STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIII, Number 3, September 1998

A NOTE ON MINIMAX RESULTS FOR CONTINUOUS FUNCTIONALS

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Dedicated to Professor Pavel Enghiş at his 70^{th} anniversary

Abstract. In this paper we extend the Willem deformation lemma for continuous functionals and we traite also the equivariant case. With the aid of this results we extend the min-max results of Ghoussoub [21]. As application we give an another proof of some multiplicity results of Corvellec [5] and we give some multiplicity results for continuous functionals which contains a large class of multiplicity results for differentiable and locally Lipschitz functionals.

1. Introduction.

In many papers is studied the critical point theory for continuous functionals, see [3], [4], [5], [2], [?] and [8]. In this paper using some results from the paper of J.-N. Corvellec, M. Degiovanni and M. Marzocchi [4] we prove the Willem deformation lemma for continuous functionals. We traite also the equivariant case. With the aid of this results we give a simplified proof and generalize some min-max results of Ghoussoub [21], Fang [6], and Ribarska-Tsachev-Krastanov [9]. Using this result we give some multiplicity results of Ghoussoub [21], which represent an another proof of some multiplicity results of Corvellec [5]. As applications for different topological index we give some minmax and multiplicity results for continuous functionals, which represent generalizations for well known results, see Fadell [19], Santos [33], Chang [16], Marzocchi [28], Goeleven-Motreanu-Panagiotoulos [23], Mironescu-Radulescu [32] and another results.

1991 *Mathematics Subject Classification.* 49J40, 58E46.
Key words and phrases. nonsmooth critical point theory, symmetric functionals.

First we recall some definitions and result from the paper of M. Degiovanni and M. Marzocchi, see [3].

Definition 1.1. Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function and $u \in X$ a fixed element. We denote by $|df|(u)$ the supremum of the $\sigma \in [0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B(u,\delta) \times [0,\delta] \to \mathbb{R}$$

such that $\forall v \in B(u,\delta)$ *for all* $t \in [0,\delta]$ we have

a) $d(\mathcal{H}(v,t),v) \leq t$

b) $f(\mathcal{H}(v,t)) \leq f(v) - \sigma t$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

Definition 1.2. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function. We define the function

$$\mathcal{G}_f : epi(f) \to \mathbb{R}$$

putting

$$epi(f) = \{(u,\xi) \in X \times \mathbb{R} : f(u) \leq \xi\} \quad and \quad \mathcal{G}_f(u,\xi) = \xi.$$

In the following $epi(f)$ will be endowed with the metric

$$d_{ep}((u,\xi),(v,\mu)) = (d(u,v)^2 + (\xi - \mu)^2)^{\frac{1}{2}}.$$

Of course $epi(f)$ is closed in $X \times \mathbb{R}$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Consequently $|d\mathcal{G}_f|(u,\xi) \leq 1$ for every $(u,\xi) \in epi(f)$.

Proposition 1.3. *Let $f : X \to \mathbb{R}$ be a continuous function and let $(u,\xi) \in epi(f)$. Then*

$$|\mathcal{G}_f|(u,\xi) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1+|df|(u)^2}}, & if \;\; f(u) = \xi and \;\; |df|(u) < \infty, \\ 1 & if \;\; f(u) < \xi or \;\; |df|(u) = \infty. \end{cases}$$

We recall a basic result from [4].

Theorem 1.4. *(Theorem 2.11, [4]) Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function, C a closed subset of X and $\delta, \sigma > 0$ such that*

$$d(u, C) \leq \delta \Longrightarrow |df|(u) > \sigma.$$

Then there exists a continuous map $\eta : X \times [0, \delta] \to X$ such that

1. $d(\eta(u, t), u)) \leq t$,
2. $f(\eta(u, t)) \leq f(u)$,
3. $d(u, C) \geq \delta \Longrightarrow \eta(u, t) = u$,
4. $u \in C \Longrightarrow f(\eta(u, t)) \leq f(u) - \sigma t$.

