Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 423–438
DOI: 10.24193/subbmath.2023.2.16

An extension of Krasnoselskii's cone fixed point theorem for a sum of two operators and applications to nonlinear boundary value problems

Lyna Benzenati and Karima Mebarki

Abstract. The purpose of this work is to establish a new generalized form of the Krasnoselskii type compression-expansion fixed point theorem for a sum of an expansive operator and a completely continuous one. Applications to three nonlinear boundary value problems associated to second order differential equations of coincidence type are included to illustrate the main results.

Mathematics Subject Classification (2010): 47H10, 54H25, 34B18.

Keywords: Fixed point, Banach space, cone, expansive mapping, sum of operators, nonlinear boundary value problem, coincidence problems.

1. Introduction

One of the main results in fixed point theory is the cone expansion and compression theorem proved by Krasnoselskii in 1964 (see, e.g., [10, 11]). It represents a powerful existence tool in studying operator equations and showing existence of positive solutions to various boundary value problems. By this result, a solution is localized in a conical shell of a normed linear space. This theorem has been recently deeply improved in various directions; see [1, 2, 3, 6, 9, 12, 13, 14] and references therein. A vector version of Krasnoselskii's fixed point theorem in cones has been given in [4, 15, 16]. In practice, the vector version allows the nonlinear term of a system to have different behaviors both in components and in variables.

Received 21 March 2020; Accepted 10 April 2020.

In this paper, we first establish some user-friendly versions of Krasnoselskii type compression-expansion fixed point theorem for a sum of an expansive operator and a completely continuous one. A vector version of the main result is also given.
Next, using the main obtained fixed-point result, we study the existence of positive solutions for three nonlinear boundary value problems associated to second order differential equations and systems of coincidence type equations.

Let X be a normed linear space with norm $\|.\|$, and let $\mathcal{P} \subset X$ be a wedge, i.e., a closed convex subset of X, $\mathcal{P} \neq \{0\}$ with $\lambda\mathcal{P} \subset \mathcal{P} \neq \{0\}$ for every $\lambda \in \mathbb{R}_+$. If in addition $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$, then $\mathcal{P}$ is a cone, and we say that $x < y$ if and only if $y - x \in \mathcal{P} \setminus \{0\}$. For two numbers $0 < r < R$, we define the conical shell $\mathcal{P}_{r,R}$ by $\mathcal{P}_{r,R} := \{x \in \mathcal{P} : r \leq \|x\| \leq R\}$.

Let $N\colon D \subset X \to X$ be a continuous operator. The operator N is said to be *bounded* if it maps bounded sets into bounded sets, *completely continuous* if it maps bounded sets into relatively compact sets, and *compact* if the set $N(D)$ is relatively compact.

Consider the operator equation

$$Nx = x,$$

where N is a given nonlinear map acting in $\mathcal{P}$.

Theorem 1.1. (Krasnoselskii's compression-expansion fixed point theorem). *Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$. Assume that $N : \mathcal{P}_{r,R} \to \mathcal{P}$ is a compact map and there exists $p \in \mathcal{P} \setminus \{0\}$ such that the following conditions are satisfied:*

$$\begin{aligned} Nx &\neq \lambda x \quad &&\text{for } \|x\| = \alpha \text{ and } \lambda > 1; \\ Nx + \mu p &\neq x \quad &&\text{for } \|x\| = \beta \text{ and } \mu > 0. \end{aligned} \tag{1.1}$$

Then N has a fixed point x in $\mathcal{P}$ with $r \leq \|x\| \leq R$.

Remark 1.2. If $\beta < \alpha$, then the conditions (1.1) represents a compression property of N upon the conical shell $\mathcal{P}_{r,R}$, while if $\beta > \alpha$, then the conditions (1.1) expresses an expansion property of N upon $\mathcal{P}_{r,R}$.

Consider a system of two operator equations

$$\begin{cases} N_1(x_1, x_2) = x_1 \\ N_2(x_1, x_2) = x_2, \end{cases}$$

where N_1, N_2 act from $\mathcal{P} \times \mathcal{P}$ to $\mathcal{P}$.

Theorem 1.3. ([16, Theorem 2.1]). *Let $(X, \|.\|)$ be a normed linear space; $\mathcal{P}_1$, $\mathcal{P}_2 \subset X$ two wedges; $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$; $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$ for $i = 1, 2$ and let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : \mathcal{P}_{r,R} = (\mathcal{P}_1)_{r_1,R_1} \times (\mathcal{P}_2)_{r_2,R_2} \to \mathcal{P}$, $N = (N_1, N_2)$, is a compact map and there exist $p_i \in \mathcal{P}_i \setminus \{0\}$, $i = 1, 2$ such that for each $i \in \{1, 2\}$ the following conditions are satisfied in $\mathcal{P}_{r,R}$:*

$$\begin{aligned} N_i x &\neq \lambda x_i \quad &&\text{for } \|x_i\| = \alpha_i \text{ and } \lambda > 1; \\ N_i x + \mu p_i &\neq x_i \quad &&\text{for } \|x_i\| = \beta_i \text{ and } \mu > 0. \end{aligned} \tag{1.2}$$

Then N has a fixed point $x = (x_1, x_2)$ in $\mathcal{P}$ such that $r_i \leq \|x_i\| \leq R_i$ for $i = 1, 2$.

A mapping $T : D \subset Y \to Y$, where (Y, d) is a metric space, is said to be *expansive* if there exists a constant $h > 1$ such that

$$d(Tx, Ty) \geq h\, d(x, y) \text{ for all } \ x, y \in D.$$

To establish our results, we need the following technical lemma concerning expansive mappings.

