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THE ASSOCIATED LOCUS OF SOME HYPERSURFACES IN $\mathbf{R}^{n+1}$

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. For a smooth hypersurface of the space $\mathbf{R}^{n+1}$ project orthogonally the origin of $\mathbf{R}^{n+1}$ on its tangent hyperplanes and call the set of all projections *the associated locus* of the given hypersurface. In this paper we are going to find the equation of the associated locus for some given hypersurfaces and to show that it is a smooth hypersurface diffeomorphic with the initial one. We will also show that in one particular case both of them, the hypersurface and its associated locus, are diffeomorphic with the n-dimensional sphere.

1. Introduction

In this section we recall a simple fact concerning homogeneous functions which will be very useful for all over this paper.

Definition 1.1. A function $f : \mathbf{R}^{n+1}\setminus\{0\} \to \mathbf{R}$ is called *homogeneous of order* $\alpha \in \mathbf{R}$ if $f(tx) = t^{\alpha} f(x)$ for all $t > 0$ and all $x \in \mathbf{R}^{n+1}\setminus\{0\}$.

Lemma 1.2. *If* $f : \mathbf{R}^{n+1}\setminus\{0\} \to \mathbf{R}$ *is a smooth homogeneous function of order* $\alpha \in \mathbf{R}^*$ *and* $c \in \mathbf{R}^*$, *then* $f^{-1}(c)$ *is either the empty set, or* $f^{-1}(c)$ *is a smooth hypersurface of* $\mathbf{R}^{n+1}$.

Example 1.3. Let α be a natural number, $\beta \in \{1, \dots, n+1\}$ and $a = (a_1, \dots, a_{n+1}) \in \mathbf{R}^{n+1}$ be such that $a_i \neq 0\ \forall\, i \in \{1, \dots, n+1\}$. Then the set

$$H_{\beta}^{\alpha} = \left\{x = (x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1}\setminus\{0\} \,\middle|\, \frac{x_1^{2\alpha}}{a_1^{2\alpha}} + \dots + \frac{x_{\beta}^{2\alpha}}{a_{\beta}^{2\alpha}} - \frac{x_{\beta+1}^{2\alpha}}{a_{\beta+1}^{2\alpha}} - \dots - \frac{x_{n+1}^{2\alpha}}{a_{n+1}^{2\alpha}} = 1\right\}$$

is a hypersurface of $\mathbf{R}^{n+1}$.

Observe that H_β^α can be also represented as $H_\beta^\alpha = f^{-1}(1)$, where

$$f : \mathbf{R}^{n+1}\setminus\{0\} \to \mathbf{R},\ f(x_1, \ldots, x_{n+1}) = \frac{x_1^{2\alpha}}{a_1^{2\alpha}} + \cdots + \frac{x_\beta^{2\alpha}}{a_\beta^{2\alpha}} - \frac{x_{\beta+1}^{2\alpha}}{a_{\beta+1}^{2\alpha}} - \cdots - \frac{x_{n+1}^{2\alpha}}{a_{n+1}^{2\alpha}}.$$

2. The associated locus of the hypersurface H_β^α

Let $x = (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$ and $\alpha \in \mathbf{R}$ be such that x_i^α exist for all $i \in \{1, \ldots, n+1\}$. Denote by x^α the vector $(x_1^\alpha, \ldots, x_{n+1}^\alpha)$ and observe that $x^2 = ||x||^2$ for all $x \in \mathbf{R}^{n+1}$ and that $(tx)^\alpha = t^\alpha x^\alpha$ for all $t > 0$. Also, if there exist the vectors $x^{\alpha\beta}$ and $(x^\alpha)^\beta$, for the real numbers α, β, then $x^{\alpha\beta} = (x^\alpha)^\beta$. Using this notation the equation of H_β^α can be rewritten as follows:

$$H_\beta^\alpha : \varphi(a^{-2\alpha}, x^{2\alpha}) = 1 \tag{1}$$

where $\varphi : \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} \to \mathbf{R}$ is the nondegenerate biliniar symmetric form given by

$$\varphi(x, y) = x_1y_1 + \cdots + x_\beta y_\beta - x_{\beta+1}y_{\beta+1} - \cdots - x_{n+1}y_{n+1}$$

for all $x = (x_1, \ldots, x_{n+1}), y = (y_1, \ldots, y_{n+1}) \in \mathbf{R}^{n+1}$.

