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V-COHOMOLOGY OF COMPLEX FINSLER MANIFOLDS

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Dedicated to Professor Pavel Enghiş at his 70^{th} anniversary

Abstract. Starting from a natural decomposition of the exterior differential of a complex Finsler manifold, we define new cohomology groups and a Dolbeault type theorem is also proved.

1. The holomorphic tangent bundle.

Let us consider a complex manifold $M, \dim_C M = n$, $(U,(z^i))$ the complex coordinates in a local chart. The complexification $T_C M$ of the tangent bundle TM is decomposed in each point z after the $(1,0)$ vector fields and their conjugates of $(0,1)$ type, $T_C M = T'M \oplus T''M$. As it is well-known ([1],[2],[8]..), $T'M$ is also a complex manifold of dimension $\dim_C T'M = 2n$ and the natural projection $\pi_T : T'M \to M$ defines on $V(T'M) = \{\xi \in T'(T'M) \ / \ \pi_{T*}(\xi) = 0\}$ a structure of holomorphic vector bundle of rank n over $T'M$. We denote by $\mathcal{V}(T'M)$ the module of its sections, called *vector fields of* $v-$type.

A given supplementary subbundle $H(T'M)$ of $V(T'M)$ in $T'M$, i.e.

$$T'(T'M) = H(T'M) \oplus V(T'M) \tag{1}$$

defines a *nonlinear complex connection,* and we denote by $\mathcal{H}(T'M)$ the module of its sections, called *vector fields of* $h-$type.

Considering also their conjugates $\overline{V(T'M)}$ and $\overline{H(T'M)}$, we obtain the following decomposition of the complexification $T_C(T'M)$ of the real tangent bundle

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$T(T'M)$,

$$T_C(T'M) = H(T'M) \oplus V(T'M) \oplus \overline{H(T'M)} \oplus \overline{V(T'M)} \quad (2)$$

The elements of the conjugates are called *vector fields of $\overline{v}$–type, and $\overline{h}$–type,* respectively.

If $(U, (z^i, \eta^i))$ are the complex local coordinates on $T'M$ and if $N_i^j(z, \eta)$ are the coefficients of the complex nonlinear connection, which are changed at local change of the local chartafter the rule,

$$N_k'^i \frac{\partial z'^k}{\partial z^j} = \frac{\partial z'^i}{\partial z^k} N_j^k - \frac{\partial^2 z'^i}{\partial z^j \partial z^k} \eta^k \quad (3)$$

then the following set of complex vector fields

$$\{\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^j \frac{\partial}{\partial \eta^j}\}, \ \{\frac{\partial}{\partial \eta^i}\}, \ \{\frac{\delta}{\delta \overline{z}^i} = \frac{\partial}{\partial \overline{z}^i} - \overline{N_i^j} \frac{\partial}{\partial \overline{\eta}^j}\}, \ \{\frac{\partial}{\partial \overline{\eta}^i}\} \quad (4)$$

are called the local *adapted bases* of $\mathcal{H}(T'M), \mathcal{V}(T'M), \overline{\mathcal{H}(T'M)}$ and $\overline{\mathcal{V}(T'M)}$, respectively. The dual adapted bases are denoted by

$$\{dz^i\}, \ \{\delta\eta^i = d\eta^i + N_j^i dz^j\}, \ \{d\overline{z}^j\}, \ \{\delta\overline{\eta}^i = d\overline{\eta}^i + \overline{N_j^i} d\overline{z}^j\} \quad (5)$$

2. Complex valued forms

Let us consider the set $F(T'M)$ of the complex valued differential forms on $T'M$ given by the direct sum,

$$F(T'M) = \bigoplus_{p,q,r,s=\overline{0.n}} F^{p,q,r,s}(T'M) \quad (6)$$

where $F^{p,q,r,s}(T'M)$ [or $F^{p,q,r,s}(U)$ for the open set U of $T'M$, or simply $F^{p,q,r,s}$ when there is no confusion danger] is the set of $(p+q+r+s)$–forms which can be non zero only when these act on p vector fields of h–type, on q vector fields of v–type, on r vector fields of $\overline{h}$–type, and on $\overline{s}$ vector fields of $\overline{v}$–type. The elements of $F^{p,q,r,s}(U)$ are called (p, q, r, s)*–forms* on U.