Lemma 1.4. *Let $(X, \|.\|)$ be a linear normed space and $D \subset X$. Assume that the mapping $T : D \to X$ is expansive with constant $h > 1$. Then the mapping $T : D \to T(D)$ is invertible and*

$$\|T^{-1}x - T^{-1}y\| \leq \frac{1}{h}\|x - y\|, \ \ \forall\, x, y \in T(D).$$

2. Main results

Theorem 2.1. *Let K be a subset of a Banach space X and $\mathcal{P} \subset X$ a wedge. Assume that $T : K \to X$ is an expansive mapping with constant $h > 1$ and $F : K \to X$ is a mapping such that $I - F : K \to \mathcal{P}$ is completely continuous one with $\mathcal{P} \subset T(K)$. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $p \in \mathcal{P} \setminus \{0\}$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$.*
Suppose that the following conditions are satisfied:

$$x \neq \lambda Tx + Fx \quad \text{for } x \in T^{-1}(\mathcal{P}),\ \|Tx\| = \alpha \ \text{ and } \lambda > 1. \tag{2.1}$$

$$x \neq Tx + Fx - \mu p \quad \text{for } x \in T^{-1}(\mathcal{P}),\ \|Tx\| = \beta \ \text{ and } \mu > 0. \tag{2.2}$$

Then $T + F$ has a fixed point x in $T^{-1}(\mathcal{P})$ such that $r \leq \|Tx\| \leq R$.

Proof. By Lemma 1.4, the operator $T^{-1} : T(K) \to K$ is a $\frac{1}{h}$-contraction. Then the operator N defined by

$$\begin{array}{rcl} N : \mathcal{P} & \to & \mathcal{P} \\ y & \mapsto & Ny = T^{-1}y - FT^{-1}y \end{array}$$

is well defined and it is completely continuous.
Claim 1. We show that Condition (2.1) implies that

$$Ny \neq \lambda y \ \text{ for } \ \|y\| = \alpha \ \text{ and } \lambda > 1.$$

On the contrary, assume the existence of $\lambda_0 > 1$ and $y_1 \in \mathcal{P}$ with $\|y_1\| = \alpha$ such that

$$Ny_1 = \lambda_0 y_1.$$

Let $x_1 := T^{-1}y_1$. Then

$$x_1 - Fx_1 = \lambda_0 Tx_1.$$

The hypotheses $y_1 \in \mathcal{P}$, $\|y_1\| = \alpha$ imply that $x_1 \in T^{-1}(\mathcal{P})$ and $\|Tx_1\| = \alpha$. Which lead to a contradiction with Condition (2.1).
Claim 2. We show that Condition (2.2) implies that

$$Ny + \mu p \neq y \text{ for } \ \|y\| = \beta \ \text{ and } \mu > 0.$$

On the contrary, assume the existence of $\mu_0 > 1$ and $y_2 \in \mathcal{P}$ with $\|y_2\| = \beta$ such that

$$y_2 - Ny_2 = \mu_0 p.$$

Let $x_2 := T^{-1}y_2$. Then

$$x_2 = Tx_2 + Fx_2 - \mu_0 p.$$

The hypotheses $y_2 \in \mathcal{P}$, $\|y_2\| = \beta$ imply that $x_2 \in T^{-1}(\mathcal{P})$ and $\|Tx_2\| = \beta$. Which lead to a contradiction with Condition (2.2).

Consequently, by Theorem 1.1, the operator N has a fixed point $y \in \mathcal{P}$ such that $r \le \|y\| \le R$. That is

$$T^{-1}y - FT^{-1}y = y.$$

Let $x := T^{-1}y$. Then $x \in T^{-1}(\mathcal{P})$, it is a fixed point of $T + F$, and

$$r \le \|Tx\| \le R.$$

□

If in addition $\mathcal{P}$ is a cone, as a consequence of Theorem 2.1, we derive the following cone compression and expansion fixed point theorems, the first in terms of the partial order relation induced by $\mathcal{P}$ and the second of norm type.

Corollary 2.2. *Let K be a subset of a Banach space X and $\mathcal{P} \subset X$ a cone. Assume that $T : K \to X$ is an expansive mapping with constant $h > 1$ and $F : K \to X$ is a mapping such that $I - F : K \to \mathcal{P}$ is completely continuous one with $\mathcal{P} \subset T(K)$. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$.*
Suppose that the following conditions are satisfied:

$$x \not> Tx + Fx \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \alpha. \tag{2.3}$$

$$x \not< Tx + Fx \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \beta. \tag{2.4}$$

Then $T + F$ has a fixed point x in $T^{-1}(\mathcal{P})$ such that $r \le \|Tx\| \le R$.

Proof. The conditions (2.1) and (2.2) of Theorem 2.1 are satisfied. Indeed, assume the contrary of Condition (2.1). Then there exist $\lambda_0 > 1$ and $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\| = \alpha$ such that

$$x_0 = \lambda_0 Tx_0 + Fx_0.$$

Thus, $Tx_0 = \frac{1}{\lambda_0}(x_0 - Fx_0) < x_0 - Fx_0$, that is $x_0 > Tx_0 + Fx_0$, which contradicts (2.3).
Assume the contrary of Condition (2.2). Then there exist $p \in \mathcal{P} \setminus \{0\}$, $\mu_0 > 0$ and $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\| = \beta$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 p.$$

Since $\mu_0 p \in \mathcal{P} \setminus \{0\}$, we obtain

$$x_1 < Tx_1 + Fx_1,$$

which contradicts (2.4). □

Corollary 2.3. *Let K be a subset of a Banach space X and $\mathcal{P} \subset X$ a cone. Assume that $T : K \to X$ is an expansive mapping with constant $h > 1$ and $F : K \to X$ is a mapping such that $I - F : K \to \mathcal{P}$ is completely continuous one with $\mathcal{P} \subset T(K)$. Let $\alpha, \beta > 0$, $\alpha \neq \beta$, $r := \min\{\alpha, \beta\}$ and $R := \max\{\alpha, \beta\}$.*
Suppose that the following conditions are satisfied:

$$\|x - Fx\| \le \|Tx\| \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \alpha. \tag{2.5}$$

$$\|x - Fx\| \ge \|Tx\| \quad \text{for } x \in T^{-1}(\mathcal{P}) \text{ with } \|Tx\| = \beta. \tag{2.6}$$

Then $T+F$ has a fixed point x in $T^{-1}(\mathcal{P})$ such that $r \leq \|Tx\| \leq R$.