Theorem 2.1. *The associated locus $\mathcal{L}_\beta^\alpha$ of H_β^α is the set*

$$\Big\{x \in \mathbf{R}^{n+1}\setminus\{0\} \ \Big|\ ||x||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}})\Big\}.$$

Proof. Denote by A_β^α the set

$$\Big\{x \in \mathbf{R}^{n+1}\setminus\{0\} \ \Big|\ ||x||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}})\Big\}$$

and consider $p^0 = (p_1^0, \ldots, p_{n+1}^0) \in H_\beta^\alpha$. The tangent hyperplane $T_{p^0}(H_\beta^\alpha)$ of H_β^α at p^0 has the following equation:

$$T_{p^0}(H_\beta^\alpha) : \sum_{i+1}^{n+1} \frac{\partial f}{\partial x_i}(p^0)(x_i - p_i^0) = 0, \text{ or, equivalently}$$

$$T_{p^0}(H_\beta^\alpha) : \sum_{i+1}^{\beta} \frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}} x_i - \sum_{i=\beta+1}^{n+1} \frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}} x_i = 1. \tag{2}$$

The parametric equations of the straight line passing through $0 \in \mathbf{R}^{n+1}$ which is orthogonal on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$ are:

$$\begin{cases} x_i = t\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, & i \in \{1,\dots,\beta\} \\ \\ x_i = -t\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, & i \in \{\beta+1,\dots,n+1\}. \end{cases} \tag{3}$$

To find the orthogonal projections of $0 \in \mathbf{R}^{n+1}$ on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$, replace the x_i, $i \in \{1,\dots,n+1\}$ from equations (3) in the equation (2) and we get:

$$\sum_{i+1}^{n+1} t\frac{(p_i^0)^{4\alpha-2}}{a_i^{4\alpha}} = 1, \text{ that is }, t = \frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle},$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbf{R}^{n+1}$. Hence, the orthogonal projection $q^0 \in \mathcal{L}^\alpha_\beta$ of $0 \in \mathbf{R}^{n+1}$ on the tangent hyperplane $T_{p^0}(H^\alpha_\beta)$ has the following coordinates

$$\begin{cases} x_i = \frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, & i \in \{1,\dots,\beta\} \\ \\ x_i = -\frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\frac{(p_i^0)^{2\alpha-1}}{a_i^{2\alpha}}, & i \in \{\beta+1,\dots,n+1\}. \end{cases} \tag{4}$$

Therefore, on the one hand, we have

$$||q^0||^2 = \sum_{i=1}^{n+1} x_i^2 = \frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle^2} \sum_{i=1}^{n+1} \frac{(p_i^0)^{4\alpha-2}}{a_i^{4\alpha}} = \frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle},$$

and on the other hand

$$\begin{cases} a_i x_i = \frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\left(\frac{p_i^0}{a_i}\right)^{2\alpha-1}, & i \in \{1,\dots,\beta\} \\ \\ a_i x_i = -\frac{1}{\langle a^{-4\alpha},(p^0)^{4\alpha-2}\rangle}\left(\frac{p_i^0}{a_i}\right)^{2\alpha-1}, & i \in \{\beta+1,\dots,n+1\}. \end{cases} \tag{5}$$

