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ON THE STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION AND ITS APPLICATIONS

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Dedicated to Professor Pavel Enghiş at his 70^{th} anniversary

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1. Introduction

To quote S.M. Ulam [22, p.63] for very general functional equations, one can ask the following question. When it is true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation? Similarly, if we replace a given functional equation by a functional inequality, when can one assert that the solutions of the inequality lie near the solutions of the strict equation?

The present paper will provide a solution of Ulam's problem for the case of the quadratic functional equation.

The quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{1}$$

clearly has $f(x)=cx^2$ as a solution with c an arbitrary constant when f is a real function of a real variable. We shall be interested in functions $f: E_1 \to E_2$ where both E_1 and E_2 are real vector spaces, and we need a few facts concerning the relation between a quadratic function and a biadditive function sometimes called its polar. This relation is explained in Proposition 1, page 166, of the book by J. Aczél and J. Dhombres [1] for the case where $E_2 = R$, but the same proof holds for functions $f: E_1 \to E_2$. It follows then that $f: E_1 \to E_2$ is quadratic if and only if there exists a unique symmetric function $B : E_1 \times E_1 \to E_2$, additive in x for fixed y, such that $f(x) = B(x,x)$. The biadditive function B, the polar of f, is given by

$$B(x,y) = \left(\frac{1}{4}\right)(f(x+y) - f(x-y)).$$

A stability theorem for the quadratic functional equation (1) was proved by F. Skof [18] for functions $f : X \to E$ where X is a normed space and E a Banach space. Her proof also works if X is replaced by an Abelian group G. In this form, the theorem was demonstrated by P.W. Cholewa [2]. A function $f : G \to E$ is called δ-quadratic if for a given $\delta > 0$ it satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta. \tag{2}$$

The statement of the Skof-Cholewa theorem follows:

Theorem 1. *If $f : G \to E$ is δ-quadratic for all x and y in G, then there exists a function $q : G \to E$ which is quadratic, i.e. q satisfies (1) for all x and y in G and also is the unique quadratic function such that $\|f(x) - q(x)\| \leq \frac{\delta}{2}$ for all x in G. The function is given by*

$$q(x) = \lim_{n\to\infty} 4^{-n} f(2^n x) \tag{3}$$

for all x in G.

The proof is ommited, as it is a special case of that of the next theorem 2 due to S. Czerwik [4]. Czerwik's main result may be stated as follows.

Theorem 2. *Let E_1 be a normed vector space, E_2 a Banach space and $\varepsilon > 0$, $p \neq 2$ be real numbers. Suppose that the function $f : E_1 \to E_2$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p). \tag{4}$$

Then there exists exactly one quadratic function $g : E_1 \to E_2$ such that

$$\|g(x) - f(x)\| \leq c + k\varepsilon\|x\|^p \tag{5}$$

for all x in E_1, where: when $p < 2$, $c = \dfrac{\|f(0)\|}{3}$, $k = \dfrac{2}{4-2^p}$ and g is given by (3) with g instead of q. When $p > 2$, $c = 0$, $k = \dfrac{2}{2^p - 4}$ and $g(x) = \lim_{n\to\infty} 4^n f(2^{-n}x)$ for *all x un E_1. Also, if the mapping $t \to f(tx)$ from R to E_2 is continuous for each fixed x in E_1, then $g(tx) = t^2 g(x)$ for all t in R.*

Proof. Case 1. $p < 2$. In (4) set $y = x \neq 0$ and divide by 4. Then use the triangle inequality to obtain

$$||4^{-1}f(2x) - f(x)|| \leq 4^{-1}||f(0)|| + 4^{-1}(2\varepsilon||x||^p). \tag{6}$$

Make the induction hypothesis

$$||4^{-n}f(2^n x) - f(x)|| \leq ||f(0)|| \sum_{k=1}^{n} 4^{-k} + 2\varepsilon||x||^p \sum_{k=1}^{n} 2^{(k-1)p}4^{-k} \tag{7}$$

which is true for $n = 1$ by (6). Assuming (7) true, replace x by $2x$ in it and divide by 4. Now combine the result with (6) to see that (7) remains true with n replaced by $n+1$, which establishes (7) for all positive integers n and all $x(\neq 0)$ in E_1. By summing the series on the right side of (7), we obtain

$$||4^{-n}f(2^n x) - f(x)|| \leq c + \frac{2\varepsilon||x||^p}{4 - 2^p} = c + k\varepsilon||x||^p, \tag{8}$$

with $k = \dfrac{2}{4 - 2^p}$. In order to prove convergence of the sequence $g_n(x) = 4^{-n}f(2^n x)$, we divide inequality (8) by 4^m and also replace x by $2^m x$ to find that

$$||g_{m+n}(x) - g_m(x)|| = ||4^{-(m+n)}f(2^{(m+n)}x) - 4^{-m}f(2^m x)|| \leq 4^{-m}c + 2^{-(2-p)m}k\varepsilon||x||^p, \tag{9}$$

which shows that the limit $g(x) = \lim_{n\to\infty} 4^{-4n}f(2^n x)$ exists for each non-zero x in E_1, since E_2 is a Banach space. By letting $n \to \infty$ in (8), we arrive at the formula (5) with $c = \dfrac{||f(0)||}{3}$ and $k = \dfrac{2}{4 - 2^p}$. To show that g is quadratic, replace x and y by $2^n x$ and $2^n y$, respectively, in (4) and divide by 4^n to get

$$||g_n(x+y) + g_n(x-y) - 2g_n(x) - 2g_n(y)|| \leq 2^{-(2-p)n}\varepsilon(||x||^p + ||y||^p).$$

Taking the limit as $n \to \infty$, we find that g satisfies (1) when x and y are different from zero. We now define $g(x)$ as $\lim_{n\to\infty} 4^{-n}f(2^n x)$ for all x in E_1; it follows that $g(0) = 0$. Thus, (1) holds for $x = y = 0$, when f is replaced by g in (1). When $y = 0$ and $x \neq 0$, we have $g(x+0) + g(x-0) - 2g(x) - 2g(0) = 0$. For

$$x \neq 0 \text{ and } y \neq 0, \quad g(x+y) + g(x-y) - 2g(x) - 2g(y) = 0,$$

and setting $y = x$, gives ys $g(2x) = 4g(x)$ for $x \neq 0$, but this last equation obviously also hods for $x = 0$. With $y = -x \neq 0$, we get $g(0) + g(2x) - 2g(x) - 2g(-x) = 0$ which reduces to $g(-x) = g(x)$, which again is clearly true for all x in E_1. Finally, for $x = 0$ and $y \neq 0$, we have $g(0+y) + g(0-y) - 2g(0) - 2g(y) = 0$. Therefore, $g : E_1 \to E_2$ is quadratic on E_1.

Case 2. $p > 2$. In (4), set $x = y = 0$ to see that $f(0) = 0$. Then replace both x and y by $\frac{x}{2}$ to obtain

$$\|f(x) - 4f(2^{-1}x)\| \leq \left(\frac{\varepsilon}{2}\right) \|x\|^p \cdot 2^{-(p-2)}. \tag{10}$$

Apply the induction hypothesis:

$$\|f(x) - 4^n f(2^{-n}x)\| \leq \left(\frac{\varepsilon}{2}\right) \|x\|^p \sum_{k=1}^{n} 2^{-k(p-2)} \tag{11}$$

for all x in E_1 and all positive integers n. In (11), replace x by $2^{-1}x$ and multiply by 4 to get

$$\|4f(2^{-1}x) - 4^{n+1} f(2^{-(n+1)}x)\| \leq \left(\frac{\varepsilon}{2}\right) \|x\|^p \sum_{k=2}^{n+1} 2^{-k(p-2)}.$$

Combine the last inequality with (10) to show that (11) remains true with n replaced by $n+1$, which completes the induction proof. Summing the series on the right side of (11), we get

$$\|f(x) - 4^n f(2^{-n}x)\| \leq k\varepsilon \|x\|^p, \tag{12}$$

where now $k = \dfrac{2}{2^p - 4}$. Putting $h_n(x) = 4^n f(2^{-n}x)$, multiplying (12) by 4^n and replacing x by $2^{-m}x$, we have

$$\|h_m(x) - h_{m+n}(x)\| = \|4^m f(2^{-m}x) - 4^{m+n} f(2^{-(m+n)}x)\| \leq 2^{-m(p-2)} k\varepsilon \|x\|^p. \tag{13}$$

This shows that $\{h_m(x)\}$ is a Cauchy sequence and this there exists $g : E_1 \to E_2$ with $g(x) = \lim_{n\to\infty} h_n(x)$ for all x in E_1. The proof that g is quadratic is similar to that in Case 1, except that here we replace x and y in (4) by $2^{-n}x$ and $2^{-n}y$ and multiply the result by 4^n.

To prove the uniqueness of the quadratic function g subject to (5), let us assume on the contrary that there is another quadratic function $h : E_1 \to E_2$ satisfying

(5), and a point y in E_1 with $a = ||g(y) - h(y)|| > 0$. Every quadratic function has a unique representation in terms of a symmetric, biadditive function. Thus, $g(x) = B(x, x)$, where $B : E_1 \times E_1 \to E_2$ is symmetric and biadditive. It follows that $g(rx) = r^2 g(x)$ for all rational numbers r. Similarly, $h(rx) = r^2 h(x)$ for rational r. Since both g and h satisfy (5),

$$||g(x)-h(x)|| = ||g(x)-f(x)+f(x)-h(x)|| \leq 2c+2k\varepsilon|| \leq 2c+2k\varepsilon||x||^p \text{ for all } x \text{ in } E_1.$$

In particular, we have for $r > 0$ that

$$r^2 a = ||g(ry) - h(ry)|| \leq 2c + 2k\varepsilon r^p ||y||^p. \tag{14}$$

In case 1, where $p < 2$, we have by (14) that

$$a \leq \frac{2c}{r^2} + \frac{2k\varepsilon||y||^p}{r^{2-p}} \text{ for rational } r > 0, \text{ so } a = 0.$$

In case 2, where $p > 2$, $c = 0$. We set $r = \dfrac{1}{s}$ in (14), so that

$$\frac{a}{s^2} = \left\| g\left(\frac{y}{s}\right) - h\left(\frac{y}{s}\right) \right\| \leq \frac{2k\varepsilon||y||^p}{s^p}.$$

Hence, $a \leq \frac{2k\varepsilon||y||^p}{s^{p-2}}$ for all $s > 0$, and again we see that $a = 0$. The proof that $g(tx) = t^2 g(x)$ for all real t will be deferred until after the:

Corollary 3. *If in theorem 2 the function f is continuous everywhere in E_1, then g is also continuous for all $x \neq 0$ in E_1. When $p > 0$, thos restriction is unecessary.*

Proof of Corollary. In case $p < 0$, we must treat $x = 0$ as a special situation since the right members of inequalities (4), (5) and (9) may become infinite as $x \to 0$. Suppose that f is continuous for all x in E_1 and that x_0 in E_1 is not zero. Set $s = \dfrac{||x_0||}{2}$. For x in the open ball $B(x_0, s) = \{x \in E_1 : ||x - x_0|| < s\}$, we have $s < ||x|| < 3s$. In inequality (9), let $n \to \infty$, so that $||f(x) - g_m(x)|| < 4^{-m}c + 2^{(2-p)m}k\varepsilon||x||^p$. For x in $B(x_0, s)$, we have $s^p > ||x||^p < (3s)^p$ when $p < 0$, while the inequalities are reversed when $p > 0$. Consequently, $g_m(x)$ converges uniformly to $g(x)$ in $B(x_0, s)$ as $m \to \infty$, and, since each function g_m is continuous in $B(x_0, s)$, it follows that the limit g is also continuous in $B(x_0, s)$. Thus, the quadratic function g

is continuous at each point $x_0 \neq 0$ in E_1. Clearly, the restriction $x_0 \neq 0$ is not needed when $p > 0$.