Proof. The conditions (2.1) and (2.2) of Theorem 2.1 are satisfied. Indeed, assume the contrary of Condition (2.1). Then there exist $\lambda_0 > 1$ and $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\| = \alpha$ such that

$$x_0 = \lambda_0 Tx_0 + Fx_0.$$

Then $x_0 - Fx_0 = \lambda_0 Tx_0$, that is

$$\|x_0 - Fx_0\| = \lambda_0 \|Tx_0\| > \|Tx_0\|,$$

which contradicts (2.5).
Assume the contrary of Condition (2.2). Then there exist $p \in \mathcal{P} \setminus \{0\}$, $\mu_0 > 0$ and $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\| = \beta$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 p.$$

$x_1 - Fx_1 = Tx_1 - \mu_0 p$ that is

$$\|x_1 - Fx_1\| < \|Tx_1\|,$$

which contradicts (2.6). □

The vector version of Theorem 2.1 is presented in the following theorem. In what follows, we shall consider two Banach spaces $(X_1, \|.\|_1), (X_2, \|.\|_2)$; two wedges $\mathcal{P}_1 \subset X_1, \mathcal{P}_2 \subset X_2$, the product space $X := X_1 \times X_2$, the corresponding wedge $\mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2$ of X. For $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$, and $r = (r_1, r_2)$, $R = (R_1, R_2)$.

Theorem 2.4. *Let $K := K_1 \times K_2$ be a subset of X.*
Assume that $T_i : K_i \subset X_i \to X_i$ be an expansive mapping with constant $h_i > 1$ and $F_i : K \to X_i$ is a mapping such that $I_i - F_i : K \to X_i$ be a completely continuous one with $\mathcal{P}_i \subset T(K_i)$, $i = 1, 2$ and $x_i - F_i(x_1, x_2) \in \mathcal{P}_i$ for $x_i \in K_i$, $i = 1, 2$.
Suppose that there exist $p_i \in \mathcal{P}_i \setminus \{0\}$, $i = 1, 2$ such that for each $i \in \{1, 2\}$ the following conditions are satisfied:

$$x_i \neq \lambda T_i x_i + F_i x \ \text{ for } x_i \in T_i^{-1}(\mathcal{P}_i),\ \|T_i x_i\| = \alpha_i \text{ and } \lambda > 1. \tag{2.7}$$

$$x_i \neq T_i x_i + F_i x - \mu p_i \ \text{ for } x_i \in T_i^{-1}(\mathcal{P}_i),\ \|T_i x_i\| = \beta_i \text{ and } \mu > 0. \tag{2.8}$$

Then $T + F = (T_1 + F_1, T_2 + F_2)$ has a fixed point $x = (x_1, x_2)$ in $T_1^{-1}(\mathcal{P}_1) \times T_2^{-1}(\mathcal{P}_2)$ such that

$$r_i \leq \|T_i x_i\| \leq R_i \ \text{ for } i = 1, 2.$$

Proof. By Lemma 1.4, for $i \in \{1, 2\}$ the operator $T_i^{-1} : T(K_i) \to K_i$ is an $\frac{1}{h_i}$-contraction. Then the operator N defined by

$$\begin{aligned} N : \mathcal{P} &\to \mathcal{P} \\ y &\mapsto N(y_1, y_2) = (N_1(y_1, y_2), N_2(y_1, y_2)) \end{aligned}$$

where

$$\begin{cases} N_1(y_1, y_2) = T_1^{-1} y_1 - F_1(T_1^{-1} y_1, T_2^{-1} y_2) \\ \\ N_2(y_1, y_2) = T_2^{-1} y_2 - F_2(T_1^{-1} y_1, T_2^{-1} y_2) \end{cases}$$

is well defined and it is completely continuous.

Claim 1. We show that Condition (2.7) implies that

$$N_i y \neq \lambda y_i \text{ for } \|y_i\| = \alpha_i \text{ and } \lambda > 1 \text{ for } i = 1, 2.$$

On the contrary, assume the existence of $\lambda_0 > 1$ and $, y^0 = (y_1^0,\ y_2^0) \in \mathcal{P}$ with $\|y_i^0\| = \alpha_i$ such that

$$N_1 y^0 = \lambda_0 y_1^0 \text{ or } N_2 y^0 = \lambda_0 y_2^0.$$

Let $x_i^0 := T_i^{-1} y_i^0$ for $i = 1, 2$. Then, we obtain

$$x_1^0 - F_1(x_1^0, x_2^0) = \lambda_0 T_1 x_1^0$$

or

$$x_2^0 - F_1(x_1^0, x_2^0) = \lambda_0 T_2 x_2^0.$$

The hypotheses $y^0 \in \mathcal{P}$, $\|y_i^0\| = \alpha_i$ imply that $x_i^0 \in T_i^{-1}(\mathcal{P}_i)$ for $i = 1, 2$ with $\|T_i x_i^0\| = \alpha_i$, which lead to a contradiction with Condition (2.7).
Claim 2. We show that condition (2.8) implies that

$$N_i y + \mu p_i \neq y_i \text{ for } \|y_i\| = \beta_i \text{ and } \mu > 0 \text{ for } i = 1, 2.$$

On the contrary, assume the existence of $\mu_0 > 0$ and $z^0 = (z_1^0, z_2^0) \in \mathcal{P}$ with $\|z_i^0\| = \beta_i$ such that

$$z_1^0 - N_1 z^0 = \mu_0 p_1 \text{ or } z_2^0 - N_2 z^0 = \mu_0 p_2.$$

Let $t_i^0 := T_i^{-1} z_i^0$ for $i = 1, 2$. Then, we obtain

$$t_1^0 = T_1 t_1^0 + F_1(t_1^0, t_2^0) - \mu_0 p_1$$

or

$$t_2^0 = T_2 t_2^0 + F_2(t_1^0, t_2^0) - \mu_0 p_2.$$

The hypotheses $z^0 \in \mathcal{P}$, $\|z_i^0\| = \beta_i$ imply that $t_i^0 \in T_i^{-1}(\mathcal{P}_i)$ for $i = 1, 2$ with $\|T_i t_i^0\| = \beta_i$, which lead to a contradiction with condition (2.8). Our result then follows from Theorem 1.3. □

Remark 2.5. Since the compact operator N in Theorems 1.1 and 1.3 may be generalized to a strict-set contraction, the conclusion of Theorems 2.1 (and its Corollaries) and Theorems 2.4 can be extended to the case of a ℓ-set contraction mapping $I - F$ $(0 < \ell < h)$ with respect to some measure of noncompactness (see [5]).

