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CONFORMAL STRUCTURES IN THE LAGRANGE GEOMEETRY OF SECOND ORDER

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. In the present paper we introduce two d-structures on $E = Osc^2 M$: the conformal metrical d-structure and the almost symplectic d-structure and we study the properties of this two d-structures.

1. Introduction

The literature on the higher order Lagrange spaces geometry highlights the theoretical and practical importance of these spaces.

Motivated by concrete problems in variational calculation, higher order Lagrange geometry has witnessed a wide acknowledgment due to the papers $[7-11]$ published by Acad.dr.R.Miron and Prof.dr.Gh.Atanasiu.

The study of higher order Lagrange spaces is grounded on the k-osculator bundle notion. The bundle space of accelerations (or 2-osculator bundle) corresponds in this study to $k = 2$, [1], [7].

Very little research has been carried out with respect to the study of the important structures in the 2-osculator bundle.

In the present paper we define the conformal metrical d-structure notion, $\hat{g}$, in the Lagrange geometry of second order and we study the properties of this structure (§2). We also introduce the conformal almost symplectic d-structure notion, $\hat{a}$, in the Lagrange geometry of second order and we study the properties of this structure (§3).

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As to the terminology as notations we use those from [12], which are essentially based on M.Matsumoto's book [6].

2. The conformal metrical d-structure in the Lagrange geometry of second order.

Let M be a real n-dimensional C^∞-manifold, $(Osc^2 M, \pi, M)$ its 2-osculator bundle, or the bundle of accelerations. The local coordinates on $E = Osc^2 M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If N is a nonlinear connection on E, with the coefficients $N_{(1)}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, $N_{(2)}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, then let $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ be an N- linear connection, D on E.

Let $L^{(2)n} = (M, L)$ be the second order Lagrange space, where $L : E \to R$ is a C^∞ differentiable regular Lagrangian of second order, whose fundamental metric d-tensor field, g_{ij}, has a constant signature on $\tilde{E} = \{(x, y^{(1)}, y^{(2)}) \in Osc^2 M$, rank $||y^{(1)i}|| = 1\}$:

$$g_{ij}(x, y^{(1)}, y^{(2)}) = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(2)i}\partial y^{(2)j}}, \tag{2.1}$$

g_{ij} is a differentiable d-tensor field on $\tilde{E}$, symmetric, covariant of order two. Let (g^{ij}) be the inverse matrix of (g_{ij}):

$$g_{ik}(x, y^{(1)}, y^{(2)})g^{kj}(x, y^{(1)}, y^{(2)}) = \delta^j_i. \tag{2.2}$$

Observation 2.1. *We can consider on E as g_{ij} any d-tensor field of type $(0,2)$ on E symmetric and nondegenerate.*

We associate to this d-structure Obata's operators:

$$\Omega^{ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j - g_{sj}g^{ir}),\ \Omega^{*ir}_{sj} = \frac{1}{2}(\delta^i_s\delta^r_j + g_{sj}g^{ir}), \tag{2.3}$$

Obata's operators have the same properties as the ones associated with a Finsler space [12].

Let $\mathcal{S}_2(E)$ be the set of all symmetric d-tensor fields of the type $(0,2)$ on E. As is easily shown, the relation for $a_{ij},\ b_{ij} \in \mathcal{S}_2(E)$ defined by:

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) \mid a_{ij} = e^{2\rho} b_{ij}, \tag{2.4}$$

is an equivalent relation on $\mathcal{S}_2(E)$.

Definition 2.1. *The equivalence class $\hat{g}$ of $\mathcal{S}_2(E)/_\sim$, to which the metric d-structure g_{ij} belongs, is called: conformal metrical d-structure on E.*

Every $g'_{ij} \in \hat{g}$ is a positive definite, symmetric d-tensor field, expressed by

$$g'_{ij} = e^{2\rho} g_{ij}. \tag{2.5}$$

We shall find the condition that in a differentiable manifold E, a given $g'_{ij} \in \mathcal{S}_2(E)$ belongs to a conformal metrical d-structure.

Lemma 2.1. *A given positive definite $g_{ij} \in \mathcal{S}_2(E)$ is a fundamental tensor field if and only if it holds:*

$$\frac{\partial g_{ij}}{\partial y^{(2)k}} y^{(2)j} = 0. \tag{2.6}$$

Theorem 2.1. *A given positive definite $g'_{ij} \in \mathcal{S}_2(E)$ belongs to a conformal metrical d-structure if and only if there exists a function $\rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E)$ satisfying:*

$$\frac{\partial g'_{ij}}{\partial y^{(2)k}} y^{(2)j} = 2 \frac{\partial \rho}{\partial y^{(2)k}} y_i^{(2)\prime}, \tag{2.7}$$

where $y_i^{(2)\prime} = g'_{ij} y^{(2)j}$.