In the adapted dual bases we have the following local expression of (p, q, r, s) –forms ω,

$$\omega = \sum \omega_{i_1 \dots i_p j_1 \dots j_q h_1 \dots h_r k_1 \dots k_s} \cdot dz^{i_1} \wedge \dots\dots \wedge dz^{i_p} \wedge$$

$$\delta\eta^{j_1} \wedge ..... \wedge \delta\eta^{j_q} \wedge d\bar{z}^{h_1} \wedge .....d\bar{z}^{h_r} \wedge \delta\bar{\eta}^{k_1} \wedge ....\delta\bar{\eta}^{k_s} \tag{7}$$

where the sum is after the indices $i_1 \prec ... \prec i_p$; $j_1 \prec ... \prec j_q$; $h_1 \prec ... \prec h_r$; $k_1 \prec ... \prec k_s$.

Now, let us consider f a complex valued differentiable function defined on $T'M$. In [2] the following operators are considered:

$$\begin{aligned} d'^h f &= \tfrac{\delta f}{\delta z^i} dz^i = (\tfrac{\partial f}{\partial z^i} - N_i^j \tfrac{\partial f}{\partial \eta^j}) dz^i \ ; \ d'^v f = \tfrac{\partial f}{\partial \eta^i} \delta\eta^i \\ d''^h f &= \tfrac{\delta f}{\delta \bar{z}^i} d\bar{z}^i = (\tfrac{\partial f}{\partial \bar{z}^i} - \overline{N}_i^j \tfrac{\partial f}{\partial \bar{\eta}^j}) d\bar{z}^i \ ; \ d''^v f = \tfrac{\partial f}{\partial \bar{\eta}^i} \delta\bar{\eta}^i \end{aligned} \tag{8}$$

and they give a natural decomposition of the exterior differential df of f.

We shall generalize these operators for any differential form. For this purpose we compute the exterior differential $d\omega$ of a $(p,q,r,s)-$form given by (7). A straightforward calculus, taking into account (5) and the properties of d, gives

$$d\omega \in F^{p+1,q,r,s} \oplus F^{p,q+1,r,s} \oplus F^{p,q,r+1,s} \oplus F^{p,q,r,s+1} \oplus F^{p+2,q-1,r,s} \oplus$$

$$F^{p+1,q-1,r+1,s} \oplus F^{p+1,q-1,r,s+1} \oplus F^{p+1,q,r+1,s-1} \oplus F^{p,q+1,r+1,s-1} \oplus F^{p,q,r+2,s-1} \tag{9}$$

Particularly, by using (8) we have

$$dF^{0,0,0,0} \subset F^{1,0,0,0} \oplus F^{0,1,0,0} \oplus F^{0,0,1,0} \oplus F^{0,0,0,1} \tag{10}$$

where $F^{0,0,0,0}$ denotes the set of complex valued differentiable functions on $T'M$. We also obtain

$$dF^{1,0,0,0} \subset F^{2,0,0,0} \oplus F^{1,1,0,0} \oplus F^{1,0,1,0} \oplus F^{1,0,0,1} \tag{11}$$

$$dF^{0,0,1,0} \subset F^{1,0,1,0} \oplus F^{0,1,1,0} \oplus F^{0,0,2,0} \oplus F^{0,0,1,1} \tag{12}$$

Now, we assume that M is a complex Finsler manifold with Finsler metric F ([2], definition 3.1) and we consider that N is the complex Rund connection on M (idem, definition 3.3). Then it is well-known that, [11]

$$\frac{\delta N_k^i}{\delta z^j} = \frac{\delta N_j^i}{\delta z^k}$$

and taking account the local expression (7) of the (p,q,r,s)-forms, the formulas (5) and the properties of the exterior differential, we give

$$dF^{0,1,0,0} \subset F^{1,1,0,0} \oplus F^{0,2,0,0} \oplus F^{0,1,1,0} \oplus F^{0,1,0,1} \oplus F^{1,0,1,0} \oplus F^{1,0,0,1} \tag{13}$$

$$dF^{0,0,0,1} \subset F^{1,0,0,1} \oplus F^{0,1,0,1} \oplus F^{0,0,1,1} \oplus F^{0,0,0,2} \oplus F^{1,0,1,0} \oplus F^{0,1,1,0} \tag{14}$$