From relations (5) it follows

$$\varphi(a^{\frac{2\alpha}{2\alpha-1}}, (q^0)^{\frac{2\alpha}{2\alpha-1}}) = \sum_{i=1}^{\beta}(a_i x_i)^{\frac{2\alpha}{2\alpha-1}} - \sum_{i=\beta+1}^{n+1}(a_i x_i)^{\frac{2\alpha}{2\alpha-1}} =$$

$$= \left(\frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle}\right)^{\frac{2\alpha}{2\alpha-1}} \varphi(a^{-2\alpha}, (p^0)^{2\alpha}) = \left(\frac{1}{\langle a^{-4\alpha}, (p^0)^{4\alpha-2}\rangle}\right)^{\frac{2\alpha}{2\alpha-1}},$$

namely

$$||q^0||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, (q^0)^{\frac{2\alpha}{2\alpha-1}}),$$

that is $q^0 \in A^{\alpha}_{\beta}$ and we just showed that $\mathcal{L}^{\alpha}_{\beta} \subseteq A^{\alpha}_{\beta}$. To prove the other inclusion, consider $x^0 = (x^0_1, \dots, x^0_{n+1}) \in A^{\alpha}_{\beta}$, that is,

$$||x^0||^{\frac{4\alpha}{2\alpha-1}} = \varphi(a^{\frac{2\alpha}{2\alpha-1}}, (x^0)^{\frac{2\alpha}{2\alpha-1}}).$$

It easy to verify that x^0 is the orthogonal projection of the origin $0 \in \mathbf{R}^{n+1}$ on the tangent hyperplane $T_{p^0}(H^{\alpha}_{\beta})$, where $p^0 = (p^0_1, \dots, p^0_{n+1})$ and its components are given by:

$$\begin{cases} p^0_i = (a_i)^{\frac{2\alpha}{2\alpha-1}} \left(\frac{x^0_i}{||x^0||^2}\right)^{\frac{1}{2\alpha-1}}, \ i \in \{1, \dots, \beta\} \\ \\ p^0_i = -(a_i)^{\frac{2\alpha}{2\alpha-1}} \left(\frac{x^0_i}{||x^0||^2}\right)^{\frac{1}{2\alpha-1}}, \ i \in \{\beta+1, \dots, n+1\}, \end{cases} \tag{6}$$

and the theorem is completely proved.□

Let us mention that the associated locus of an ellipsoid appears in [Ca, pp. 90,91] as an exercise.

Theorem 2.2. *The associated locus $\mathcal{L}^{\alpha}_{\beta}$ of H^{α}_{β} is a smooth hypersurface of* $\mathbf{R}^{n+1}$

Proof. According to theorem 2.1 we have succesively

$$\mathcal{L}^{\alpha}_{\beta} = \left\{x \in \mathbf{R}^{n+1}\setminus\{0\} \,\middle|\, ||x||^{\frac{4\alpha}{2\alpha-1}} = \varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\right)\right\} =$$

$$= \left\{x \in \mathbf{R}^{n+1}\setminus\{0\} \,\middle|\, \frac{\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\right)}{||x||^{\frac{4\alpha}{2\alpha-1}}} = 1\right\} = g^{-1}(1),$$

where

$$g : \mathbf{R}^{n+1}\setminus\{0\} \to \mathbf{R},\ g(x) = \frac{\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\right)}{||x||^{\frac{4\alpha}{2\alpha-1}}}.$$

For $t > 0$ and $x \in \mathbf{R}^{n+1}\setminus\{0\}$ we have:

$$g(tx) = \frac{\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, (tx)^{\frac{2\alpha}{2\alpha 1}}\right)}{||tx||^{\frac{4\alpha}{2\alpha-1}}} = \frac{\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, t^{\frac{2\alpha}{2\alpha-1}} \cdot x^{\frac{2\alpha}{2\alpha-1}}\right)}{t^{\frac{4\alpha}{2\alpha-1}}||x||^{\frac{4\alpha}{2\alpha-1}}} =$$

$$= \frac{t^{\frac{2\alpha}{2\alpha-1}}\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\right)}{t^{\frac{4\alpha}{2\alpha-1}}||x||^{\frac{4\alpha}{2\alpha-1}}} = t^{\frac{2\alpha}{1-2\alpha}} \cdot \frac{\varphi\left(a^{\frac{4\alpha}{2\alpha-1}}, x^{\frac{4\alpha}{2\alpha-1}}\right)}{||x||^{\frac{4\alpha}{2\alpha-1}}} = t^{\frac{2\alpha}{1-2\alpha}} \cdot g(x).$$