3. Applications

3.1. Example 1

Consider the following nonlinear boundary value problem

$$\begin{cases} -\frac{d^2}{dt^2} f(t, x(t)) = g(t) h(x(t)),\ 0 < t < 1 \\ \\ x(0) = x(1) = 0, \end{cases} \tag{3.1}$$

where $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function defined by:

$$f(t, u) = u^3 + a(t) u,\ a \in \mathcal{C}^2([0, 1], \mathbb{R}_+), \text{ with } \min_{t \in [0,1]} a(t) > 1,$$

$g \in \mathcal{C}([0, 1], \mathbb{R}_+)$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous increasing function.

Problem (3.1) is equivalent to the integral equation

$$f(t,x(t)) = \int_0^1 G(t,s)g(s)h(x(s))ds,\ t\in[0,1], \tag{3.2}$$

where G is the corresponding Green's function defined in $[0,1]\times[0,1]$ by:

$$G(t,s) = \begin{cases} t(1-s), & if \quad 0\le t\le s\le 1, \\ s(1-t), & if \quad 0\le s\le t\le 1. \end{cases} \tag{3.3}$$

The Green function satisfies the following properties:

$$\begin{array}{rcl} 0\le G(t,s) & \le & G(s,s),\ \forall\,(t,s)\in[0,1]\times[0,1], \\ G(t,s) & \ge & \frac{1}{4}G(s,s),\ \forall(t,s)\in[\frac{1}{4},\frac{3}{4}]\times[0,1]. \\ \int_0^1 G(t,s)\,ds & \le & \frac{1}{8},\ \forall\,t\in[0,1]. \\ \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)\,ds & \ge & \frac{1}{16},\ \forall\,t\in[\frac{1}{4},\frac{3}{4}]. \end{array}$$

We will set

$$A := \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)\,ds,$$

$$B := \frac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)g(s)\,ds, \quad \text{for some } t_0\in[0,1].$$

We let

$(\mathcal{C}_0)$ $1 < a_0 := \min_{t\in[0,1]} a(t) \le a^0 := \max_{t\in[0,1]} a(t).$

Assume that the following assumptions hold for some positive reals α,β with $\alpha\neq\beta$:

$(\mathcal{C}_1)$ $Ah\left(\frac{1}{a_0}\alpha\right)\le\alpha,$

$(\mathcal{C}_2)$ $Bh\left(\frac{1}{4}\beta_0\right)\ge\beta$, where $\beta_0=\beta_0(\beta)>0$ such that $\beta_0^3+a^0\beta_0=\beta$.

Remark 3.1. From the properties of Green's function, we get

$$\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)\,ds \le \frac{1}{8}\max_{t\in[0,1]} g(t)$$

and

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)g(s)\,ds \ge \frac{1}{16}\min_{t\in[\frac{1}{4},\frac{3}{4}]} g(t).$$

Then, for the conditions $(\mathcal{C}_1)$ and $(\mathcal{C}_2)$ to be satisfied it is enough that constants α and β satisfy

$$\frac{1}{8}\max_{t\in[0,1]} g(t)h\left(\frac{1}{a_0}\alpha\right)\le\alpha \ \text{ and } \ \frac{1}{16}\min_{t\in\in[\frac{1}{4},\frac{3}{4}]} g(t)h\left(\frac{1}{4}\beta_0\right)\ge\beta.$$

Now we state our main result

Theorem 3.2. *Let Assumptions ($\mathcal{C}_0$)-($\mathcal{C}_2$) be satisfied. Then the nonlinear boundary value problem has a solution x which belongs to $\mathcal{C}([0,1],\mathbb{R}_+)$.*

Proof. Consider the Banach space $X = \mathcal{C}([0,1])$ normed by $\|x\| = \max_{t\in[0,1]} |x(t)|$, the set

$$K = \{x \in X \mid x(t) \geqslant 0, \forall\, t \in [0,1]\}$$

and the positive cone $\mathcal{P}$

$$\mathcal{P} = \left\{ x \in X : x \geq 0 \text{ on } [0,1] \text{ and } x(t) \geq \frac{1}{4}\|x\| \text{ for } \frac{1}{4} \leq t \leq \frac{3}{4} \right\}.$$

Define the operators $T : K \to K$ and $F : K \to X$ by

$$Tx(t) = x(t)^3 + a(t)x(t)$$

$$Fx(t) = x(t) - \int_0^1 G(t,s)g(s)h(x(s))\, ds,$$

respectively, for $t \in [0,1]$. Then the integral equation (3.2) is equivalent to the operational equation $x = Tx + Fx$. We check that all assumptions of Theorem 2.1 are satisfied.

(a) The operator $T : K \to K$ is surjective and it is expansive with constant $a_0 > 1$.

(b) Using the Arzela-Ascoli compactness criteria, we can show that $I - F$ maps bounded sets of K into relatively compact sets. In view of the sup-norm and the continuity of functions G, g and h, it is easily checked that $I - F$ is continuous. Therefore, the operator $I - F : K \to \mathcal{P}$ is completely continuous.

(c) Assume the existence of $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\| = \alpha$ and $\lambda_0 > 1$ such that

$$x_0 = \lambda_0 Tx_0 + Fx_0,$$

Then, $\lambda_0 Tx_0 = x_0 - Fx_0 = \int_0^1 G(.,s)g(s)h(x_0(s))\, ds$ on $[0,1]$.

