z)}{G(f^n z)}.$$

As n is being fixed we use (4.5) to obtain:

$$\lim_{\substack{z \to x \\ z \in B_\theta \setminus J}} \frac{G(z)}{\tau_\theta^2(z)} = e^{u(x) - u^1(f^n x)}. \tag{4.11}$$

Let us take now $n_2 > n_1$ and two corresponding branches $f_{\theta_2}^{-n_2}$, $f_{\theta_1}^{-n_1}$. If $x \in B_{\theta_1} \cap B_{\theta_2} \cap f^{-n_1}(X) \cap f^{-n_2}(X)$ by (4.11) we can write:

$$\lim_{\substack{z \to x \\ z \in B_{\theta_1} \cap B_{\theta_2} \setminus J}} \frac{\tau_{\theta_1}^2(z)}{\tau_{\theta_2}^2(z)} = e^{u^1(f^{n_2}(x)) - u^1(f^{n_1}(x))}. \tag{4.12}$$

let us denote by $x_2 := f^{n_2}(x)$, $x_1 = f^{n_1}(x)$. Then $x_1, x_2 \in B \cap X$ and $f^{n_2 - n_1}(x_1) = x_2$. By the definition of u^1 it follows that $u^1(x_1) = u^1(x_2)$. Consequently (4.12) becomes:

$$\lim_{\substack{z \to x \\ z \in B_\theta \setminus J}} \frac{\tau_{\theta_1}^2(z)}{\tau_{\theta_2}^2(z)} = 1 \text{ for } \omega \text{ a.e. } x \in B_{\theta_1} \cap B_{\theta_2} \cap J.$$

Now we can apply Grishin's lemma (Lemma B) to obtain that $\Delta \tau_{\theta_1}^2 = \Delta \tau_{\theta_2}^2$ on $B_{\theta_1} \cap B_{\theta_2}$ and hence the function $\tau_{\theta_1}^2 - \tau_{\theta_1}^2$ is harmonic in $B_{\theta_1} \cap B_{\theta_2}$ and it vanishes on $B_{\theta_1} \cap B_{\theta_2} \cap J$.

Now , either $\tau_{\theta_1}^2 \equiv \tau_{\theta_1}^2$ or $B_{\theta_1} \cap B_{\theta_2} \cap J$ is covered by a finite number of real analytic curves. It's not hard to see that if the latter happens the whole J can be covered by a finite number of real analytic curves so this will be the case for any pair of θ_1, θ_2 for which $B_{\theta_1} \cap B_{\theta_2}$ is not empty.

Furthermore, without loss of generality (see [BPV],[LyV] or [Vo]) we can consider the situation when the curves are disjoint. Let $*$ be a holomorphic symmetry with respect to these curves. Instead of τ_θ^2 we are going to work with

$$\tau_\theta^3(z) \stackrel{\text{def}}{=} \tau_\theta^2(z) + \tau_\theta^2(z^*).$$

The advantage is that now $\tau_{\theta_1}^3 \equiv \tau_{\theta_2}^3$ in $B_{\theta_1} \cap B_{\theta_2}$.

In any case we obtain a function τ^4 on $\mathcal{O}$ such that $\tau^4_{|B_\theta} = \tau^3_\theta$ (or τ^2_θ if the first possibility "$\tau^2_{\theta_1} = \tau^2_{\theta_2}$" always occurs).

It is clear that our function τ^4 has the automorphic property on $f^{-1}\mathcal{O}$.

Since J is totally disconnected there is a number $N > 0$ such that $f^{-N}U \subset B$. In the last step we are going to extend τ^4 to the whole U.

Step III: extension of τ to U

Consider $z \in U$ which is not a critical value of f^N. Choose a topological disc free from critical values of f^N and containing both z and p.

Let V_N, be the component of of $f^{-N}V$, containing the point p and contained in B. Then the map $f^{-N} : V \to V_N$ is univalent and we can define:

$$\tau^5(z) = d^N \tau^4(f^{-N}(z)), \quad z \in V$$

It is clear that τ^5 does not depend on V since $\tau^5_{|V \cap B} = \tau^4$. We extend τ^5 to the critical values of f^N by continuity.

Because τ^5 is a positive subharmonic function, harmonic on $U \setminus J$ and vanishing on J we only need to check the automorphic property.

To do that let us denote by B_1 an arbitrary component of $f^{-1}B$. We are going to show that $\tau^5_{|B_1} = \tau^4$. Since τ^4 was automorphic on B_1 this proves that τ^5 is automorphic on V_i where V_i contains B_1.

Let us pick $z \in B_1$ and an appropriate univalent branch f_θ^{-N} of f^{-N}. Put $z_1 = f_\theta^{-N} z$ and by our definition we have $\tau^5(z) = d^N \tau^4(z_1)$. On the other hand observe that $f^{N+1} z_1 = fz \in B$.

By the automorphic property of τ^4 we have:

$$\tau^4(z_1) = \frac{1}{d^{N+1}} \tau^4(f^{N+1} z_1) = \frac{1}{d^{N+1}} \tau^4(fz).$$

It follows that $\tau^5(z) = \frac{1}{d} \tau^4(fz)$ and consequently, by the definition of τ^4 in Step II we have: $\tau^5(z) = \tau^4(z)$ for $z \in B_1$. This shows that $\tau^5_{|V_i}$ is automorphic. As B_1 was arbitrary we obtain $\tau^5_{|V_i}$ is automorphic for any i.

This concludes our construction and proves the theorem.

Final Remarks:

Our first remark is that our result holds true for general GPL where the Julia set is disconnected (whithout being totally disconnected). The reason for this is that an invariant, ergodic measure with positive entropy is supported on the 'totally disconnected' part of the the Julia set. This fact follows from arguments used in [PUZ] or [Z].

After this paper has been written the author has found out about the work of Anna Zdunik [Zd] where similar problems are discussed in the setting of polynomial-like maps. The approach in [Zd] is different. It is based on a very elegant idea (similar to the one in [MR] to apply the Perron-Frobenius operator to subharmonic functions. In this way the author constructs an invariant measure absolutely continuous with respect to harmonic measure and then relate this invariant measure to the measure of maximal entropy. This can be applied also in the case of generalized polynomial-like maps and our result follows by the method in [Zd]. We think however that our approach might be useful in treating similar problems and is worthy of future developpement.

Aknowledgements:

I would like to thank my advisor Alexander Volberg for suggesting the problem and for many helpful conversations on the subject of this paper. I would like to thank also Manfred Denker for his kind explanations about the central limit theorem and to Pekka Koskela for his interest in this work and for stimulating conversations on generelized polynomial-like maps.

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