Therefore g is a smooth homogeneous function of order $\frac{2\alpha}{1-2\alpha}$. Because $(a_1, 0, \dots, 0) \in g^{-1}(1)$, it follows that $g^{-1}(1) \neq \emptyset$, that is, according to lemma 1.2, $\mathcal{L}^{\alpha}_{\beta} = g^{-1}(1)$ is a smooth hypersurface of $\mathbf{R}^{n+1}$.□

The hypersurface H^{α}_{n+1} and its associated locus $\mathcal{L}^{\alpha}_{n+1}$ will be simply denoted by H^{α} and $\mathcal{L}^{\alpha}$ respectively. The equation of H^{α} is:

$$H^{\alpha} : \langle a^{-\alpha}, x^{2\alpha}\rangle = 1.$$

Corollary 2.3. *The associated locus of H^{α} is given by*

$$\mathcal{L}^{\alpha} = \left\{x \in \mathbf{R}^{n+1}\setminus\{0\} \,\middle|\, ||x||^{\frac{4\alpha}{2\alpha-1}} = \langle a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\rangle\right\}$$

and it is a smooth hypersurface of $\mathbf{R}^{n+1}$.

3. The diffeomorphism between H^{α}_{β} and $\mathcal{L}^{\alpha}_{\beta}$

Theorem 3.1. *The mappings $\chi : H^{\alpha}_{\beta} \to \mathcal{L}^{\alpha}_{\beta}$, $\chi_1 : \mathcal{L}^{\alpha}_{\beta} \to H^{\alpha}_{\beta}$ given by*

$$\chi(x) = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle}\left(\frac{x_1^{2\alpha-1}}{a_1^{2\alpha}}, \dots, \frac{x_{\beta}^{2\alpha-1}}{a_{\beta}^{2\alpha}}, -\frac{x_{\beta+1}^{2\alpha-1}}{a_{\beta+1}^{2\alpha}}, \dots, -\frac{x_{n+1}^{2\alpha-1}}{a_{n+1}^{2\alpha}}\right),\ x = (x_1, \dots, x_{n+1}) \in H^{\alpha}_{\beta}$$

$$\chi_1(x) = \frac{1}{||x||^{\frac{2}{2\alpha-1}}}\left(a_1^{\frac{2\alpha}{2\alpha-1}} \cdot x_1^{\frac{1}{2\alpha-1}}, \dots, a_{\beta}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{\beta}^{\frac{1}{2\alpha-1}}, -a_{\beta+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{\beta+1}^{\frac{1}{2\alpha-1}}, \dots, -a_{n+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{n+1}^{\frac{1}{2\alpha-1}}\right),$$

for all $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^{\alpha}_{\beta}$, are well defined and they are inverse to each other.

Proof. Indeed on the one hand, for $x = (x_1, \dots, x_{n+1}) \in H^{\alpha}_{\beta}$, we have:

$$||\chi(x)||^{\frac{4\alpha}{2\alpha-1}} = \left(||\chi(x)||^2\right)^{\frac{2\alpha}{2\alpha-1}} = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{4\alpha}{2\alpha-1}}}\left(\sum_{i=1}^{n+1}\frac{x_i^{4\alpha-2}}{a_i^{4\alpha}}\right)^{\frac{2\alpha}{2\alpha-1}} =$$

$$= \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{4\alpha}{2\alpha-1}}}\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2\alpha}{2\alpha-1}} = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2\alpha}{2\alpha-1}}},$$

and on the other hand,

$$\varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, \chi(x)^{\frac{2\alpha}{2\alpha-1}}\right) = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2\alpha}{2\alpha-1}}}\varphi\left(a^{\frac{2\alpha}{2\alpha-1}-\frac{4\alpha^2}{2\alpha-1}}, x^{2\alpha}\right) =$$

$$= \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2\alpha}{2\alpha-1}}}\varphi\left(a^{-2\alpha}, x^{2\alpha}\right) = \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2\alpha}{2\alpha-1}}}.$$

Therefore

$$||\chi(x)||^{\frac{4\alpha}{2\alpha-1}} = \varphi\left(a^{\frac{2\alpha}{2\alpha-1}}, \chi(x)^{\frac{2\alpha}{2\alpha-1}}\right) \text{ for all } x \in H^{\alpha}_{\beta},$$

that is $\chi(x) \in \mathcal{L}^{\alpha}_{\beta}$ for all $x \in H^{\alpha}_{\beta}$, which means that the mapping χ is well defined.