So

$$\alpha < \lambda_0\|Tx_0\| = \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)h(x_0(s))\, ds. \tag{3.4}$$

On the other hand, we have

$$\|x_0\| = \|T^{-1}Tx_0\| \leq \frac{1}{a_0}\,\|Tx_0\| = \frac{1}{a_0}\alpha,$$

where $\frac{1}{a_0} < 1$ is the Liptchiz constant of T^{-1}, which implies that

$$0 \leq x_0(t) \leq \frac{1}{a_0}\alpha \text{ for } t \in [0,1].$$

Since the function h is increasing, we get

$$0 \leq h(x_0(t)) \leq h\left(\frac{1}{a_0}\alpha\right) \text{ for } t \in [0,1].$$

Thus, for all $t \in [0,1]$, we obtain

$$\begin{aligned}\int_0^1 G(t,s)g(s)h(x_0(s))\,ds &\leq h\left(\tfrac{1}{a_0}\alpha\right)\int_0^1 G(t,s)g(s)\,ds\\ &\leq \|\int_0^1 G(.,s)g(s)\,ds\|\,h\left(\tfrac{1}{a_0}\alpha\right)\\ &\leq Ah\left(\tfrac{1}{a_0}\alpha\right) \leq \alpha.\end{aligned}$$

By passage to the maximum, we obtain

$$\max_{t\in[0,1]}\int_0^1 G(t,s)g(s)h(x_0(s))\,ds \leq \alpha,$$

which leads to a contradiction with (3.4).

(d) Assume the existence of $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\| = \beta$ and $\mu_0 > 0$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 y_0,$$

where $y_0 \in \mathcal{P}$ with $y_0(t) > 0$ on $[0,1]$. Then

$$\int_0^1 G(.,s)g(s)h(x_1(s))\,ds = x_1 - Fx_1 = Tx_1 - \mu_0 y_0 < Tx_1 \text{ on } [0,1].$$

Since for all $t \in [0,1]$, $(Tx_1)(t) \leq \|Tx_1\| = \beta$, we get

$$\int_0^1 G(t,s)g(s)h(x_1(s))\,ds < (Tx_1)(t) \leq \beta,\ \forall\, t \in [0,1]. \tag{3.5}$$

On the other hand, from the property of Green's function G, for all $t \in [\frac{1}{4},\frac{3}{4}]$, we have

$$\begin{aligned}\int_0^1 G(t,s)g(s)h(x_1(s))\,ds &\geq \tfrac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)g(s)h(x_1(s))\,ds\\ &\geq \tfrac{1}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)g(s)h(x_1(s))\,ds.\end{aligned}$$

Since $\|Tx_1\| = \beta$ there exists $t_1 \in [0,1]$ such that $(Tx_1)(t_1) = \beta$. That is

$$(x_1(t_1))^3 + a(t_1)x_1(t_1) = \beta \leq (x_1(t_1))^3 + a^0 x_1(t_1),$$

where $a^0 = \max_{t\in[0,1]} a(t)$. Let $\beta_0 = \beta_0(\beta) > 0$ such that $\beta_0^3 + a^0\beta_0 = \beta$. So $x_1(t_1) \geq \beta_0$, which implies that $\|x_1\| \geq \beta_0$. Hence $x_1(s) \geq \frac{1}{4}\beta_0$, $\forall s \in [\frac{1}{4},\frac{3}{4}]$, which gives

$$h(x_1(s)) \geq h\left(\frac{1}{4}\beta_0\right).$$

Thus

$$\int_0^1 G(t,s)g(s)h(x_1(s))\,ds \geq \frac{1}{4}\,h\left(\frac{1}{4}\beta_0\right)\int_{\frac{1}{4}}^{\frac{3}{4}} G(t_0,s)g(s)\,ds = Bh\left(\frac{1}{4}\beta_0\right) \geq \beta,$$

which leads to a contradiction with (3.5). Therefor Theorem 2.1 applies and assure that Problem (3.1) has at least one positive solution $x \in \mathcal{C}([0,1])$ such that

$$r \leq \|Tx\| \leq R,$$

where $r = \min(\alpha,\beta)$ and $R = \max(\alpha,\beta)$. □

3.2. Example 2

Consider the following second-order nonlinear boundary value problem posed on the positive half-line

$$\begin{cases} -\frac{d^2}{dt^2}f(t,x(t)) + k^2 f(t,x(t)) = g(t)h(t,x(t)),\ t \in (0,+\infty). \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} x(t) = 0, \end{cases} \tag{3.6}$$

where k is a positive real parameter and $f : [0,+\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function defined by:

$$f(t,u) = u^3 + a(t)u,\ a \in \mathcal{C}^2([0,+\infty),\mathbb{R}_+).$$

The functions $g : [0,+\infty) \to \mathbb{R}_+$ and $h : [0,+\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.
Problem (3.6) is equivalent to the integral equation

$$f(t,x(t)) = \int_0^{+\infty} G(t,s)g(s)h(s,x(s))ds, \tag{3.7}$$

where G is the corresponding Green's function defined by:

$$G(t,s) = \frac{1}{2k}\begin{cases} e^{-ks}(e^{kt} - e^{-kt}), & if \quad 0 < t \leq s < \infty, \\ e^{-kt}(e^{ks} - e^{-ks}), & if \quad 0 < s \leq t < \infty. \end{cases}$$

The Green function G satisfies the following useful estimates:

$$G(t,s) \leq G(s,s) \leq \tfrac{1}{2k},\ \ \forall\, t,s \in [0,+\infty).$$

$$G(t,s)e^{-\mu t} \leq G(s,s)e^{-ks},\ \ \forall\, t,s \in [0,+\infty),\ \forall\, \mu \geq k.$$

$$G(t,s) \geq \Lambda G(s,s)e^{-ks},\ \forall\,(0 < \gamma < \delta),\ \ \forall\, t \in [\gamma,\delta]\,,\ \forall\, s \in [0,+\infty),$$

where

$$0 < \Lambda = \min(e^{-k\delta}, e^{k\gamma} - e^{-k\gamma}) < 1.$$

Assume that the following conditions are satisfied

$$(\mathcal{H}_0)\ \ 1 < a_0 :\ = \inf_{t\in[0,+\infty)} a(t) \leq a^0 :\ = \sup_{t\in[0,+\infty)} a(t).$$

($\mathcal{H}_1$) $h : [0, +\infty) \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and satisfies the polynomial growth condition:

$$\exists\, d > 0 : d \neq 1,\ 0 \leq h(t, x) \leq b(t) + c(t)x^d,\ \forall (t, x) \in [0, +\infty) \times \mathbb{R}_+,$$

where the functions $b, c \in \mathcal{C}([0, +\infty), \mathbb{R}_+)$.