Proof. Let g'_{ij} belongs to a conformal metrical d-structure. Since g'_{ij} satisfies (2.5), we obtain (2.7) from Lemma 2.1. Conversly, if there exists a function ρ satisfying (2.7), then $g_{ij} = e^{-2\rho} g'_{ij}$ satisfies (2.6). □

Obata's operators are defined for $g'_{ij} \in \hat{g}$ by putting $(g'^{ij}) = (g'_{ij})^{-1}$. Since equation (2.5) is equivalent to

$$g'^{ij} = e^{-2\rho} g^{ij}, \tag{2.8}$$

We have:

Proposition 2.1. *Obata's operators depend on the conformal metrical d-structure $\hat{g}$, and do not depend on its representative $g'_{ij} \in \hat{g}$.*

3. The conformal almost symplectic d-structure in the Lagrange geometry of second order.

Let M be a real n-dimensional C^∞-manifold, $(Osc^2 M, \pi, M)$ its 2-osculator bundle, or the bundle of accelerations. The local coordinates on the total space $E = Osc^2 M$ are denoted by $(x^i, y^{(1)i}, y^{(2)i})$.

If N is a nonlinear connection on E with the coefficients $N_{(1)}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, $N_{(2)}{}^i{}_j(x^i, y^{(1)i}, y^{(2)i})$, then let $D\Gamma(N) = (L^i_{jk}, C_{(1)}{}^i_{jk}, C_{(2)}{}^i_{jk})$ be an N- linear connection, D, on E.

We consider on E an almost symplectic d-structure, defined by a d-tensor field of the type $(0,2)$, let us say $a_{ij}(x, y^{(1)}, y^{(2)})$, skewsymmetric

$$a_{ij}(x, y^{(1)}, y^{(2)}) = -a_{ji}(x, y^{(1)}, y^{(2)}), \tag{3.1}$$

and nondegenerate:

$$det||a_{ij}(x, y^{(1)}, y^{(2)})|| \neq 0,\ \forall y^{(1)} \neq 0,\ \forall y^{(2)} \neq 0, \tag{3.2}$$

We asociate to this d-structure Obata's operators:

$$\Phi^{ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j - a_{sj} a^{ir}), \Phi^{*ir}_{sj} = \frac{1}{2}(\delta^i_s \delta^r_j + a_{sj} a^{ir}), \tag{3.3}$$

where (a^{ij}) is the inverse matrix of (a_{ij}):

$$a_{ij} a^{jk} = \delta^k_i. \tag{3.4}$$

Obata's operators have the same properties as ones associated with a Finsler space [14].

Let $L^{(2)n} = (M, L)$ be the second order Lagrange space, where $L : E \to R$ is a C^∞-differentiable, regular Lagrangian of second order.

Let $\mathcal{A}_2(E)$ be the set of all skewsymmetric d-tensor fields of the type $(0, 2)$ on E. As is easily shown, the relation for $a_{ij},\ b_{ij} \in \mathcal{A}_2(E)$ defined by

$$a_{ij} \sim b_{ij} \Leftrightarrow \exists \rho(x, y^{(1)}, y^{(2)}) \in \mathcal{F}(E) | a_{ij} = e^{2\rho} b_{ij} \tag{3.5}$$

is an equivalent relation on $\mathcal{A}_2(E)$.

Definition 3.1. *The equivalence class, $\hat{a}$, of $\mathcal{A}_2(E)/_\sim$, to which the d-tensor field a_{ij} belongs, is called conformal almost symplectic d-structure on E.*

Every $a'_{ij} \in \hat{a}$ is a skewsymmetric and nondegenerate d-tensor field expressed by:

$$a'_{ij} = e^{2\rho} a_{ij}. \tag{3.6}$$

Obata's operators are defined for $a'_{ij} \in \hat{a}$ by putting $(a'^{ij}) = (a'_{ij})^{-1}$. Since equation (2.6) is equivalent to

$$(a'^{ij}) = e^{-2\rho} a^{ij}. \tag{3.7}$$

We have:

Proposition 3.1. *Obata's operators depend on the conformal almost symplectic d-structure $\hat{a}$, and do not depend on its representative $a'_{ij} \in \hat{a}$.*

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