Proof of Theorem 2 (concluded). We have seen that $g)rx) = r^2 g(x)$ when r is a rational number. To prove that g is homogeneous of degree two for all real numbers as well, it is sufficient to prove that the map $t \to g(tx)$ is continuous in t for fixed x in E_1. By hypothesis, the map $t \to f(tx)$ is continuous in t for fixed x in E_1. Apply corollary 3 to the case where $E_1 = R$ to show that in case $x \neq 0$ and $t_0 \geq 0$ then $t \to g(tx)$ is continuous at $t = t_0$. Thus, $g(t_0 x) = t_0^2 g(x)$ for all $t_0 \neq 0$, $x \neq 0$. But this equality is obvious both $x = 0$ and $t_0 = 0$. Therefore, $g(tx) = t^2 g(x)$ for all real t and x in E_1.

Remark. S. Czerwik [4] proved that $g(tx) = t^2 g(x)$ under the weaker assumption that $tf(tx)$ was Borel measurable in t for each fixed x.

The exclusion of the case $p = 2$ in Theorem 2 is necessary as shown by the counterexample to be cited below, due to Czerwik [4; pp.63-64]. It is a modification of the example of Gajda [8].

Let $\phi : R \to R$ be defined by

$$\phi(x) = \begin{cases} \mu x^2 & \text{for } |x| < 1 \\ \mu & \text{for } |x| ge 1, \end{cases}$$

where $\mu > 0$, and put, for all x in R,

$$f(x) = \sum_{n=0}^{\infty} 4^{-n} \phi(2^n x).$$

Then f is bounded by $\dfrac{4\mu}{3}$ on R and satisfies the condition

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq 32\mu(x^2 + y^2) \tag{A}$$

for all x and y in R, as it will now be shown. If $x^2 + y^2$ is either 0 or $\geq \dfrac{1}{4}$, then the left side of (A) is less than 8μ, so (A) is true. Now suppose that $0 < x^2 + y^2 < \frac{1}{4}$. Then there exists a positive integer k such that

$$4^{-k-1} \leq x^2 + y^2 < 4^{-k}, \tag{B}$$

so that $4^{k-1}x^2 < 4^{-1}$, $4^{k-1}y^2 < 4^{-1}$ and $2^{k-1}x, 2^{k-1}y, 2^{k-1}(x \pm y)$ all belong to the interval $(-1,1)$. Hence, for $n = 0, 1, \ldots, k-1$, $2^n x, 2^n y, 2^n(x \pm y)$ all belong to this same interval and

$$\phi(2^n(x+y)) + \phi(2^n(x-y)) - 2\phi(2^n x) - 2\phi(2^n y) = 0 \text{ for } n = 0, 1, \ldots, k-1.$$

From the definition of f and from the inequality (B), we have

$$\begin{aligned} &|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \\ &\leq \sum_{n=0}^{\infty} 4^{-n} |\phi(2^n(x+y)) + \phi(2^n(x-y)) - 2\phi(2^n x) - 2\phi(2^n y)| \\ &\leq \sum_{n=k}^{\infty} 6\mu 4^{-n} = (2\mu)4^{1-k} < 32(x^2+y^2)\mu, \end{aligned}$$

thus f satisfies (A).

Suppose now that there exists a quadratic function $g : R \to R$ and a constant $\beta > 0$ such that $|f(x) - g(x)| < \beta x^2$ for all x in R. Since f is bounded for all x, it follows that g is bounded on any open interval containing the origin, so that g has the form $g(x) = \eta x^2$ for x in R, where η os a constant (see, e.g. S. Kurepa [11]). Thus, we have

$$|f(x)| \leq (\beta + |\eta|)x^2, \text{ for } x \text{ in } R. \tag{C}$$

Let k be a positive integer with $k\mu > \beta + |\eta|$. If $x \in (0, 2^{1-k})$, then $2^n x \in (0,1)$ for $n \leq k-1$, and, for $x \in R$, we have

$$f(x) = \sum_{n=0}^{\infty} 4^{-n}\phi(2^n x) \geq \sum_{n=0}^{k-1} \mu 4^{-n}(2^n x)^2 = k\mu x^2 > (\beta + |\eta|)x^2,$$

which contradicts (C). □

Remark. Theorem 2 can be generalized without difficulty to the situation where the right side of inequality (4) is replaced by $\varepsilon H(\|x\|, \|y\|)$, in which $H : R_+ \times R_+ \to R_+$ is positive homogeneous of degree $p \neq 2$, i.e. $H(cs, ct) = c^p H(s,t)$ when c, s and t are all positive. This statement is an analog for approximately quadratic functions of a theorem of Rassias and Šemrl [13] for approximately additive functions.

The relationship between a quadratic function and its biadditive polar is basic in the methods used by F. Skof and S. Terracini [19] in their study of the stability of the quadratic functional equation for functions defined on restricted domains in R with values in a Banach space E. They proved the following stability theorem for symmetric biadditive functions based on results from Skof [17].

Theorem 3. *Denote the set $[0,r) \times [0,r)$ in R^2 by S, where $r > 0$, and let E be a Banach space. Suppose that $\phi : S \to E$ is symmetric and δ-biadditive, i.e. $\phi(x,y) = \varphi(y,x)$ for (x,y) in E and $\|\phi(x,t+u)-\phi(x,t)-\phi(x,u)\| < \delta$ for all x in $[0,r)$ and t,u, and $t+u$ in $[0,r)$ and some $\delta > 0$. Then there exists at least one function $F : S \to E$ which is symmetric, biadditive and such that $\|\phi(x,y) - F(x,y)\| < 9\delta$ for (x,y) in S.*

Proof. We will refer to the proof of the theorem of Skof [17]. By hypothesis, for each y in $[0,r)$, the function of $x : \phi_y(x) = \phi(x,y)$ is δ-additive on the set $T(r) = \{(x',x'') \in R^2 : \ x', x'' \in [0,r), \ x'+x'' \in [0,r)\}$. Following the proof of extension II above, we define the function $\phi_y^* : R_+ \to E$ for fixed y by $\phi_y^*(x) = n\phi_y\left(\frac{r}{2}\right) + \phi_y(\mu)$ for $x = \frac{nr}{2} + \mu$, $n = 1,2,\ldots$ and $0 \le \mu < \frac{r}{2}$. Thus, we have for x in $[0,r)$ that

$$\|\phi_y(x) - \phi_y^*(x)\| < \delta. \tag{15}$$

This function is extended to R by putting $\phi_y^*(x) = -\phi_y^*(-x)$ when $x < 0$. It follows that, for $y \in [0,r)$, $\phi_y^*(x)$ is 2 δ-additive on R^2 and hence, by theorem 1.1 above, there is a unique additive function

$$G_y^*(x) = \lim_{n\to\infty} 2^{-n}\phi_y^*(2^n x) \tag{16}$$

for all x in R and y in $[0,r)$ such that

$$\|\phi_y^*(x) - G_y^*(x)\| \le 2\delta, \quad x \in R, \ y \in [0,r). \tag{17}$$

Now define $G(x,y) = G_y^*(x)$ when $(x,y) \in S$. $G(x,y)$ is additive in the first variable. We will show that it is 2 δ-additive in its second variable on the set $T(r) = \{(y,z) \in R^2 : \ y,z \in [0,r), \ y+z \in [0,r)\}$. Indeed, fix $x \in [0,r)$ and $y,z \in [0,r)$ with $y+z \in [0,r)$. Put $2^n x = k_n\frac{r}{2} + \mu_n$, where $\mu_n \in \left[0,\frac{r}{2}\right)$, and k_n is a positive integer, so that $k_n = \left(\frac{2}{r}\right)(2^n x - \mu_n)$. Then we have

$$G(x,y+z) - G(x,y) - G(x,z) = \lim_{n\to\infty} 2^{-n}(\phi_{y+z}^*(2^n x) - \phi_y^*(2^n x) - \phi_z^*(2^n x)) =$$

$$= \lim_{n\to\infty} 2^{-n}(\phi_{y+z}(\mu_n)-\phi_y(\mu_n)-\phi_z(\mu_n)) + \lim_{n\to\infty} 2^{-n}k_n\left(\phi_{y+z}\left(\frac{r}{2}\right)-\phi_y\left(\frac{r}{2}\right)-\phi_z\left(\frac{r}{2}\right)\right) =$$
$$= 2\left(\frac{x}{r}\right)\left(\phi_{y+z}\left(\frac{r}{2}\right)-\phi_y\left(\frac{r}{2}\right)-\phi_z\left(\frac{r}{2}\right)\right).$$

Hence,

$$\|G(x,y+z)-G(x,y)-G(x,z)\| < 2\delta \text{ when } x\in[0,r) \text{ and } y,z\in T(r). \tag{18}$$

Next we extend G to a function $H : [0,r)\times R \to E$. With x in $[0,r)$ and $y\in R_+$, let $y = \frac{nr}{2}+\mu$, where $\mu\in\left[0,\frac{r}{2}\right)$ and $n=1,2,\ldots$, and define H by

$$H(x,y) = nG\left(x,\frac{r}{2}\right)+G(x,\mu) \tag{19}$$

and for $y<0$ put

$$H(x,y) = -H(x,-y). \tag{20}$$

By extension II, we find that H is 4 δ-additive in the second variable and

$$\|H(x,y)-G(x,y)\| < 2\delta \text{ for } (x,y) \text{ in } S. \tag{21}$$

Also, for each R, H is additive in the first variable on the set

$$T(r) = \{x',x'')\in R^2 :\ x',x''\in[0,r),\ x'+x''\in[0,r)\},$$

since G has this property.

For each fixed x in $[0,r)$, it follows by Hyers's stability theorem [9] (see also [12]-[16]) that the function

$$F(x,y) = \lim_{n\to\infty} 2^{-n}H(x,2^n y) \tag{22}$$

is additive in y and satisfies

$$\|F(x,y)-H(x,y)\| \le 4\delta,\quad (x,y)\in[0,r)\times R. \tag{23}$$

F is also additive in x on $T(r)$ by (22) since H is. By (15), (17), (21) and (23), we obtain the required inequality $\|\phi(x,y)-F(x,y)\| < 9\delta$ when $(x,y)\in S$.

In order to prove the symmetry of F, observe that since φ is symmetric, $\|F(x,y)-F(y,x)\| < 18\delta$. For $y=0$, we have $F(x,0)=0=(F(0,x)$ for $x\in[0,r)$. For a given $y\in(0,r)$ set, $h_y(x) = F(y,x)-F(x,y)$, $x\in[0,r)$. Noe $h_y(x)$ is additive on $T(r)$ and bounded, so it is the restriction to $[0,r)$ of a function of x of the form $h_y(x) = a(y)x$. But $h_y(y) = 0$ for all $y \in [0,r)$, so $a(y) \equiv 0$. □

In proving the next theorems of Skof and Terracini [19], we will make use of the following sets: $K(r) = \{(x,y) \in R^2 : 0 \le y \le x,\ x+y < r\}$ and $D(r) = \{(x,y) \in R^2 : |x+y| < r,\ |x-y| < r\}$.