($\mathcal{H}_2$) Assume the integrals

$$\begin{cases} M_1 & := \int\limits_0^\infty e^{-ks} b(s)G(s, s)g(s)ds \\ M_2 & := \int\limits_0^\infty e^{(d\theta - k)s} c(s)G(s, s)g(s)ds \end{cases}$$

are convergent and satisfy

$$\exists\, R > 0,\ M_1 + M_2 \frac{1}{a_0^d} R^d \leq R.$$

($\mathcal{H}_3$) There exists r with $0 < r < R$ such that

$$\Lambda \int\limits_\gamma^\delta e^{-ks} G(s, s)g(s)h(s, u)\, ds \geq r e^{\theta\delta} \quad \text{for all} \quad u \geq \Lambda r_0,$$

where $r_0 = r_0(r) > 0$ such that $r_0^3 + a^0 r_0 = r$.

Now we state our main result.

Theorem 3.3. *Let Assumptions ($\mathcal{H}_0$)-($\mathcal{H}_3$) be satisfied. Then the nonlinear boundary value problem (3.6) has at least one positive solution.*

Proof. Given a real parameter $\theta \geq k$ and consider the weighted Banach space

$$X = \left\{ x \in \mathcal{C}([0, +\infty), \mathbb{R}) : \sup_{t \in [0, +\infty)} \{e^{-\theta t}|x(t)|\} < \infty \right\}$$

normed by

$$\|x\|_\theta = \sup_{t \in [0, +\infty)} \{e^{-\theta t}|x(t)|\}.$$

Consider the set

$$K = \{x \in X \mid x(t) \geqslant 0, \forall\, t \in [0, +\infty)\}.$$

For arbitrary positive real numbers $0 < \gamma < \delta$, let $\mathcal{P}$ the positive cone defined in X by

$$\mathcal{P} = \left\{ x \in X :\ x \geq 0 \text{ on } [0, +\infty) \text{ and } \min_{t \in [\gamma, \delta]} x(t) \geq \Lambda \|x\|_\theta \right\}.$$

Define the operators $T : K \to K$ and $F : K \to X$ by:

$$Tx(t) = x(t)^3 + a(t)x(t)$$

$$Fx(t) = x(t) - \int\limits_0^{+\infty} G(t, s)g(s)h(s, x(s))\, ds,$$

respectively, for $t \in [0, +\infty)$.Then the integral equation (3.7) is equivalent to the operational equation $x = Tx + Fx$. We check that all assumptions of Theorem 2.1 are satisfied:

(a) The operator $T : K \to K$ is surjective and it is expansive with constant $a_0 > 1$.

(b) Using the properties of Green function G and appealing to the Zima compactness criteria (see [17, 18]), we can show that the operator $I - F : K \to \mathcal{P}$ is completely continuous (see [7, 8]).

(c) Assume the existence of $x_0 \in T^{-1}(\mathcal{P})$ with $\|Tx_0\|_\theta = R$ and $\lambda_0 > 1$ such that

$$x_0 = \lambda_0 Tx_0 + Fx_0,$$

Then, $\lambda_0 Tx_0 = x_0 - Fx_0 = \int_0^{+\infty} G(., s)g(s)h(s, x_0(s))\, ds$ on $[0, +\infty)$.

So

$$R < \lambda_0 \|Tx_0\|_\theta = \|(I - F)x_0\|_\theta. \tag{3.8}$$

On the other hand, we have

$$\|x_0\|_\theta = \|T^{-1}Tx_0\|_\theta \le \frac{1}{a_0} \|Tx_0\|_\theta = \frac{1}{a_0} R,$$

where $\frac{1}{a_0} < 1$ is the Liptchiz constant of T^{-1}. Thus, by Assumptions $(\mathcal{H}_1), (\mathcal{H}_2)$ and the properties of function G, for all $t \in [0, +\infty)$, we obtain

$$\begin{aligned}
|(I - F)x_0(t)|e^{-\theta t} &= \int_0^{+\infty} e^{-\theta t} G(t, s)g(s)h(s, x_0(s))\, ds \\
&\le \int_0^{+\infty} e^{-ks} G(s, s)g(s)[b(s) + c(s)|x_0(s)|^d]\, ds \\
&\le \int_0^{+\infty} e^{-ks} G(s, s)g(s)b(s)\, ds \\
&\quad + \|x_0\|_\theta^d \int_0^{+\infty} e^{(d\theta - k)s} G(s, s)g(s)c(s)\, ds \\
&\le M_1 + M_2 \|x_0\|_\theta^d \\
&\le M_1 + \tfrac{1}{a_0^d} R^d \le R.
\end{aligned}$$

By passage to the supremum over t, we get

$$\sup_{t \in [0, +\infty)} \{|(I - F)x_0(t)|e^{-\theta t}\} \le M_1 + M_2 \|x_0\|_\theta^d \le R,$$

which leads to a contradiction with (3.8).