From (9)-(14) it result the following

Proposition 01 *If M is a complex Finsler manifold endowed with the complex Rund connection then we have*

$$dF^{p,q,r,s} \subset F^{p+1,q,r,s} \oplus F^{p,q+1,r,s} \oplus F^{p,q,r+1,s} \oplus F^{p,q,r,s+1} \oplus$$

$$F^{p+1,q-1,r+1,s} \oplus F^{p+1,q-1,r,s+1} \oplus F^{p+1,q,r+1,s-1} \oplus F^{p,q+1,r+1,s-1} \quad (15).$$

From the above decomposition (15) it follows that we can define eight morphisms of complex vector spaces if we consider the different components, namely

$$d'^h : F^{p,q,r,s} \to F^{p+1,q,r,s} \;;\; d'^v : F^{p,q,r,s} \to F^{p,q+1,r,s}$$
$$d''^h : F^{p,q,r,s} \to F^{p,q,r+1,s} \;;\; d''^v : F^{p,q,r,s} \to F^{p,q,r,s+1}$$
$$\partial_1 : F^{p,q,r,s} \to F^{p+1,q-1,r+1,s} \;;\; \partial_2 : F^{p,q,r,s} \to F^{p+1,q-1,r,s+1}$$
$$\partial_3 : F^{p,q,r,s} \to F^{p+1,q,r+1,s-1} \;;\; \partial_4 : F^{p,q,r,s} \to F^{p,q+1,r+1,s-1}$$

We remark that these operators and the classical operators d' , d'' that appear in the decomposition $d = d' + d''$ of the differential on a complex manifold, are related by the following relations

$$d' = d'^h + d'^v + \partial_3 + \partial_4 \;;\; d'' = d''^h + d''^v + \partial_1 + \partial_2 \tag{16}$$

Moreover, by equalizing the terms of the same type in the relation

$$d^2 = (d'^h + d'^v + \partial_3 + \partial_4 + d''^h + d''^v + \partial_1 + \partial_2)^2 = 0$$

we obtain:

$$(d'^h)^2 = 0, \quad (d'^v)^2 = 0, \quad (d''^h)^2 = 0, \quad (d''^v)^2 = 0$$
$$(\partial_1)^2 = 0, \quad (\partial_2)^2 = 0, \quad (\partial_3)^2 = 0, \quad (\partial_4)^2 = 0$$
$$d'^h d'^v + d'^v d'^h = 0, \quad d''^v d'^v + d'^v d''^v = 0, \quad d''^v d''^h + d''^h d''^v = 0$$

$$d'^h\partial_2+\partial_2 d'^h=0,\quad d'^v\partial_4+\partial_4 d'^v=0,\quad d''^h\partial_1+\partial_1 d''^h=0$$
$$d'^h\partial_3+\partial_3 d'^h=0,\quad d''^v\partial_2+\partial_2 d''^v=0,\quad d''^h\partial_4+\partial_4 d''^h=0$$
$$\partial_1\partial_2+\partial_2\partial_1=0,\quad \partial_1\partial_3+\partial_3\partial_1=0,\quad \partial_3\partial_4+\partial_4\partial_3=0$$
$$d'^h d''^v+d''^v d'^h+d'^v\partial_2+\partial_2 d'^v=0,\; d'^v d''^h+d''^h d'^v+d''^v\partial_4+\partial_4 d''^v=0$$
$$d'^v\partial_3+\partial_3 d'^v+d'^h\partial_4+\partial_4 d'^h=0,\quad d''^v\partial_1+\partial_1 d''^v+d''^h\partial_2+\partial_2 d''^h=0$$
$$d''^h\partial_3+\partial_3 d''^h+\partial_1\partial_4+\partial_4\partial_1=0,\quad d'^h\partial_1+\partial_1 d'^h+\partial_3\partial_2+\partial_2\partial_3=0$$
$$d'^h d''^h+d''^h d'^h+\partial_1 d'^v+d'^v\partial_1+\partial_2\partial_4+\partial_4\partial_2+\partial_3 d''^v+d''^v\partial_3=0 \quad .$$

By the same argument we have

$$d''^v(\omega\wedge\theta)=d''^v\omega\wedge\theta+(-1)^{\deg\omega}\omega\wedge d''^v\theta \tag{17}$$

for any $\omega\in F^{p,q,r,s}$, $\theta\in F^{p',q',r',s'}$ and similar equalities for the other operators defined above.