Theorem 4. *Let $f : [0,r) \to E$ (a Banach space) be δ-quadratic on $K(r)$ for some $\delta > 0$. Then there exists a quadratic function $q : R \to E$ such that*

$$\|f(x) - q(x)\| < \frac{79\delta}{2} \text{ for } x \in [0,r) \tag{24}$$

Proof. Since f is δ-quadratic on $K(r)$,

$$\|f(0)\| < \frac{\delta}{2}, \quad \|f(2x) + f(0) - 4f(x)\| < \delta,$$

and thus

$$\|f(2x) - 4f(x)\| < \frac{3\delta}{2} \text{ for } x \in \left[0, \frac{r}{2}\right). \tag{25}$$

Extend the function f to the interval $(-r,0)$ by defining the extension φ as $\phi(x) = f(x)$ for $x \in [0,r)$ and $\phi(x) = f(-x)$ for $x \in (-r,0)$. We will show that φ is δ-quadratic on $D(r)$. For brevity, put $\mu(x,y) = \|\phi(x+y) + \phi(x-y) - 2\phi(x) - 2\phi(y)\|$, so that $\mu(x,y) < \delta$ on $K(r)$ by hypothesis. On the set $K_1(r) = \{(x,y) \in R^2 : 0 \le x \le y,\ y+x < r\}$, we have $\mu(x,y) = \|f(x+y) + f(-x+y) - 2f(x) - 2f(y)\| < \delta$ since $(y,x) \in K(r)$. On $K_2(r) = \{(x,y) \in R^2 : x < 0,\ y \ge 0,\ y-x < r\}$, we have $\mu(x,y) = \mu(-x,y) < \delta$ since $(-x,y) \in K_1(r) \cup K(r)$ and φ is even. Finally, if $(x,y) \in D(r)$ with $y < 0$, then $(x,-y) \in K(r) \cup K_1(r) \cup K_2(r)$, so again $\mu(x,y) < \delta$, as was asserted.

Next, define the auxiliary function $h : D(r) \to E$ by putting $4h(x,y) = \phi(x+y) - \phi(x-y)$. Clearly, $h(x,y) = h(y,x)$ for all $(x,y) \in D(r)$. When $y \in \left[0, \frac{r}{2}\right)$, h is δ-additive with respect to x on $T\left(\frac{r}{2}\right) = \left\{(u,v) \in R^2 : u,v \in \left[0, \frac{r}{2}\right),\ u+v \in \left[0, \frac{r}{2}\right)\right\}$, and also in y by the interchange of x and y. From the definition of h, it follows that

$$4(h(u+v,y) - h(u,y) - h(v,y)) =$$

$$= [\phi(u+v+y)+\phi(u-v-y)-2\phi(u)-2\phi(v+y)]-[\phi(u-y-v)+\phi(u-y+v)-2\phi(u-y)-2\phi(v)]-$$

$$-[\phi(u+y)+\phi(u-y)-2\phi(u)-2\phi(y)]+[\phi(v+y)+\phi(v-y)-2\phi(v)-2\phi(y)]$$

when $(u, v+y), (u-y, v), (u, y)$ and (v, y) are points of the set $D(r)$ where φ is δ-quadratic. Hence, $\|h(u+v, y) - h(u, y) - h(v, y)\| < \delta$.

Thus, h satisfies the hypothesis of Theorem 3, with $\frac{r}{2}$ in place of r, therefore exists a function $F : \left[0, \frac{r}{2}\right) \times \left[, \frac{r}{2}\right) \to E$ which is symmetric and biadditive such that

$$\|h(x, y) - F(x, y)\| < 9\delta \text{ when } (x, y) \in \left[0, \frac{r}{2}\right) \times \left[, \frac{r}{2}\right). \tag{26}$$

For $x \in \left[0, \frac{r}{2}\right)$, we have

$$\|h(x,x)-f(x)\| = 4^{-1}\|f(2x)-f(0)-4f(x)\| \leq 4^{-1}\|f(2x)+f(0)-4f(x)\|+4^{-1}\|2f(0)\| < \frac{\delta}{2}$$

so that

$$\|h(x, x) - f(x)\| < \frac{delta}{2} \text{ for } x \in \left[0, \frac{r}{2}\right). \tag{27}$$

The function $F(x, x)$ is quadratic on $K\left(\frac{r}{2}\right)$. According to a theorem of Skof [18], it may be extended to a function $q : R \to E$ which is quadratic and such that $q(x) = F(x, x)$ when $x \in \left[0, \frac{r}{2}\right)$. Thus, when $x \in \left[0, \frac{r}{2}\right)$, we have $\|f(x) - F(x, x)\| \leq \|f(x) - h(x, x)\| + \|h(x, x) - F(x, x)\|$, so, by (26) and (27),

$$\|f(x) - q(x)\| < \left(\frac{19}{2}\right)\delta, \quad x \in \left[0, \frac{r}{2}\right). \tag{28}$$

Now for $x \in \left[\frac{r}{2}, r\right]$, i.e. $\frac{x}{2} \in \left[\frac{r}{4}, \frac{r}{2}\right]$; taking account of (25) and (28), we have

$$\|f(x) - q(x)\| \leq \left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| + \left\|4f\left(\frac{x}{2}\right) - 4q\left(\frac{x}{2}\right)\right\| < \frac{79\delta}{2},$$

and the theorem is proved. □

With the help of theorem 4, Skof and Terracini obtained the following

Theorem 5. *Denote the set* $\{(x, y) \in R^2 : |x+y| < r, |x-y| < r\}$ *by* $D(r)$. *Let* E *be a Banach space and suppose that* $f : (-r, r) \to E$ *is* δ*-quadratic on* $D(r)$ *for some* $\delta > 0$. *Then there exists a function* $q : R \to E$ *which is quadratic and satisfies the inequality* $\|f(x) - q(x)\| < \frac{81\delta}{2}$ *when* $-r < x < r$.

Proof. Observe that for a given $y \in (-r, r)$, with $(x, y) \in D(r)$ and $(x, -y) \in D(r)$ we have $\|f(y) - f(-y)\| = 2^{-1}\|(f(x+y) + f(x-y) - 2f(x) - 2f(y)) - (f(x-y) + f(x+y) - 2f(x) - 2f(-y))\| < \delta$. Denote by f_0 the restriction of f to $[0, r)$ and apply theorem 4 to obtain $\|f_0(x) - q(x)\| < k\delta$, where $k = \frac{79}{2}$. For $x \in (-r, 0)$, if f is an even function, we have

$$\|f(x) - q(x)\| \leq \|f(x) - f(-x)\| + \|f(-x) - q(-x)\| < (1+k)\delta = \frac{81\delta}{2}. \quad \square$$

The next topic on the stability of a certain type of conditional Cauchy equation is intriguing because it turns out to be q srot of hybrid between approximately additive and approximately quadratic mappings.

Functionals which are approximately additive on A-orthogonal vectors

Given a complex Hilbert space X, let $A : X \to X$ be a bounded selfadjoint linear operator whose range AX has dimension > 2. A functional $\phi : X \to C$ is said to be **additive on A-orthogonal pairs** if x, y in X with $(Ax, y) = 0$ implies that $\phi(x + y) = \phi(x) + \phi(y)$. Such functionals were studied by F. Vajzovic [23]. On page 80 of this reference, he proved the following:

Theorem 6. *If $\phi : X \to C$ is continuous and additive on A-orthogonal pairs, then there exists a unique scalar β and unique vectors u and v in X such that*

$$\phi(x) = (x, u) + (v, x) + \beta(Ax, x).$$

Here and in the remainder of this section, the inner product of the Hilbert space X will be denoted by means of parentheses.

H. Drljevic and Z. Mavar [6] considered a stability problem for such functionals as follows. Using the concept of approximate additivity of Th.M. Rassias [12], these authors defined a functional $\phi : X \to C$ to be **approximately additive on A0orthogonal pairs** if there exist constants $\theta > 0$ and p in the interval $[0, 1)$ such that

$$|\phi(x+y) - \phi(x) - \phi(y)| \leq \theta \left[|(Ax, x)|^{\frac{p}{2}} + |(Ay, y)|^{\frac{p}{2}} \right] \tag{29}$$

for all x, y in X for which $(Ax, y) = 0$. Their main theorem is:

Theorem 7. *Let X be a complex Hilbert space and $A : X \to X$ be a bounded linear selfadjoint operator whose range AX has dimension > 2. Suppose that $\phi : X \to C$ is approximately additive on orthogonal pairs, so that ϕ satisfies (29) for some $\theta > 0$ and some p in $[0, 1)$, and also that $\phi(tx)$ is continuous in the scalar t for each fixed x and all t in C.*

Then there exists a unique continuous functional $\phi : X \to C$ which is additive on A-orthogonal pairs and satisfies the inequality

$$|\phi(x) - \psi(x)| \le \varepsilon(p, \theta)|(Ax, x|^{\frac{p}{2}}$$

for all x in X, where $\varepsilon(p, \theta)$ is a constant.

Moreover, by Theorem 6, ψ is of the form

$$\psi(x) = (x, u) + (v, x) + \beta(Ax, x),$$

where the vectors u, v and the scalar β are constants.

Proof. We decompose ϕ into its odd and even parts, putting:

$$G(x) = \frac{\phi(x) - \phi(-x)}{2} \text{ and } H(x) = \frac{\phi(x) + \phi(-x)}{2}.$$

It is easy to see by use of (29) that both G and H are approximately additive on A-orthogonal pairs with the same constants θ and p as those that appear in (29).

Properties of the odd functional G

As just stated, we have

$$|G(x + y) - G(x) - G(y)| \le \theta \left[|(Ax, x)|^{\frac{p}{2}} + |(Ay, y)|^{\frac{p}{2}}\right] \tag{30}$$

for all x, y in which satisfy $(Ax, y) = 0$.

In the trivial case in which $(Ax, x) = 0$ for some x in X, we have $|G(2x) - 2G(x)| = 0$, so that $\dfrac{G(2x)}{2} = G(x)$ and it follows that, for all $n \in N$,

$$\frac{G(2^n x)}{2^n} = G(x) \text{ when } (Ax, x) = 0. \tag{31}$$

Lemma 8. *Let $(Ax, x) \ne 0$ for some fixed x. Then there exists a y in X such that $(Ay, y) \ne 0$ and $(Ax, y) = 0$.*

Proof. Suppose that, contrary to the Lemma, $(Ay, y) = 0$ for each y in the hyperplane $Y = \{y \in X : (Ax, y) = 0\}$. Then, for each pair y_1, y_2 in Y, we have $(A(y_1 + y_2), y_1 + y_2) = 0$, so that $(Ay_1, y_2) + (Ay_2, y_1) = 0$. Now replace y_2 by iy_2 in the last equality to get $-i(Ay_1, y_2)+i(Ay_2, y_1) = 0$, that is $-(Ay_1, y_2)+(Ay_2, y_1) = 0$, to see that $(Ay_2, y_1) = 0$ for each pair y_1, y_2 in Y. Since $(Ax, x) \neq 0$, it follows that x does not belong to Y. Thus, every z in X may be written in the form $z = \gamma x + y$ for some complex number γ and some y in Y. So for all z in X, $Az = \gamma Ax + Ay$ and $(Az, y) = (Ax, y) + (Ay, y) = 0$. Therefore, $Az = \gamma' Ax$, which is contrary to the hypothesis that $\dim(Ax) > 2$. □

Remark. Since A is selfadjoint it follows that (Ax, x) and (Ay, y) are always real. If x and y are those of Lemma 8, then, by multiplying y by an appropriate positive real number if necessary, we may assume that $(Ay, y)\pm(Ax, x)$, with $(Ax, y) = 0$.