(d) Assume the existence of $x_1 \in T^{-1}(\mathcal{P})$ with $\|Tx_1\|_\theta = r$ and $\mu_0 > 0$ such that

$$x_1 = Tx_1 + Fx_1 - \mu_0 y_0,$$

where $y_0 \in \mathcal{P}$ with $y_0(t) > 0$ on $[0, +\infty)$. Then

$$\int_0^{+\infty} G(t, s)g(s)h(s, x_1(s))\, ds = x_1 - Fx_1 = Tx_1 - \mu_0 y_0 < Tx_1.$$

Since for all $t \in [0,+\infty)$, $|(Tx_1)(t)|e^{-\theta t} \le \|Tx_1\|_\theta = r$, we get

$$\int_0^{+\infty} G(t,s)g(s)h(s,x_1(s))\,ds < (Tx_1)(t) \le re^{\theta\delta},\ \forall\, t \in [\gamma,\delta]. \tag{3.9}$$

On the other hand, $\|Tx_1\|_\theta = r$ implies one of the following cases:
Case 1. There exists $t_1 \in [0,+\infty)$ such that $|(Tx_1)(t_1)|e^{-\theta t_1} = r$. That is

$$(e^{-\theta t_1}x_1(t_1))^3 + a(t_1)e^{-\theta t_1}x_1(t_1) = r \le (e^{-\theta t_1}x_1(t_1))^3 + a^0 e^{-\theta t_1}x_1(t_1),$$

where $a^0 = \sup\limits_{t\in[0,+\infty)} a(t)$. Let $r_0 = r_0(r) > 0$ such that $r_0^3 + a^0 r_0 = r$.
Thus, $e^{-\theta t_1}x_1(t_1) \ge r_0$, which implies that $\|x_1\|_\theta \ge r_0$. Hence $x_1(s) \ge \Lambda\, r_0$, $\forall s \in [\gamma,\delta]$.
Case 2. $\lim\limits_{t\to+\infty} |(Tx_1)(t)|e^{-\theta t} = r$. That is

$$\begin{aligned} &\lim_{t\to+\infty}(e^{-\theta t}x_1(t))^3 + \lim_{t\to+\infty} a(t) \lim_{t\to+\infty} e^{-\theta t}x_1(t) = r \\ \le\ &\lim_{t\to+\infty}(e^{-\theta t}x_1(t))^3 + a^0 \lim_{t\to+\infty} e^{-\theta t}x_1(t). \end{aligned}$$

Thus, there exists $r_0 = r_0(r) > 0$ such that

$$\lim_{t\to+\infty} e^{-\theta t}x_1(t) \ge r_0,$$

which gives $\|x_1\|_\theta \ge r_0$.
Consequently, from Assumption $(\mathcal{H}_2)$ and the properties of Green function G, for all $t \in [\gamma,\delta]$, we have

$$\begin{aligned} \int_0^{+\infty} G(t,s)g(s)h(s,x_1(s))\,ds &\ge \Lambda \int_0^{+\infty} e^{-ks}G(s,s)g(s)h(s,x_1(s))\,ds \\ &\ge \Lambda \int_\gamma^\delta e^{-ks}G(s,s)g(s)h(s,x_1(s))\,ds \\ &\ge re^{\theta\delta}, \end{aligned}$$

which leads to a contradiction with (3.9). Then Theorem 2.1 applies. Therefore, Problem (3.6) has at least one solution $x \in K$ such that

$$r \le \|Tx\| \le R.$$ □

3.3. Example 3

In the following example, we will use the Theorem 2.4 to study the existence of positive solutions to a boundary value problem for a system of differential equations of the second order. A study that allows the nonlinear term of our system to have different behaviors both in components and in variables, and it gives a kind of localization of each component of a solution.

Consider the following nonlinear boundary value problem for system of two differential equations with Dirichlet condition

$$\begin{cases} -\frac{d^2}{dt^2}f_1(t,x_1(t)) = g_1(t)h_1(x_1(t),x_2(t)),\ 0<t<1 \\ -\frac{d^2}{dt^2}f_2(t,x_2(t)) = g_2(t)h_2(x_1(t),x_2(t)),\ 0<t<1 \\ x_1(0)=x_1(1)=0, \\ x_2(0)=x_2(1)=0, \end{cases} \tag{3.10}$$

where for $i\in\{1,2\}$, $f_i:[0,1]\times\mathbb{R}_+\to\mathbb{R}_+$ are continuous functions defined by:

$$f_i(t,u)=u^3+a_i(t)u,\ a_i\in\mathcal{C}^2([0,1],\mathbb{R}_+).$$

$g_i\in\mathcal{C}([0,1],\mathbb{R}_+)$ and $h_i:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ are continuous increasing functions with respect to its two variables.

The system (3.10) is equivalent to the integral system

$$\begin{cases} f_1(t,x_1(t)) = \int_0^1 G(t,s)g_1(s)h_1(x(s))ds,\ t\in[0,1] \\ f_2(t,x_2(t)) = \int_0^1 G(t,s)g_2(s)h_2(x(s))ds,\ t\in[0,1], \end{cases} \tag{3.11}$$

where $x=(x_1,x_2)$ and G is the corresponding Green's function given in (3.3). We will set

$$\begin{aligned} A_i &:= \max_{t\in[0,1]}\int_0^1 G(t,s)g_i(s)\,ds, \\ B_i &:= \frac{1}{4}\int_{\frac14}^{\frac34} G(t_i^0,s)g_i(s)\,ds, \quad \text{for some } t_i^0\in[0,1]. \end{aligned}$$

In what follows we consider $i\in\{1,2\}$ and let

($\mathbf{C}_0$) $1<a_i^0:=\min_{t\in[0,1]}a_i(t)\le b_i^0:=\max_{t\in[0,1]}a_i(t)$.

Assume that the following assumptions hold for some α_i,β_i with $\alpha_i\neq\beta_i$:

($\mathbf{C}_1$) $A_ih_i(\frac{1}{a_1^0}\alpha_1,\frac{1}{a_2^0}\alpha_2)\le\alpha_i$,

($\mathbf{C}_2$) $B_ih_i(\frac14\beta_1^0,\frac14\beta_2^0)\ge\beta_i$, where $\beta_i^0=\beta_i^0(\beta_i)>0$ such that $(\beta_i^0)^3+b_i^0\beta_i^0=\beta_i$.