From (17) and from the linearity of d''^v we deduce that if $\omega\in F^{p,q,r,s}$ is locally given by (7) then

$$d''^v\omega=\sum\frac{\partial\omega_{i_1\ldots i_p j_1\ldots j_q h_1\ldots h_r k_1\ldots k_s}}{\partial\overline{\eta}^i}\delta\overline{\eta}^i\wedge dz^{i_1}\wedge\ldots..\wedge dz^{i_p}$$

$$\wedge\delta\eta^{j_1}\wedge\ldots..\wedge\delta\eta^{j_q}\wedge d\overline{z}^{h_1}\wedge\ldots..d\overline{z}^{h_r}\wedge\delta\overline{\eta}^{k_1}\wedge\ldots.\delta\overline{\eta}^{k_s} \tag{18}$$

where the sum is after the indices $i_1\prec\ldots\prec i_p$; $j_1\prec\ldots\prec j_q$; $h_1\prec\ldots\prec h_r$; $k_1\prec\ldots\prec k_s$.

We know ([8],proposition 1.1) that the local bases $\{\frac{\partial}{\partial\overline{\eta}^i}\}$, $\{\frac{\partial}{\partial\overline{\eta}'^i}\}$ of $\overline{V(T'M)}$, corresponding to a change of complex coordinates $\{z^i,\eta^i\}\to\{z'^i,\eta'^i\}$ on $T'M$, are related by

$$\frac{\partial}{\partial\overline{\eta}^i}=\frac{\partial\overline{z}'^j}{\partial\overline{z}^i}\frac{\partial}{\partial\overline{\eta}'^i} \tag{19}$$

The formulas (19) prove that if f is a complex valued differentiable function defined on $T'M$ then the condition

$$\frac{\partial f}{\partial\overline{\eta}^i}=0 \quad ; \quad i=1,2\ldots n \tag{20}$$

is independent with respect to this change. Moreover, the of form $\omega \in F^{p,q,r,0}$ is d''^{v}−closed (i.e. $d''^{v}\omega = 0$) if and only if its local components satisfy the conditions (20). We denote by $\Phi^{p.q.r}$ the sheaf of germs of these forms.

Another property of the operator d''^{v} is a Grothendieck-Dolbeault type lemma, namely

Theorem 1 *Let ω be a d''^{v}-closed (p,q,r,s)−form defined on a neighborhood U on $T'M$ and $s \geq 1$. Then there exists a $(p,q,r,s-1)$−form θ defined on some neighborhood $U' \subset U$ and such that $d''^{v}\theta = \omega$ on U'.*

Proof. We use an argument inspired by the paper [12]. Let be h the index defined by the condition that the form ω, given by (7), does not contains $d\overline{z}^{h+1}, \ldots.., d\overline{z}^{n}$. We shall prove the assertion by using the induction on h.

For $h = 0$ we have

$\omega = \omega_{i_1 \ldots i_p j_1 \ldots j_q k_1 \ldots k_s} dz^{i_1} \wedge \ldots.. \wedge dz^{i_p} \wedge \delta\eta^{j_1} \wedge \ldots.. \wedge \delta\eta^{j_q} \wedge \delta\overline{\eta}^{k_1} \wedge \ldots.\delta\overline{\eta}^{k_s}$

and then

$d''^{v}\omega = d''\omega$ (modulo the therms containing $d\overline{z}^{1}, \ldots., d\overline{z}^{n}$).

But $d''^{v}\omega = 0$ and if we consider $\overline{z}^{1}, \ldots., \overline{z}^{n}$ like parameters, then the Grothendieck-Dolbeault lemma can be applied and therefore there exists a $(p,q,0,s-1)$−form θ with the property

$\omega = d''\theta$ (*modulo* $d\overline{z}^{1}, \ldots., d\overline{z}^{n}$)

on some neighborhood $U' \subset U$. Hence we have

$$\omega = d''\theta + \sum_{i=1,n} \lambda_i \wedge d\overline{z}^{i} \tag{21}$$

where λ_i are $(p,q,0,s-1)$−forms. Now because $\omega \in F^{p,q,0,s}$, from (21) we obtain $\omega = d''^{v}\theta$ and the assertion is proved for $h = 0$.