Lemma 9. *Assume that* $(Ax, x) \neq 0$, $(Ax, y) = 0$ *and* $(Ay, y) = (Ax, x)$. *Then the limit* $\hat{G}(x) = \lim_{n\to\infty} \frac{G(2^n x)}{2^n}$ *exists.*

Proof. From the assumptions of the lemma, we see that $(A(x+y), x-y) = 0$, i.e. $x + y$ and $x - y$ form an A-orthogonal pair. Hence, by (30),

$$|G(x+y+x-y)-G(x+y)-G(x-y)| \leq \theta \left[|(A(x + y), x + y)|^{\frac{p}{2}} + |(A(x - y), x - y)|^{\frac{p}{2}}\right],$$

that is

$$|G(2x) - G(x + y) - G(x - y)| \leq 2\theta \cdot 2^{\frac{p}{2}}|(Ax, x)|^{\frac{p}{2}}.$$

From this inequality and (30), using the oddness of G, we obtain

$$|G(2x) - 2G(x)| \leq |G(2x) - G(x + y) - G(x - y)| + |G(x + y) - G(x) - G(y)|+$$

$$+|G(x - y) - G(x) - G(-y)| \leq$$

$$\leq 2\theta \left(2^{\frac{p}{2}}|(Ax, x)|^{\frac{p}{2}} + |(Ax, x)|^{\frac{p}{2}} + |(Ax, x)|^{\frac{p}{2}}\right) = 2\theta \left(2 + 2^{\frac{p}{2}}\right) |(Ax, x)|^{\frac{p}{2}},$$

or

$$\left|\frac{G(2x)}{2} - G(x)\right| \leq \theta \left(2 + 2^{\frac{p}{2}}\right) |(Ax, x)|^{\frac{p}{2}}. \tag{32}$$

By mathematical induction, as in the proof of the theorem of Th.M. Rassias [12], we find that

$$|2^{-n}G(2^n x) - G(x)| \leq \theta \left(2 + 2^{\frac{p}{2}}\right) |(Ax, x)|^{\frac{p}{2}} \sum_{k=0}^{n-1} 2^{k(p-1)},$$

or by summing the series indicated,

$$|2^{-n}G(2^n x) - G(x)| \leq \theta |(Ax, x)|^{\frac{p}{2}} \frac{2\left(2 + 2^{\frac{p}{2}}\right)}{2 - 2^p}. \tag{33}$$

In order to show that $\left\{\dfrac{G(2^n x)}{2^n}\right\}$ is a Cauchy sequence, let m and n be integer with $m > n > 0$. Then

$$\left|\frac{G(2^m x)}{2^m} - \frac{G(2^n x)}{2^n}\right| = 2^{-n} \left|\frac{G(2^m x)}{2^{m-n}} - G(2^n x)\right|.$$

Use inequality (33) with x replaced by $2^n x$ and n by $m - n$ to find that

$$\left|\frac{G(2^m x)}{2^m} - \frac{G(2^n x)}{2^n}\right| \leq \theta 2^{-n} |(A(2^n x), 2^n x)|^{\frac{p}{2}} \frac{2\left(2 + 2^{\frac{p}{2}}\right)}{2 - 2^p} =$$

$$= 2^{n(p-1)} \theta |(Ax, x)|^{\frac{p}{2}} \frac{2\left(2 + 2^{\frac{p}{2}}\right)}{2 - 2^p}.$$

Since $p - 1 < 0$, the above sequence is a Cauchy sequence and so converges for each x in X. We put $\hat{G}(x) = \lim\limits_{n\to\infty} \dfrac{G(2^n x)}{2^n}$. □

Lemma 10. *If $(Ax, x) \neq 0$, $(Ax, y) = 0$ and $(Ay, y) = -(Ax, x)$, then the limit $\lim\limits_{n\to\infty} 2^{-n}G(2^n x)$ exists.*

Proof. From the hypothesis it follows that $A(x \pm y, x \pm y) = 0$ when the $\pm$ signs are consistent. Hence, by (31),

$$\frac{G[2^n(x \pm y)]}{2^n} = G(x \pm y), \quad n \in N.$$

In the inequality (30), replace x by $2^n x$, y by $2^n y$ and divide the result by 2^n to get

$$\left|\frac{G[2^n(x + y)]}{2^n} - \frac{G(2^n x)}{2^n} - \frac{G(2^n y)}{2^n}\right| \leq 2\theta \cdot 2^{n(p-1)} |(Ax, x)|^{\frac{p}{2}}.$$

In this last inequality, replace y by $-y$ to obtain

$$\left|\frac{G[2^n(x - y)]}{2^n} - \frac{G(2^n x)}{2^n} + \frac{G(2^n y)}{2^n}\right| \leq 2\theta \cdot 2^{n(p-1)} |(Ax, x)|^{\frac{p}{2}},$$

$$\left| G(x+y) - \frac{G(2^n x)}{2^n} - \frac{G(2^n y)}{2^n} \right| \le 2\theta \cdot 2^{n(p-1)} |(Ax,x)|^{\frac{p}{2}},$$

and

$$\left| G(x-y) - \frac{G(2^n x)}{2^n} + \frac{G(2^n y)}{2^n} \right| \le 2 \cdot 2^{n(p-1)} \theta |(Ax,x)|^{\frac{p}{2}}.$$

From the last two inequalities, we find that

$$\left| \frac{[G(x+y) + G(x-y)]}{2} - \frac{G(2^n x)}{2^n} \right| \le 2 \cdot 2^{n(p-1)} \theta |(Ax,x)|^{\frac{p}{2}}.$$

Hence, the limit

$$\lim_{n\to\infty} \frac{G(2^n x)}{2^n} = \frac{G(x+y) + G(x-y)}{2} \text{ exists.} \quad \square \tag{34}$$

From (31), and Lemmas 9 and 10, it follows that the limit $\hat{G}(x) = \lim_{n\to\infty} \frac{G(2^n x)}{2^n}$ exists for each x in X.

Lemma 11. *$\hat{G}$ is additive on A-orthogonal pairs.*

Proof. Assume that x, y in X satisfy $(Ax, y) = 0$. Using (30), we find that

$$\left| \frac{G[2^n(x+y)]}{2^n} - \frac{G(2^n x)}{2^n} - \frac{G(2^n y)}{2^n} \right| \le 2^{n(p-1)} \theta \left[|(Ax,x)|^{\frac{p}{2}} + |(Ay,y)|^{\frac{p}{2}} \right].$$

Since $p < 1$, the right member of this inequality approaches 0 as $n \to \infty$, so that

$$\hat{G}(x+y) - \hat{G}(x) - \hat{G}(y) = 0. \quad \square$$

Lemma 12. *The functional $\hat{G} : X \to C$ satisfies $\hat{G}(ax) = a\hat{G}(x)$ for each real a and each x in X.*

Proof. We first demonstrate that

$$\hat{G}(ax + bx) = \hat{G}(ax) + \hat{G}(bx) \text{ for all } a, b \text{ in } R \text{ and all } x \text{ in } X. \tag{35}$$

Case 0. Note that, if, for some x, we have $(Ax, x) = 0$, then x, x is an orthogonal pair, so by Lemma 11 we have $G(ax + bx) = G(ax) + G(bx)$.

Assume that $(Ax, x) \neq 0$. Then we know by previous results that there exists y in X such that $(Ay, y) \neq 0$ and $(Ax, y) = 0$ and that we may assume that $(Ay, y) = \pm(Ax, x)$.

Case 1. Let $(Ay, y) \neq 0$, $(Ax, y) = 0$ and $(Ay, y) = (Ax, x)$.

Then $(A(x+y), x-y) = 0$, so that $x+y, x-y$ as well as x, y are orthogonal pairs. Hence, for real numbers a and b, we have

$$\hat{G}[a(x+y)+b(x-y)] = \hat{G}[a(x+y)] + \hat{G}[b(x-y)] = \hat{G}(ax) + \hat{G}(ay) + \hat{G}(bx) - \hat{G}(by).$$

Moreover,

$$\hat{G}[a(x+y)+b(x-y)] = \hat{G}[(a+b)x + (a-b)y] = \hat{G}[(a+b)x] + \hat{G}[(a-b)y].$$

It follows that

$$\hat{G}[(a+b)x] + \hat{G}[(a-b)y] = \hat{G}(ax) + \hat{G}(bx) + \hat{G}(ay) - \hat{G}(by). \tag{36}$$

Now interchange a and b to obtain

$$G[(a+b)x] - G[(a-b)y] = G(ax) + G(bx) + G(by) - G(ay). \tag{37}$$

By adding (36) and (37) and then dividing the result by 2, we have $G(ax + bx) = G(ax) + G(bx)$ in Case 1.

Case 2. Assume that $(Ay, y) \neq 0$, $(Ax, y) = 0$ and $(Ay, y) = -(Ax, x)$.

Then, as we have seen before, $(A(x \pm y), x \pm y) = 0$, so that $x+y, x+y, x-y, x-y$ as well as x, y are orthogonal pairs. Thus, we have

$$\hat{G}[a(x+y)+b(x+y)] = \hat{G}[a(x+y)] + \hat{G}[b(x+y)] = \hat{G}(ax) + \hat{G}(bx) + \hat{G}(ay) + \hat{G}(by),$$

$$\hat{G}[a(x+y)+b(x+y)] = \hat{G}[(a+b)x + (a+b)y] = \hat{G}[(a+b)x] + \hat{G}[(a+b)y],$$

so that

$$\hat{G}[(a+b)x] + \hat{G}[(a+b)y] = \hat{G}(ax) + \hat{G}(bx) + \hat{G}(ay) + \hat{G}(by). \tag{38}$$

In (38), replace y by $-y$ to get

$$\hat{G}[(a+b)x] - \hat{G}[(a+b)y] = \hat{G}(ax) + \hat{G}(bx) - \hat{G}(ay) - \hat{G}(by), \tag{39}$$

where we have used the fact that G is odd. From (38) and (39), it follows that $\hat{G}(ax+bx) = \hat{G}(ax) + \hat{G}(bx)$ in Case 2. Thus, this equality holds for all $x \in X$ when a and b are real numbers.

In order to complete the proof of Lemma 12, let a mapping $\Phi_x(t) = \hat{G}(tx)$, t in R, be defined from R into C. By what has just been proved, we have

$$\Phi_x(a+b) = \Phi_x(a) + \Phi_x(b)$$

for all real a and b. Put $\Phi_{xn}(t) = \dfrac{G(2^n tx)}{2^n}$, so that

$$\Phi_x(t) = \lim_{n\to\infty} \Phi_{xn}(t).$$

Each of the functions $\Phi_{xn}(t)$ is continuous in t for all t by a hypothesis of Theorem 7. Hence, $\Phi_x(t) : R \to C$ is measurable in t since it is a limit of continuous functions. Since $\Phi_x(t)$ is both additive and measurable in t, it follows that $\Phi_x(a) = a\Phi_x(1)$ for a in R and each x in X. That is, $\hat{G}(ax) = a\hat{G}(x)$ for each a in R and each x in X. □

As to the estimate of the difference $\hat{G}-G$, we have shown that, when $(Ax,x) = 0$, this difference is zero according to (31). In Case 1, where $(Ax,x) \neq 0$, $(Ax,y) = 0$ and $(Ax,x) = (Ay,y)$, it follows from (33) that

$$|\hat{G}(x) - G(x)| \leq \varepsilon_1(p,\theta) \text{ for all } x \text{ in } X \tag{40}$$

where $\varepsilon_1(p,\theta) = \dfrac{2\theta\left(2+2^{\frac{p}{2}}\right)}{2-2^p}$.