Our main existence result on system (3.10) is

Theorem 3.4. *Let Assumptions* ($\mathbf{C}_0$)*-*($\mathbf{C}_2$) *be satisfied. Then the system (3.10) has a solution $x=(x_1,x_2)$ which belongs to $C([0,1],\mathbb{R}_+)\times C([0,1],\mathbb{R}_+)$.*

Proof. We apply Theorem 2.4. Here $X_1=X_2=C[0,1]$ with norm

$$\|u\|_\infty=\max_{t\in[0,1]}|u(t)|,$$

and

$$K_1=K_2=\{u\in C[0,1]:\ u(t)\ge 0 \text{ for all } t\in[0,1]\};$$

$$\mathcal{P}_1=\mathcal{P}_2=\left\{u\in u\in C[0,1]: u\ge 0 \text{ on } [0,1] \text{ and } u(t)\ge\frac14\|u\| \text{ for } \frac14\le t\le\frac34\right\}.$$

Define the operators $T_i:K_i\to K_i$ and $F_i:K_1\times K_2\to X_i$, for $i=1,2$, by:

$$T_ix_i(t)=x_i(t)^3+a_i(t)x_i(t)$$

$$F_ix(t) = x_i(t) - \int_0^1 G(t,s)g_i(s)h_i(x(s))\,ds,$$

respectively, for $t \in [0,1]$.

Then, the integral system (3.11) is equivalent to the operator equation

$$(x_1, x_2) = (T_1x_1 + F_1(x_1,x_2), T_2x_2 + F_2(x_1,x_2)),$$

According to Theorem 2.4 and in a way similar to the one used to show Theorem 3.2, we can easily show that the system (3.10) has at least one positive solution $x = (x_1, x_2)$ which belongs to $C[0,1] \times C[0,1]$ such that

$$r_i \le \|T_ix_i\| \le R_i,$$

where $r_i = \min(\alpha_i, \beta_i)$ and $R_i = \max(\alpha_i, \beta_i)$ for $i = 1, 2$. □

Acknowledgments. The authors are grateful to Professor Radu Precup, Babeş–Bolyai University, Romania, for his valuable remarks and suggestions. The authors acknowledge support of "Direction Générale de la Recherche Scientifique et du Développement Technologique (DGRSDT)", MESRS, Algeria.

References

[1] Anderson, D.R., Avery, R.I., *Fixed point theorem of cone expansion and compression of functional type*, J. Difference Equ. Appl., **8**(2002), no. 11, 1073-1083.

[2] Avery, R.I., Anderson, D.R., Krueger, R.J., *An extension of the fixed point theorem of cone expansion and compression of functional type*, Comm. Appl. Nonlinear Anal., **13**(2006), no. 1, 15-26.

[3] Benzenati, L., Mebarki, K., *Multiple positive fixed points for the sum of expansive mappings and k-set contractions*, Math. Methods Appl. Sci., **42**(2019), no. 13, 4412-4426.

[4] Benzenati, L., Mebarki, K., Precup, R., *A vector version of the fixed point theorem of cone compression and expansion for a sum of two operators*, Nonlinear Studies, **27**(2020), no. 3, 563-575.

[5] Deimling, K., *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, 1985.

[6] Djebali, S., Mebarki, K., *Fixed point index theory for perturbation of expansive mappings by k-set contractions*, Topol. Methods Nonlinear Anal., **54**(2019), no. 2, 613-640.

[7] Djebali, S., Mebarki, K., *Multiple positive solutions for singular BVPs on the positive half-line*, Comput. Math. Appl., **55**(2008), 2940-2952.

[8] Djebali, S., Moussaoui, T., *A class of second order bvps on infinite intervals*, Electron. J. Qual. Theory Differ. Equ., **4**(2006), 1-19.

[9] Feng, M., Zhang, X., Ge, W., *Positive fixed point of strict set contraction operators on ordered Banach spaces and applications*, Abstr. Appl. Anal., (2010), vol. Article ID 439137, 13 pages, doi:10.1155/2010/439137.

[10] Guo, D., Lakshmikantham, V., *Nonlinear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering, **vol. 5**, Academic Press, Boston, Mass, USA, 1988.

[11] Krasnosel'skii, M.A., *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[12] Kwong, M.K., *On Krasnoselskii's cone fixed point theorem*, Fixed Point Theory Appl., 2008(2008), Article ID 164537, 18 pp.

[13] Kwong, M.K., *The topological nature of Krasnoselskii's cone fixed point theorem*, Nonlinear Anal., **69**(2008), 891-897.

[14] O'Regan, D., Precup, R., *Compression-expansion fixed point theorem in two norms and applications*, J. Math. Anal. Appl., **309**(2005), no. 2, 383-391.

[15] Precup, R., *A vector version of Krasnoselskii's fixed point theorem in cones and positive periodic solutions of nonlinear systems*, J. Fixed Point Theory Appl., **2**(2007), 141-151.

[16] Precup, R., *Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications*, in Mathematical Models in Engineering, Biology, and Medicine, AIP Conference Proceedings 1124, Melville–New York, 2009, 284-293.

[17] Zima, M., *On a Certain Boundary Value Problem. Annales Societas Mathematicae Polonae*, Series I: Commentationes Mathematicae XXIX, 1990, 331-340.

[18] Zima, M., *On positive solutions of boundary value problems on the half-line*, J. Math. Anal. Appl., **259**(2001), no. 1, 127-136.

Lyna Benzenati
Laboratory of Applied Mathematics, Faculty of Exact Sciences,
Bejaia University, 06000 Bejaia, Algeria
e-mail: benzenatilyna@gmail.com

Karima Mebarki
Laboratory of Applied Mathematics, Faculty of Exact Sciences,
Bejaia University, 06000 Bejaia, Algeria
e-mail: mebarqi_karima@hotmail.fr