Analogously, for $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^\alpha_\beta$, we get:

$$\varphi(a^{-2\alpha}, \chi_1(x)^{2\alpha}) = \frac{1}{||x||^{\frac{4\alpha}{2\alpha-1}}}\varphi\big(a^{-2\alpha+\frac{4\alpha^2}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big) = \frac{1}{||x||^{\frac{4\alpha}{2\alpha-1}}}\varphi\big(a^{\frac{2\alpha}{2\alpha-1}}, x^{\frac{2\alpha}{2\alpha-1}}\big) = 1,$$

that is $\chi_1(x) \in H^\alpha_\beta$ for all $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^\alpha_\beta$, and the mapping χ_1 is well defined.

For $x = (x_1, \dots, x_{n+1}) \in \mathcal{L}^\alpha_\beta$ we also have:

$$(\chi \circ \chi_1)(x) = \chi(\chi_1(x)) =$$

$$= \chi\Big(\frac{1}{||x||^{\frac{2}{2\alpha-1}}}\Big(a_1^{\frac{2\alpha}{2\alpha-1}} \cdot x_1^{\frac{1}{2\alpha-1}}, \dots, a_\beta^{\frac{2\alpha}{2\alpha-1}} \cdot x_\beta^{\frac{1}{2\alpha-1}}, -a_{\beta+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{\beta+1}^{\frac{1}{2\alpha-1}}, \dots, -a_{n+1}^{\frac{2\alpha}{2\alpha-1}} \cdot x_{n+1}^{\frac{1}{2\alpha-1}}\Big)\Big) =$$

$$= \frac{1}{\frac{1}{||x||^{\frac{8\alpha-4}{2\alpha-1}}}\langle a^{-4\alpha+4\alpha}, x^2\rangle} \cdot \frac{1}{||x||^2}(x_1, \dots, x_{n+1}) = x = id_{\mathcal{L}^\alpha_\beta}(x).$$

On the other hand, for $x = (x_1, \dots, x_{n+1}) \in H^\alpha_\beta$, we have

$$(\chi_1 \circ \chi)(x) = \chi_1(\chi(x)) = \chi\Big(\frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle}\Big(\frac{x_1^{2\alpha-1}}{a_1^{2\alpha}}, \dots, \frac{x_\beta^{2\alpha-1}}{a_\beta^{2\alpha}}, -\frac{x_{\beta+1}^{2\alpha-1}}{a_{\beta+1}^{2\alpha}}, \dots, -\frac{x_{n+1}^{2\alpha-1}}{a_{n+1}^{2\alpha}}\Big)\Big) =$$

$$= \frac{1}{\frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{2}{2\alpha-1}}}\Big(\sum_{i=1}^{n+1}\frac{x_i^{4\alpha-2}}{a_i^{4\alpha}}\Big)^{\frac{1}{2\alpha-1}}} \cdot \frac{1}{\langle a^{-4\alpha}, x^{4\alpha-2}\rangle^{\frac{1}{2\alpha-1}}}(x_1, \dots, x_{n+1}) = x = id_{H^\alpha_\beta}(x). \square$$

Corollary 3.2. *The mappings χ and χ_1 are diffeomorphisms between H^α_β and $\mathcal{L}^\alpha_\beta$.*

The next theorem can be proved in a completely analogous way.

Theorem 3.3. *The mapping $h : H^\alpha \to S^n$, $h(x) = \frac{x}{||x||}$ is a diffeomorphism and $h^{-1} : S^n \to H^\alpha$ is given by $h^{-1}(x) = \frac{x}{\langle a^{-2\alpha}, x^{2\alpha}\rangle^{1/2\alpha}}$.*

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