We now turn to Case 2, where $(Ax,x) \neq 0$, $(Ax,y) = 0$ and $(ay,y) = -(Ax,x)$. In this Case, $\hat{G}(x) = \frac{(x+y)-G(x-y)}{2}$, and, by (34) and (30), we obtain

$$\left|\lim_{n\to\infty}\frac{G(2^n x)}{2^n} - G(x)\right| = \left|\frac{G(x+y)+G(x-y)}{2} - G(x)\right| =$$

$$= \left|\frac{G(x+y)-G(x)-G(y)}{2} + \frac{G(x-y)-G(x)-G(-y)}{2}\right| \leq$$

$$\leq 2^{-1}|G(x+y)-G(x)-G(y)| + 2^{-1}|G(x-y)-G(x)-G(-y)| \leq$$

$$\leq \theta|(Ax,x)|^{\frac{p}{2}} + \theta|(Ax,x)|^{\frac{p}{2}} = 2\theta|(Ax,x)|^{\frac{p}{2}}.$$

Therefore, in all three cases, and hence for all x in X, we have

$$|\hat{G}(x) - G(x)| \leq \varepsilon_1(p,\theta)(Ax,x)^{\frac{p}{2}} \tag{41}$$

where $\varepsilon_1(p,\theta) = 2\theta\dfrac{2+2^{\frac{p}{2}}}{2-2^p} > 2\theta$.

Since by Lemma 12 the mapping $\hat{G}$ is homogeneous of degree one with respect to real numbers, Drljevic and Mavar (1982) concluded that this property may be substituted for the continuity of the functional φ in Vajzovic's result cited above as Theorem 6. Thus, they found that the odd function $\hat{G}$, which was shown above to be additive on orthogonal pairs, is of the form

$$\hat{G}(x) = (x, u) + (v, x), \tag{42}$$

and hence is continuous and additive on X.

Properties of the even functional H

Consider the even functional $H(x) = \dfrac{\phi(x) + \phi(-x)}{2}$. By (29), it follows immediately that when $(Ax, y) = 0$, then

$$|H(x+y) - H(x) - H(y)| \leq \theta \left[|(Ax,x)|^{\frac{p}{2}} + |(Ay,y)|^{\frac{p}{2}} \right]. \tag{43}$$

Clearly $\phi(0) = 0$ implies that $H(0) = 0$.

Lemma 13. *For each x in X, the limit $\hat{H}(x) = \lim\limits_{n\to\infty} 4^{-n} H(2^n x)$ exists.*

Proof. Case 0. Suppose that $(Ax, x) = 0$ for some x in X. Then, as before, in considering the functional G, we have $h(2x) = 2H(x)$, and $(Ax, -x) = 0$, so by (43) with y replaced by $-x$ it follows that $h(x) = 0$. Thus, $H(2x) = 0$ and $\dfrac{H(2^n x)}{4^n} = 0$ hold for all $n \in N$, and we have $\hat{H}(x) = 0$ in this case.

Suppose that $(Ax, x) \neq 0$ for some x. By Lemma 8, we know that there exists y in $Y = \{y : \ (Ax, y) = 0\}$ with $(Ay, y) \neq 0$.

Case 1. $(Ax, y) = 0$, $(Ax, x) \neq 0$ and (Ax, x) and (Ay, y) have the same sign. As we have seen previously, we may assume that $(Ax, x) = (Ay, y)$, so that $(A(x+y), x-y) = 0$. This in turn implies that

$$|H(2x) - H(x+y) - H(x-y)| \leq \theta \cdot 2^{1+\frac{p}{2}} |(Ax,x)|^{\frac{p}{2}}. \tag{44}$$

Notice that

$$H(y) - H(x) = \left| H\left(\frac{y+x}{2} + \frac{y-x}{2}\right) - H\left(\frac{y+x}{2}\right) - H\left(\frac{y-x}{2}\right) \right| +$$

$$+ \left| H\left(\frac{y+x}{2}\right) + H\left(\frac{y-x}{2}\right) - H\left(\frac{y+x}{2} + \frac{x-y}{2}\right) \right| \leq$$

$$\leq \left| H\left(\frac{y+x}{2}+\frac{y-x}{2}\right) - H\left(\frac{y+x}{2}\right) - H\left(\frac{y-x}{2}\right)\right| +$$
$$+ \left| H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) - H\left(\frac{x+y}{2}+\frac{x-y}{2}\right)\right| \leq$$
$$\leq 2\theta \left[\left| \left(A\left(\frac{x+y}{2}\right), \frac{x+y}{2}\right)\right|^{\frac{p}{2}} + \left| \left(A\left(\frac{x-y}{2}\right), \frac{x-y}{2}\right)\right|^{\frac{p}{2}} \right],$$

or

$$|H(y) - H(x)| \leq 2^2\theta \cdot 2^{-\frac{p}{2}} |(Ax, x)|^{\frac{p}{2}} = \theta \cdot 2^{2-\frac{p}{2}} |(Ax, x)|^{\frac{p}{2}}. \tag{45}$$

By using (44) and (45), we have

$$|H(2x) - 4H(x)| = |H(2x) - H(x+y) - H(x-y) + H(x+y) + H(x-y) - 4H(x)| \leq$$
$$\leq |H(2x) - H(x+y) - H(x-y)| +$$
$$+ |H(x+y) - H(x) - H(y) + H(x-y) - H(x) - H(-y) + 2(H(y) - H(x))| \leq$$
$$\leq |H(2x) - H(x+y) - H(x-y)| +$$
$$+ |H(x+y) - H(x) - H(y) + H(x-y) - H(x) - H(-y) + 2(H(y) - H(x))| \leq$$
$$\leq \theta \cdot 2^{1+\frac{p}{2}} |(Ax, x)|^{\frac{p}{2}} + 4\theta |(Ax, x)|^{\frac{p}{2}} + \theta \cdot 2^{3-\frac{p}{2}} |(Ax, x)|^{\frac{p}{2}}.$$

Therefore,

$$|H(2x) - 4H(x)| \leq 2\theta \left(2 + 2^{\frac{p}{2}} + 2^{2-\frac{p}{2}}\right) |(Ax, x)|^{\frac{p}{2}}. \tag{46}$$

Let us put

$$\mu(p, \theta) = \theta \left(1 + 2^{\frac{p}{2}-1} + 2^{1-\frac{p}{2}}\right). \tag{47}$$

Divide inequality (46) by 4 and use the abbreviation (47) to obtain

$$\left| \frac{H(2x)}{4} - H(x) \right| \leq \mu(p, \theta) |(Ax, x)|^{\frac{p}{2}}. \tag{48}$$

In (48), replace x by $2x$ and divide the result by 4 to get

$$\left| \frac{H(2^2 x)}{4^2} - \frac{H(2x)}{4} \right| \leq 2^{p-2} \mu(p, \theta) |(Ax, x)|^{\frac{p}{2}}.$$

Combining the last two inequalities, we have

$$\left| \frac{H(2^2 x)}{4^2} - H(x) \right| \leq \mu(p, \theta) |(Ax, x)|^{\frac{p}{2}} (1 + 2^{p-2}).$$

By mathematical induction, we find that

$$\left|\frac{H(2^n x)}{4^n} - H(x)\right| \leq \mu(p,\theta)|(Ax,x)|^{\frac{p}{2}} \sum_{k=0}^{n} 2^{k(p-2)} \text{ for all } n \text{ in } N.$$

By summing the series indicated, we may write

$$\left|\frac{H(2^n x)}{4^n} - H(x)\right| \leq \frac{4\mu(p,\theta)|(Ax,x)|^{\frac{p}{2}}}{4-2^p}. \tag{49}$$

Using a method similar to that used above in the case of the functional G, we find that the sequence $\left\{\frac{H(2^n x)}{4^n}\right\}$ converges in Case 1.

Case 2. $(Ax,y) = 0$, while (Ay,y) and (Ax,x) have opposite signs. Again we put $Y = \{y : \ (Ax,y)\} = 0$. Without loss of generality, we may assume that $(Ax,x) > 0$ and that $(Ay',y') \leq 0$ for each y' in Y. We can also find a y in Y such that $(Ay,y) = -(Ax,x)$. Let P be a projection of the space X onto Y parallel to the vector Ax. Then $Ay = \alpha(y)Ax + PAy$, so that $(PAy,y) = (Ay,y)$. We note that the last equality holds if y is replaced by any y' in Y. Let $Z = \{z : \ z \in Y, \ (PAy,z) = 0\}$. Id $(PAz,z) = 0$ for all z in Z, then since $Z \subset Y$m it follows that $(Az,z) = (PAz,z) = 0$ for all z in Z. Thus, Az is perpendicular to z, so that $PAz = \beta(z)PAy$. Clearly, $x \notin Y$. Also, from $(Ay,y) = -(Ax,x) \neq 0$, it follows that $y \notin Z$. Moreover, x and y are linearly independent. For, iff $\alpha x + \beta y = 0$ for some scalars α and β, then $\alpha Ax + \beta Ay = 0$ and $\alpha PAx + \beta PAy = 0$, that is $\beta PAy = 0$. Hence, $\beta(PAy,y) = \beta(Ay,y) = 0$, so $\beta = 0$. Hence, $\alpha x = 0$ so $\alpha = 0$, since $x \neq 0$.

Thus, $y' \in Y$ can be written in the form $y' = \alpha x + \beta y + z$, with z in Z, and so $PAy' = \alpha PAx + \beta PAy + PAz = \beta PAy + PAz = (\beta + \beta(z))PAy$. It follows that $(PAz,z) = 0$ for all z in Z implies that

$$PAy' = (\beta + \beta(z))PAy \text{ for all } y' \in Y.$$

For each u in X, we write

$$Au = \alpha(u)Ax + PAu \tag{50}$$

Let $u = \alpha_1 x + \beta_1 y + z$. Then $PAu = \alpha_1 PAx + \beta_1 PAy + PAz$. As shown above, $PAz = \beta(z)PAy$, so that $PAu = (\beta_1 + \beta(z))PAy$ for all u in X. Thus, (50) becomes $A(u) = \alpha(u)Ax + (\beta_1 + \beta(z))PAy$ for all u in X, which is a contradiction to the hypothesis that the dimension of $A(X)$ is greater than two. Therefore, there exists a z' in Z such that $(PAz', z') \neq 0$. We may choose z in Z so that $(PAz, z) = (PAy, y)$, or $(Az, z) = (Ay, y)$. Also, $(Ay, z) = 0$, for we have $Ay = \alpha(y)Ax + PAy$ from the definition of P, so that $(Ay, z) = (\alpha(y)Ax, z) + (PAy, z)$. The first term of the right side of the last equality vanishes because $z \in Y$, and the second term vanishes by the definition of Z.