We assume the result valid for the indices $h_0 \leq h-1$ and we prove it for h, i.e. for (p,q,r,s)−forms which do not contain $d\overline{z}^{h+1}, \ldots, d\overline{z}^{n}$. Such a form is expressed as follows

$$\omega = d\overline{z}^{h} \wedge \alpha + \beta \tag{22}$$

where $\alpha \in F^{p,q,r-1,s}$, $\beta \in F^{p,q,r,s}$ and do not contain $d\overline{z}^h, d\overline{z}^{h+1}, ..., d\overline{z}^n$. Therefore by using (17) we have

$$d''^v\omega = -d\overline{z}^h \wedge d''^v\alpha + d''^v\beta = 0$$

hence

$$d''^v\alpha = 0 \;\; ; \;\; d''^v\beta = 0$$

and by applying the induction hypothesis it follows that on some neighborhood $U' \subset U$ there are two forms $\alpha^* \in F^{p,q,r-1,s-1}$, $\beta^* \in F^{p,q,r,s-1}$ such that

$$\alpha = d''^v\alpha^* \quad ; \quad \beta = d''^v\beta^* \tag{23}$$

Now, from (22),(23) and taking into account (17) we obtain

$$\omega = d''^v(-d\overline{z}^h \wedge \alpha^* + \beta^*)$$

Q.E.D.

Let $\mathcal{F}^{p,q,r,s}$ be the sheaf of germs of $(p,q,r,s)-$forms and we denote by $i : \Phi^{p,q,r} \to \mathcal{F}^{p,q,r,0}$ the natural inclusion. The sheaves $\mathcal{F}^{p,q,r,s}$ are fine and taking into account the Theorem 1. it follows that the sequence of sheaves

$$0 \to \Phi^{p,q,r} \xrightarrow{i} \mathcal{F}^{p,q,r,0} \xrightarrow{d''^v} \mathcal{F}^{p,q,r,1} \ldots\ldots \xrightarrow{d''^v} \mathcal{F}^{p,q,r,s} \xrightarrow{d''^v} \mathcal{F}^{p,q,r,s+1} \ldots\ldots$$

is a fine resolution of $\Phi^{p,q,r}$ and we denote by $H^s(M, \Phi^{p,q,r})$ the cohomology groups of M with coefficients in the sheaf $\Phi^{p,q,r}$, called *$v-$cohomology groups of M*. then we obtain a de Rham type theorem, namely

Theorem 2. *The $v-$cohomology groups of the complex Finsler manifold M are given by*

$$H^s(M, \Phi^{p,q,r}) \approx Z^{p,q,r,s} \;/\; d''^v F^{p,q,r,s-1}(T'M)$$

where $Z^{p,q,r,s}$ is the space of d''^v- closed $(p,q,r,s)-$forms globally defined on $T'M$.

References

[1] M.Abate, G.Patrizio, *Finsler Metrics-A Global Approach*, Lecture Notes in Math.,1591,Springer-Verlag, 1994.

[2] T.Aikou, *On Complex Finsler Manifolds,* Rep.of. Kagoshima Univ. No.24, 1991, p. 9-25,

[3] G.Gheorghiev, V.Oproiu, *Varietati Diferentiabile Finit si Infinit Dimensionale,* Ed.Acad. Romane, Vol.2,1979,

[4] M.Fukui,*Complex Finsler manifolds*, J.Math.Kyoto Univ. 29-4(1987), p.609-620,

[5] S.Kobayashi,*Negative vector bundles and complex Finsler structures*, Nagoya Math.J.,57(1975),p.153-166,

[6] S.Kobayashi, H.Wu, *Complex Differential Geometry,* DMV Seminar Band 3,Birkhauser Verlag,Basel-Boston,1983,

[7] R.Miron, M.Anastasiei, *The Geometry of Lagrange Spaces; Theory and Applications,* Kluwer Acad.Publ.,no59, FTPH,1994,

[8] G.Munteanu, *Complex Lagrange Geometry*, Balkan J.of Geom.and its Appl., Vol 3, no1, 1998.

[9] G.Pitis, *A class of invariants in Finsler geometry of vector bundles.* Proc.Romanian-Japanese Coll.on Finsler Geom., Univ.of Brasov ,1984,p.131-137.

[10] H.I.Royden, *Complex Finsler Spaces*,Contemporary Math. Am.Math.Soc., 49(1986), p.119-123.

[11] H.Rund, *The curvature theory of direction-dependent connection on complex manifolds.* Tensor N.S.,24,(1972),p.182-188.

[12] I.Vaisman, *Varietes riemanniennes feuilletees.* Czechoslovak Math.J., 21, (96)(1971),p.46-75.