Thus, we have an element y in X with $(Ay, y) \neq 0$ and an element z in X satisfying $(Ay, z) = 0$ and $(Az, z) = (Ay, y)$. So we can use the results of Case 1, replacing x by y and y by z, to conclude that the sequence $\left\{\frac{H(2^n y)}{4^n}\right\}$ converges. On the other hand, since $(Ay, y) = -(Ax, x)$ and $(Ax, y) = 0$ implies that $(A(x \pm y), x \pm y) = 0$, it follows from Case 0 that $\frac{H(2^n(x \pm y))}{2^n} = 0$ for $n \in N$. For the A-orthogonal pair x, y, we have $|H(x + y) - H(x) - H(y)| \leq 2\theta|(Ax, x)|^{\frac{p}{2}}$. Since $2^n x, 2^n y$ is also A-orthogonal pair with $(A(2^n y), 2^n y) = -(A(2^n x, 2^n x))$, we have

$$\left|\frac{H(2^n(x + y))}{4^n} - \frac{H(2^n x)}{4^n} - \frac{H(2^n y)}{4^n}\right| \leq 2\theta \cdot 2^{n(p-2)}|(Ax, x)|^{\frac{p}{2}},$$

or

$$\left|\frac{H(2^n x)}{4^n} + \frac{H(2^n y)}{4^n}\right| \leq 2\theta \cdot 2^{n(p-2)}|(Ax, x)|^{\frac{p}{2}}.$$

Since the sequence $\left\{\frac{H(2^n y)}{4^n}\right\}$ converges, the same is true for the sequence $\left\{\frac{H(2^n x)}{4^n}\right\}$. From the results of Cases 0,1 and 2, we conclude that the sequence $\left\{\frac{H(2^n x)}{4^n}\right\}$ converges for each x in X. □

Lemma 14. *The functional $\hat{H}$ defined by $\hat{H} = \lim_{n\to\infty} \frac{H(2^n x)}{4^n}$ has the following properties:*

(1) $\hat{H}$ is additive on A-orthogonal pairs.

(2) $\hat{H}(x) = 0$ if $(Ax, x) = 0$.

(3) If $(Ax, x) = (Ay, y) \neq 0$ and $(Ax, y) = 0$, then $\hat{H}(x) = \hat{H}(y)$, while, if $(Ax, x) = -(Ay, y)$ and $(Ax, y) = 0$, then $\hat{H}(x) = -\hat{H}(y)$.

Proof. For (1), let x, y in X satisfy $(Ax, y) = 0$. Then, by (43), it follows that

$$\left|\frac{H(2^n(x+y))}{4^n} - \frac{H(2^n x)}{4^n} - \frac{H(2^n y)}{4^n}\right| \leq 2^{n(p-2)}\theta\left[|(Ax,x)^{\frac{p}{2}} + |(Ay,y)|^{\frac{p}{2}}\right].$$

Taking the limit as $n \to \infty$, we get $\hat{H}(x+y) - \hat{H}(x) - \hat{H}(y) = 0$. For (2), from the definition of $\hat{H}$ in terms of H, it is clear that $\hat{H}(0) = 0$. Now let $(Ax, x) = 0$. Then, by the last inequality with $y = -x$, we have

$$\left|\frac{H(0)}{4^n} - \frac{H(2^n x)}{4^n} - \frac{H(-2^n x)}{4^n}\right| \leq 0 \text{ for all } n \in N,$$

so $\hat{H}(0) - 2\hat{H}(x) = 0$ and $\hat{H}(x) = 0$. For (3), let $(Ax, x) = (Ay, y) \neq 0$ and $(Ax, y) = 0$. Then $(A(x+y), x-y) = 0$, so, using the A-orthogonal pairs $\frac{x+y}{2}$ and $\frac{x-y}{2}$, we get

$$\hat{H}(x) = \hat{H}\left(\frac{x+y}{2} + \frac{x-y}{2}\right) = \hat{H}\left(\frac{x+y}{2}\right) + \hat{H}\left(\frac{x-y}{2}\right),$$

while, using $\frac{x+y}{2}$ and $\frac{y-x}{2}$, we get $\hat{H}(y) = \hat{H}\left(\frac{x+y}{2} + \frac{y-x}{2}\right) = \hat{H}\left(\frac{x+y}{2}\right) + \hat{H}\left(-\frac{x-y}{2}\right) = \hat{H}(x)$, since $\hat{H}$ is even. On the other hand, suppose that $(Ax, x) = -(Ay, y)$ and $(Ax, y) = 0$. Then $(A(x+y), x+y) = 0$. Using the properties (2) and (1) above, we have $0 = \hat{H}(x+y) = \hat{H}(x) + \hat{H}(y)$, so that $\hat{H}(x) = -\hat{H}(y)$ in this case. □

Lemma 15. *For all x in X and all complex numbers a we have*

$$\hat{H}(ax) = |a|^2 \hat{H}(x).$$

Proof. From property (3) of Lemma 14, we conclude that the functional $\hat{H}$ is a function of (Ax, x). Let $\hat{H}(x) = \Gamma((Ax, x))$. Given x in X, we can find a y in X such that $(Ax, y) = 0$ and $(Ax, x) = \pm(Ay, y)$.

Case 1. $(Ax, x) = (Ay, y) > 0$.

Since $\hat{H}$ is additive in orthogonal pairs, it follows that, for real a and b, $\hat{H}(ax + by) = \hat{H}(ax) + H(by)$. By the definition of the function Γ, putting $u = a^2(Ax, x)$ and $v = b^2(Ax, x) = b^2(Ay, y)$, we have

$$\Gamma(u+v) = \Gamma(u) + \Gamma(v) \text{ for all } u, v \geq 0. \tag{51}$$

We may extend the function to all real numbers in a well known way, so that (51) will hold for all real u and v. Now define the mapping $\phi : R \to R$ by $\psi(a) = \hat{H}(ax) = \Gamma((a(ax), ax)) = \Gamma(a^2(Ax, x))$. Then, for real numbers a, b,

$$\phi(a+b) = \Gamma((a+b)^2(Ax,x)) = \Gamma(a^2(Ax,x) + 2ab(Ax,x) + b^2(Ax,x)),$$

$$\phi(a-b) = \Gamma((a-b)^2(Ax,x)) = \Gamma(a^2(Ax,x) - 2ab(Ax,x) + b^2(Ax,x)).$$

Using (51), we obtain

$$\psi(a+b) + \psi(a-b) = 2\phi(a) + 2\psi(b). \tag{52}$$

Since ψ is measurable, by a known theorem of S. Kurepa [11], it may be written as $\psi(a) = \alpha(x)a^2$, or $\hat{H}(ax) = \Gamma(a^2(Ax,x))$. Put $a = 1$ to get $\hat{H}(x) = \alpha(x)$; so that $\hat{H}(ax) = a^2\hat{H}(x)$ for x in X and a in R. Now suppose that $a = r\exp(i\omega)$ is a complex number ($r = |a|$). Then $(A(ax), ax) = (A(r\exp(i\omega)x, r\exp(i\omega)x) = r^2(Ax,x))$. Thus, $\hat{H}(ax) = \Gamma(A(ax), ax) = \Gamma(r^2(Ax,x))$ or $\hat{H}(ax) = |a|^2\hat{H}(x)$ for all x in X and all complex numbers a, in Case 1.

Case 2. $(Ax, x) = -(Ay, y)$ for all y in Y.

As before, we can find a z in Y such that $(Az, z) = (Ay, y)$ and $(Ay, z) = 0$. Hence, by the result of Case 1, we have

$$\hat{H}(ay) = |a|^2\hat{H}(y).$$

Since, by the condition of Case 2, we have $\hat{H}(x) = -\hat{H}(y)$ by Lemma 14, (3), it follows that

$$\hat{H}(ax) = -\hat{H}(ay) = -|a|^2\hat{H}(y) = |a|^2\hat{H}(x).$$

Therefore, $\hat{H}(ax) = |a|^2\hat{H}(x)$ holds for all x in X and all complex numbers a. □

From inequality (49), it follows that

$$|\hat{H}(x) - H(x)| \leq \varepsilon_2(p,\theta)|(Ax,x)|^{\frac{p}{2}}, \text{ with } \varepsilon_2(p,\theta) = \frac{4\mu(p,\theta)}{4-2^p}. \tag{53}$$

On the basis of the property $\hat{H}(ax) = |a|^2\hat{H}(x)$ of $\hat{H}$, the authors conclude that the conclusion of Theorem 6 above holds, so that H is of the form

$$\hat{H}(x) = (x, c) + (d, x) + \beta(Ax, x),$$

where the constant vectors c, d in X and the complex constant β are uniquely determined by the functional $\hat{H}$. Thus, $\hat{H}$ is continuous. Also, since $\hat{H}$ is an even functional, it follows that $\hat{H}(x) = \beta(Ax, x)$.

To complete the proof of Theorem 7, we note that, by (41) and (53), we have

$$|\phi(x)-[\hat{G}(x)+\hat{H}(x)]| = |G(x)+H(x)-\hat{G}(x)-\hat{H}(x)| \leq |G(x)-\hat{G}(x)|+|H(x)-\hat{H}(x)| \leq$$

$$\leq \varepsilon(p,\theta)|(Ax, x)|^{\frac{p}{2}}, \text{ where } \varepsilon(p,\theta) = \varepsilon_1(p,\theta) + \varepsilon_2(p,\theta).$$

Therefore, the required functional of Theorem 7 is by (42):

$$\psi(x) = \hat{G}(x) + \hat{H}(x) = (x, u) + (v, x) + \beta(Ax, x).$$

To prove the uniqueness of ψ, suppose on the contrary that there is another functional $\psi_1 \neq \psi$ which is continuous, additive on A-orthogonal pairs and which satisfies

$$\psi_1(x) - \phi(x)| \leq \varepsilon'|(Ax, x)|^{\frac{p}{2}} \text{ for some constant } \varepsilon' > 0 \text{ and all } x \in X.$$

Since ψ_1 is continuous and additive on A-orthogonal pairs, it follows that it is of the form $\psi_1(x) = (x, c) + (d, x) + \gamma(Ax, x)$, where c, d are constant vectors in X and $\gamma \in C$. Then

$$|\psi(x) - \psi_1(x)| \leq |\psi(x) - \phi(x)| + |\phi(x) - \psi_1(x)| \leq (\varepsilon(p,\theta) + \varepsilon')|(Ax, x)|^{\frac{p}{2}},$$

that is, for all x in X, we have

$$|(x, u-c) + (v-d, x) + (\beta-\gamma)(Ax, x)| \leq (\varepsilon(p,\theta) + \varepsilon')(Ax, x)^{\frac{p}{2}}.$$

In this last inequality, replace x with nx to obtain

$$|(nx, u-c) + (v-d, nx) + (\beta-\gamma)n^2(Ax, x)| \leq (\varepsilon(p,\theta) + \varepsilon')n^p|(Ax, x)|^{\frac{p}{2}}. \quad (54)$$

Divide (54) by n^2 to get

$$|n^{-1}(x,u-c)+n^{-1}(v-d,x)+(\beta-\gamma)(Ax,x)|\leq(\varepsilon(p,\theta)+\varepsilon')n^{p-2}|(Ax,x)|^{\frac{p}{2}},$$

and, letting $n\to\infty$, we obtain $\beta=\gamma$. Thus, (54) now becomes

$$|(nx,u-c)+(v-d,nx)|\leq(\varepsilon(p,\theta)+\varepsilon')n^{p}|(Ax,x)|^{\frac{p}{2}}.$$

Divide this last inequality by n and then let $n\to\infty$ to get $(x,u-c)+(v-d,x)=0$. Now, if we first put $x=u-c$ and second put $x=i(u-c)$, to obtain $||u-c||^2+(v-d,u-c)=0$ and $i||u-c||^2-i(v-d,u-c)=0$, we find that $u=c$ and $v=d$. The uniqueness property of the functional ψ has been proved. □

Comments

In their paper, Drljevic and Mavar [6,p.171], stated without proof the following:

Theorem 16. *Let X be a Banach space and h a functional on X such that $h(tx)$ is continuous in the scalar t for each fixed x in X. Let $\theta\geq 0$ and $p\in[0,2)$ be real numbers such that*

$$|h(x+y)+h(x-y)-2h(x)-2h(y)|\leq\theta(||x||^p+||y||^p)\text{ for each } x,y \text{ in } X.$$

Then there exists a unique quadratic functional h_1 on X such that

$$|h(x)-h_1(x)|\leq\frac{4\theta}{4-2^p}||x||^p.$$

This anticipated, in the case of functionals, one of the results of Czerwik (see Case 1 of Theorem 2 above).

Later, Drljevic [7] proved the following:

Theorem 17. *Let X be a complex Hilbert space of dimension ≥ 3, $A:X\to X$ a bounded self-adjoint linear operator with $\dim AX\geq 2$, and let the real numbers $\theta\geq 0$ and $p\in[0,2)$ be given. Suppose that $h:X\to C$ is continuous and satisfies the inequality $|h(x+y)+h(x-y)-2h(x)-2h(y)-2h(y)|\leq\theta\left[|(Ax,x)|^{\frac{p}{2}}+|(Ay,y)|^{\frac{p}{2}}\right]$ whenever $(Ax,y)=0$.*

Then the limit $h_1(x)=\lim\limits_{n\to\infty}\dfrac{h(2^nx)}{4^n}$ exists for each $x\in X$ and the functional h_1 is continuous and satisfies $h_1(x+y)+h_1(x-y)=2h_1(x)+2h_1(y)$ whenever

$(Ax, y) = 0$. *Moreover, there exists a real number* $\varepsilon > 0$ *such that* $|h(x) - h_1(x)| \leq \varepsilon|(Ax, x)|^{\frac{p}{2}}$.

The methods of proof of this theorem are based in part on those explained above in the proof of Theorem 7.

Approximately homogeneous mappings

This topic has been studied by S. Czerwik [5] and by Jacek and Jozef Tabor [21]. We begin with a presentation of Czerwik's work. The following notations will be used. R *denotes the set of all real numbers,* R_+ *the set of non-negative reals and* R_0 *the set of non-zero reals. For each* α *in* R *and each* p *in* R_0*, we define* $U_p = \{\alpha \in R : \alpha^p \text{ exists in } R\}$.

Lemma 18. *Let* X *be a real vector space and* Y *a real normed space. Given* $f : X \to Y$, p *in* R_0 *and* $h : R \times X \to R_+$ *which satisfy the inequality*

$$\|f(\alpha x) - \alpha^p f(x)\| \leq h(\alpha, x) \tag{55}$$

for all (α, x) *in* $U_p \times X$*, then the inequality*

$$\|f(\alpha^n x) - \alpha^{np} f(x)\| \leq \sum_{s=0}^{n-1} |\alpha|^{sp} h(\alpha, \alpha^{n-s-1} x) \tag{56}$$

holds for all $n \in N$ *and* $(\alpha, x) \in U_p \times X$.

Proof. *We use (56) as an induction hypothesis. Note that it is true for* $n =$ *by (55). In (56), replace* x *by* αx *to get*

$$\|f(\alpha^{n+1} x) - \alpha^{np} f(\alpha x)\| \leq \sum_{s=0}^{n-1} |\alpha|^{sp} h(\alpha, \alpha^{n-s} x).$$

Now multiply (55) by α^{np} *to obtain*

$$\|\alpha^{np} f(\alpha x) - \alpha^{(n+1)p} f(x)\| \leq |\alpha|^{np} h(\alpha, x).$$

Combine the last two inequalities to find that

$$\|f(\alpha^{n+1} x) - \alpha^{(n+1)p} f(x)\| \leq \sum_{s=0}^{n} |\alpha|^{sp} h(\alpha, \alpha^{n-s} x),$$

which completes the induction proof. □

Theorem 19. *Let the assumptions of Lemma 18 be satisfies, and let Y be a Banach space. Suppose that for some β in U_p with $\beta \neq 0$ the series*

$$\sum_{n=1}^{\infty} |\beta|^{-np} h(\beta, \beta^n x) \tag{57}$$

converges for each x in X, and that

$$\liminf_{n\to\infty} |\beta|^{-np} h(\alpha, \beta^n x) = 0 \tag{58}$$

for each (α, x) in $U_p \times X$. Then there exists a unique mapping $g : X \to Y$ such that $g(\alpha x) = \alpha^p g(x)$ for each (α, x) in $U_p \times X$ and which satisfies

$$||g(x) - f(x)|| \leq \sum_{n=1}^{\infty} |\beta|^{-np} h(\beta, \beta^{n-1} x) \tag{59}$$

for each x in X.

Proof. For $n \in N$, set

$$g_n(x) = \beta^{-np} f(\beta_n x), \quad x \in X. \tag{60}$$

From (56), we get when $n \in N$ and $x \in X$:

$$||g_n(x) - f(x)|| \leq \sum_{s=1}^{n} |\beta|^{-sp} h(\beta, \beta^{s-1} x). \tag{61}$$

In order to show that the $g_n(x)$ form a Cauchy sequence, we note that, if in (56) we replace x by $\beta^n x$ and n by $n - m$ where $n > m$, we have

$$||g_n(x) - g_m(x)|| \leq |\beta|^{-np} ||f(\beta^n x) - \beta^{(n-m)p} f(\beta^m x)|| =$$

$$= ||\beta|^{-np} ||f(\beta^{n-m}(\beta^m x) - \beta^{(n-m)p} f(\beta^m x))|| \leq$$

$$\leq |\beta|^{-np} \sum_{s=0}^{n-m-1} |\beta|^{sp} h(\beta, \beta^{n-s-1}) = \sum_{s=0}^{n-m-1} |\beta|^{(s-n)p} h(\beta, \beta^{n-s-1} x).$$

This inequality may be written as $||g_n(x) - g_m(x)|| \leq \sum_{k=m+1}^{n} |\beta|^{-kp} h(\beta, \beta^{k-1} x)$, and, by hypothesis, it follows that $\{g_n(x)\}$ is a Cauchy sequence for each x in X. From (60), (55) and (58), we obtain

$$||g(\alpha x) - \alpha^p g(x)|| = \lim_{n\to\infty} ||\beta^{-np}[f(\alpha\beta^n x) - \alpha^p f(\beta^n x)] \leq$$

$$\leq \lim_{n\to\infty} |\beta|^{-np} h(\alpha, \beta^n x) = 0.$$

Thus, g is a p-homogeneous mapping when $\alpha \in U_p$. Also, from (61), we get (59).

It remains to prove that g is the unique p-homogeneous mapping that satisfies (59). Suppose that there are two such mappings, say g_1 and g_2. Then, for $m \in N$,

$$\|g_1(x) - g_2(x)\| = |\beta|^{-mp}\|g_1(\beta^m x) - g_2(\beta^m x)\| \leq$$

$$\leq |\beta|^{-mp}[\|g_1(\beta^m x) - f(\beta^m x)\| + \|g_2(\beta^m x) - f(\beta^m x)\|] \leq$$

$$\leq |\beta|^{-mp} \cdot 2\sum_{s=1}^{n} |\beta|^{-sp} h(\beta, \beta^{s+m-1}) \leq 2 \sum_{k=m+1}^{\infty} |\beta|^{-kp} h(\beta, \beta^{k-1}x).$$

Consequently, since the series (57) converges, it follows that $g_2 = g_1$. □

Corollary 20. *Let the assumptions of the Lemma 18 be satifies with* $h(\alpha, x) = \delta + |\alpha|^p \varepsilon$ *for given positive numbers* δ *and* ε*, and let* Y *be a Banach space. Then there is a unique p-homogeneous mapping* $g : X \to X$ *such thart*

$$\|g(x) - f(x)\| \leq \varepsilon \text{ for all } x \text{ in } X. \tag{62}$$

Proof. Assume that $p > 0$. By Theorem 19, for every $\beta = m \in N$, $\beta \geq 2$, there exists a p-homogeneous mapping

$$g_m(x) = \lim_{n \to \infty} m^{-np} f(m^n x), \quad x \in X,$$

such that

$$\|g_m(x) - f(x)\| \leq \sum_{n=1}^{\infty} m^{-np} h(m, m^{n-1}x) \leq \sum_{n=1}^{\infty} m^{-np}(\delta + m^p \varepsilon)$$

or

$$\|g_m(x) - f(x)\| \leq \frac{\delta + m^p \varepsilon}{m^p - 1}, \quad x \in X. \tag{63}$$

Now we shall show that, for each pair $m > 1$, $r > 1$ in N, we have $g_m = g_r$. By (63), for $n \in N$,

$$\|g_m(x) - g_r(x)\| = 2^{-np}\|g_m(2^n x) - g_r(2^n x)\| \leq 2^{-np}\left(\frac{\delta + m^p \varepsilon}{m^p - 1} + \frac{\delta + r^p \varepsilon}{r^p - 1}\right).$$

Thus, since $p > 0$, if we let $n \to \infty$, we get $g_m = g_r$. We pit $g(x) = g_2(x)$, $x \in X$. By (63) we have $\|g(x) - f(x)\| \leq \frac{\delta + m^p \varepsilon}{m^p - 1}$, and now, letting $m \to \infty$, we find that $\|g(x) - f(x)\| \leq \varepsilon$.

In the case where $p < 0$, we can take $\beta = \dfrac{1}{m}$ and say $q = -p$ and carry out a similar proof. □

Example. Take $f(x) = \sin x$ for x in R. Then $|\sin(\alpha x) - \alpha^p \sin x| < 1 + |\alpha|^p$ for (α, x) in $Up \times R$. This shows that not all cases under Corollary 20 are superstable.

Corollary 21. *Let the assumptions of Lemma 18 be satisfied with* $h(\alpha, x) = \delta + |\alpha|^p \varepsilon$, δ, ε *in* R_+, *and let* Y *be a Banach space. Then, if either* δ *or* ε *is zero,* $f(\alpha x) = \alpha^p f(x)$ *for all* (α, x) *in* $(U_p \setminus \{0\}) \times X$.

Proof. Suppose that $\delta = 0$. Then

$$\|f(\alpha x) - \alpha^p f(x)\| \leq |\alpha|^p \varepsilon \text{ for } (\alpha, x) \text{ in } U_p \times X. \tag{64}$$

Putting $x = \dfrac{y}{\alpha}$ with α in $U_p \setminus \{0\}$, we get $\left\| f(y) - \alpha^p f\left(\frac{y}{\alpha}\right) \right\| \leq |\alpha|^p \varepsilon$. Assume that $p > 0$. Then $f(y) = \lim_{\alpha \to 0} \alpha^p f\left(\frac{y}{\alpha}\right)$ for y in X. Therefore, for (β, x) in $(U_p \setminus \{0\}) \times X$, we have $f(\beta x) = \lim \alpha^p f\left(\dfrac{\beta x}{\alpha}\right) = \lim \beta^p \left(\dfrac{\alpha}{\beta}\right)^p f\left(\dfrac{\beta x}{\alpha}\right) = \beta^p f(x)$, so the corollary is verified for $\delta = 0$ and $p > 0$. If $p < 0$, then, from (64) we get $\lim\limits_{|\alpha| \to \infty} \alpha^p f\left(\dfrac{y}{\alpha}\right) = f(y)$, and as before we find that the corollary holds for $\delta = 0$ and $p < 0$.

On the other hand, suppose that $\varepsilon = 0$. Then $\|\alpha^{-p} f(\alpha x) - f(x)\| \leq |\alpha|^{-p} \delta$ for (α, x) in $(U_p \setminus \{0\}) \times X$. Hence, when $p > 0$, $f(x) = \lim\limits_{|\alpha| \to \infty} \alpha^{-p} f(\alpha x)$, and when $p < 0$, $f(x) = \lim\limits_{\alpha \to 0} \alpha^{-p} f(\alpha x)$. As before, it is easily shown that Corollary 21 holds in these cases as well. □

Czerwik [5] remarked that the problem remained open for $p = 0$ except when $X = Y = R$.

Jacek and Josef Tabor [21] have used a different definition of approximately homogeneous mappings from that of S. Czerwik. Jozef Tabor [20], in connection with his study of approximately linear mappings has already proved that every mapping from one real normed space X to anothe Y which for a given $\varepsilon > 0$ satisfies $\|f(\alpha x) -$

$\alpha f(x)\| \leq \varepsilon$ for all real α and x in X is homogeneous (see Corollary 1 of J. Tabor [20]).

In the seminar of R. Ger (Katowice, October 1992), K. Baron asked if the conclusion still holds if ε in the above inequality is replaced by $\varepsilon|\alpha|$. In this particular case, it turns out that these two conditions are equivalent. However, Baron's question led Jacek Tabor and Jozef Tabor to consider some generalizations of the inequality $\|f(\alpha x) - \alpha f(x)\| \leq \varepsilon|\alpha|$, which lead to the results given below. They began with a very general statement:

Lemma 22. *Consider a set X, a Hausdorff topological space Y and mappings $g_1 : X \to X$, $g_2 : Y \to Y$ and $f : X \to Y$. Suppose that g_2 is continuous on Y. Then the following two conditions are equivalent:*

(i) $g_2(f(x)) = f(g_1(x))$ for all x in X.

(ii) There exists a sequence of mappings $f_n : X \to Y$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ and } \lim_{n\to\infty} g_2(f_n(x)) = f(g_1(x)), \quad x \text{ in } X.$$

Proof. Observe that (i) implies (ii) because we may put $f_n = f$ for n in N. Suppose that (ii) holds. Since g_2 is continuous we get $f(g_1(x)) = \lim_{n\to\infty} g_2(f_n(x)) = g_2(f(x))$ for x in X. □

Definition. Given a set X and a semigroup G with unit I, we say that G acts on X if there is a mapping $\phi : G \times X \to X$ such that $\phi(\beta, \phi(\alpha, x)) = \phi(\beta\alpha, x)$ for α, β in G, x in X, where $\phi(1, x) = x$. In what follows, we shall write $\phi(\alpha, x)$ as a multiplication, e.g., $\alpha \circ x$ or $\alpha * x$.

Notation. R_+ denotes the non-negative real numbers, K denotes the field of either real of complex numbers and $0^0 = 1$.

Theorem 23. *Given a set X, a metric space (Y, d) and a semigroup G with identity acting on X (denoted $\alpha \circ x$) and also on Y (denoted $\alpha * y$). Assume that for each α in G the mapping $y \to \alpha * y$ is continuous in y for all y in Y. For a given* *mapping* $g : G \times X \to R_+$ *suppose that* $f : X \to Y$ *satisfies the inequality*

$$d(f(\alpha \circ x), \alpha * f(x)) \leq g(\alpha, x). \tag{65}$$

Assume also that there exists a sequence of invertible elements α_n *in* G *such that, for* α *in* G *and* x *in* X, *we have*

$$\lim_{n \to \infty} g(\alpha\alpha_n, (\alpha_n)^{-1} \circ x) = 0. \tag{66}$$

Then $f(\alpha \circ x) = \alpha * f(x)$ *for all* α *in* G *and* x *in* X.

Proof. In (65), replace α by $\alpha\alpha_n$ and x by $(\alpha_n)^{-1} \circ x$:

$$d(f(\alpha \circ x), \alpha\alpha_n * (\alpha_n^{-1} \circ x)) \leq g(\alpha\alpha_n, (\alpha_n)^{-1} \circ x).$$

Hence, by (66), we have

$$\lim_{n \to \infty} \alpha\alpha_n * f(\alpha_n^{-1} \circ x) = f(\alpha \circ x). \tag{67}$$

Taking $\alpha = 1$ in (67), we obtain

$$\lim_{n \to \infty} \alpha_n * f(\alpha_n^{-1} \circ x) = f(x) \text{ for } x \text{ in } X. \tag{68}$$

For an arbitrary α in G, put $g_1(x) = \alpha\circ$, $g_2(y) = \alpha * y$ and $f_n(x) = \alpha_n * f(\alpha_n^{-1} \circ x)$. By (68) and (67), f_n satisfies condition (ii) of Lemma 22. By this lemma, we get $f(g_1(x)) = g_2(f(x))$, that is $f(\alpha \circ x) = \alpha * f(x)$ for α in G and x in X. □

Corollary 24. *Let* X *be a normed space, where* $L(X)$ *denotes the semigroup of continuous linear operators on* X *with composition as the binary operation, and let* p_1, p_2 *be non-negative real numbers with* $p_1 \neq p_2$. *Let* $k : X \to R_+$ *be a mapping such that*

$$k(Ax) \leq ||A||^{p_2} k(x) \textit{ for } A \textit{ in } L(X), \quad x \textit{ in } X. \tag{69}$$

Suppose that $f : X \to X$ *satisfies the inequality*

$$||f(Ax) - Af(x)|| \leq ||A||^{p_1} k(x) \textit{ for } A \textit{ in } L(X), \quad x \textit{ in } X. \tag{70}$$

Then there exists an α *in* K *such that* $f(x) = \alpha x$.

Proof. Put $g(A,x) = ||A||^{p_1}k(x)$ for A in $L(X)$, x in X, and $A_n = \alpha_n I$, where $I =$ the identity map and

$$\alpha_n = \begin{cases} 1/n & \text{if } p_1 > p_2, \\ n & \text{if } p_1 < p_2. \end{cases}$$

By (70), the inequality (65) is satisfied. By (69) we have for x in X

$$g(AA_n, A_n^{-1}x) = ||AA_n||^{p_1}k(A_n^{-1}x) \leq ||A||^{p_1}||A_n||^{p_1}||A_n^{-1}||^{p_2}k(x) \leq ||A||^{p_1}|\alpha_n|^{p_1-p_2}.$$

Thus, $g(AA_n, A_n^{-1}x) \to 0$ as $n \to \infty$, so condition (66) holds. Hence, by Theorem 23, we have

$$f(Ax) = Af(x) \text{ for all } A \text{ in } L(X) \text{ and } x \text{ in } X. \tag{71}$$

It remains to prove that $f(x) = \alpha x$ for some α in K. In (71), put $x = 0$ and $A = 2I$ to see that $f(0) = 2f(0)$, so that $f(0) = 0 = \alpha 0$ for α in K. Suppose that, contrary to the statement in question, there exists an x in X, $x \neq 0$ such that $f(x) \neq \alpha x$ for each α in K. Then x and $f(x)$ ate linearly independent, so that there exists an A in $L(X)$ with $Af(x) = 0$ and $Ax = x$. Hence, by (71), $f(x) = 0$, a contradiction, and we conclude that for each x in X there exists an α such that $f(x) = \alpha x$. Now we must show that α does not depend on x. Let x_1, x_2 in X satisfy $x_1 \neq 0$, $x_2 \neq 0$ and $x_1 \neq x_2$, with $f(x_1) = \alpha_1 x_1$ and $f(x_2) = \alpha_2 x_2$. Take an A in $L(X)$ such that $Ax_1 = x_2$. Then, by (71), $\alpha_1 x_2 = \alpha_1 Ax_1 = A(\alpha_1 x_1) = A(f(x_1)) = f(Ax_1) = f(x_2) = \alpha_2 x_2$ and so $\alpha_1 = \alpha_2$. □

Corollary 25. *Let K be the real or complex field and let p, p_1 and p_2 be non-negative real numbers with $p_1 \neq p_2$. With X a vector space over K and Y a normed vector space over K, let $k : X \to R_+$ be a mapping such that*

$$k(\alpha x) \leq |\alpha|^{\beta_2}k(x) \textit{ for all } \alpha \textit{ in } K \textit{ and } x \textit{ in } X. \tag{72}$$

If a mapping $f : X \to Y$ satisfies the inequality

$$||f(\alpha x) - |\alpha|^p f(x)|| \leq |\alpha|^{p_1}k(x), \tag{73}$$

then

$$f(\alpha x) = |\alpha|^p f(x), \quad \alpha \text{ in } K, x \text{ in } X. \tag{74}$$

Proof. Here, K acting on X means the usual multiplication by scalars, but K acting on Y will be defined by $\alpha * y = |\alpha|^p y$ for α in K and y in Y. We put $g(\alpha, x) = |\alpha|^{p_1} k(x)$. Then (73) implies that (65) is satisfied. Again we take $\alpha = \frac{1}{n}$ if $p_1 > p_2$ and $\alpha_n = n$ if $p_1 < p_2$. Then, by (72), we have for α in K, x in X, $g(\alpha\alpha_n, \alpha_n^{-1}x) \leq |\alpha|^{p_1}|\alpha_n|^{p_1 - p_2} k(x) \to 0$ as $n \to \infty$. Thus, condition (66) holds. By Theorem 23, we have

$$f(\alpha x) = \alpha * f(x) = |\alpha|^p f(x) \text{ for } \alpha \text{ in } K \text{ and } x \text{ in } X. \quad \square$$

In a similar way, the authors proved:

Corollary 26. *If* $k(\alpha x) < |\alpha|^{p_2} k(x)$ *and* $\|f(\alpha x) - \alpha f(x)\| \leq |\alpha|^{p_1} k(x)$, *then* $f(\alpha x) = \alpha f(x)$.

These authors also generalized these results to the case where Y is a topological vector space over K, and where the domain of f is a subset X_1 of X which is closed under multiplication by scalars, with a similar substitution for Y. Their generalization of Corollary 25 reads as follows:

Theorem 27. *Let* X *be a vector space over* K, Y *a topological vector space over* K *and let* X_1 *and* Y_1 *be subsets of* X *and* Y, *respectively, such that* $KX_1 \subset X$ *and* $KY_1 \subset Y$. *We are given a bounded set* $V \subset Y$, *a mapping* $g : K \times Y_1 \to K$ *and a sequence of non-zero elements* α_n *of* K *such that*

$$\lim_{n\to\infty} g(\alpha\alpha_n^{-1}x) = 0 \text{ for } \alpha \in K, \ x \in X_1.$$

Suppose that the mapping $f : X_1 \to Y_1$ *satisfies the condition*

$$f(\alpha x) - |\alpha|^p f(x) \in g(\alpha, x)V \text{ for all } \alpha \text{ in } K \text{ and } x \text{ in } X_1.$$

Then

$$f(\alpha x) = |\alpha|^p f(x) \text{ for all } \alpha \text{ in } K \text{ and } x \text{ in } X_1.$$

Comments

It is interesting to compare the results of S. Czerwik and of J. and J. Tabor on the subject of approximately homogeneous mappings, which were clearly arrived at independently. Consider the case where α is a real and non-negative and where X is a real vector space, Y a Banach space and let $f : X \to Y$ satisfy $\|f(\alpha x) - \alpha^p f(x)\| \leq h(\alpha, p, x)$. The Tabors looked at cases where h (their g) was constant ($h = \varepsilon$) or where h has a sub-homogeneity property. In both cases, superstability resulted. However, in Corollary 20, together with the Example which follows, Czerwik showed that, if h is the sum of a non-zero constant and a particular homogeneous function, superstability fails. On the other hand, the Tabors succeeded in generalizing their results to more general spaces.

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