In the following, for every $c \in \mathbb{R}$ we use the next notations:

$$K_c(f) = \{x \in X : |df|(x) = 0 \text{ and } f(x) = c\};$$

$$f^c = \{x \in X : f(x) \leq c\} :$$

$$f_c = \{x \in X : f(x) \geq c\}.$$

2. Willem deformation lemma

In this section we extend the Willem deformation lemma for continuous functionals.

Theorem 2.1. *Let (X, d) be a complete metric space, $f : X \to \mathbb{R}$ a continuous function, C a closed subset of X and $c \in \mathbb{R}$ a real number. Let ε and $\delta > 0$ two number such that we have:*

$$\forall \;\; u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta} : \;\; \text{we have } |df|(u) > \varepsilon. \tag{2.1}$$

Then there exists two real numbers $\varepsilon' \in (0, \varepsilon)$ and $\lambda > 0$ and a continuous map $\eta : X \times [0, 1] \to X$ such that:

a) $d(\eta(u, t), u) \leq \lambda t$,

b) $f(\eta(u, t)) \leq f(u)$,

c) *if* $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta} : \eta(u, t) = u, \;\; \forall \;\; t \in [0, 1]$

d) $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$,

e) $\forall \; t \in]0, 1]$ *and* $\forall \; u \in f^c \cap C$ *we have* $f(\eta(t, u)) < c$.

Proof. First, we suppose that the function $f : X \to \mathbb{R}$ is Lipschitz continuous with constant 1. We consider the set:

$$C^* := \{u \in X | c - t_1 \leq f(u) \leq c + t_1, d(u, C) \leq 2\delta - \delta_1\}, \tag{2.2}$$

where $\delta_1 + t_1 < 2\varepsilon, \delta_1 < 2\delta$ and $\delta_1, t_1 > 0$, for example $\delta_1 := \min\{\varepsilon, \delta\}$ and $t_1 = \frac{\varepsilon}{2}$. Obvious the set C^* is a closed subset of X. We observe that from the relation $d(u, C^*) \leq \delta_1$ we get:

$$u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon) \cap C_{2\delta}. \tag{2.3}$$

Indeed, because $|f(v) - f(u)| \leq 1 \cdot d(u, v)$ for $\forall u, v \in X$ we obtain

$$-d(u, v) \leq f(u) - f(v) \leq d(u, v), \quad \forall v \in C^*.$$

Using this relation and the fact $d(u, C^*) \leq \delta_1$ we get

$$c - (t_1 + \delta_1 \leq f(u) \leq c + (t_1 + \delta_1).$$

Because $\delta_1 + t_1 < 2\varepsilon$ we obtain $u \in f^{-1}([c - 2\varepsilon, c + 2\varepsilon])$. It is easy to verify that $d(u, C^*) \leq \delta_1$ implies $u \in C_{2\delta}$.

Because $\varepsilon > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ from the relation (2.3) we obtain $|df|(u) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$. Now we can apply Proposition 2.4, for $C^*, \delta_1 \quad \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$ and we get a continuous function $\eta' : X \times [0, \delta] \to \mathbb{R}$ which satisfied the conditions 1)-4) from Theorem 2.4. Without loss of generality, we assume that $\lambda = \delta_1$, and define the function $\eta : X \times [0, 1] \to \mathbb{R}$ by $\eta(u, t) = \eta'(u, \lambda t)$. The properties a) and b) are obvious. Let $u \notin f^{-1}([c - 2\varepsilon, c + 2\varepsilon]) \cap C_{2\delta}$ since f is a Lipschitz function with constant 1, we have $d(u, C^*) > \delta_1$ and using Proposition 2.4, a) we get $\eta(u, t) = u$.

For the proof of d) let $\varepsilon' = \min\{t_1, \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\}$ and we distinguish two cases:

2.4) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) \geq c - \varepsilon'$ it follows that $u \in C^*$, hence we have

$$f(\eta(u, 1)) \leq f(u) - \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \leq c + \varepsilon - \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \leq c - \varepsilon'.$$

2.5) If $u \in f^{c+\varepsilon'} \cap C$ and $f(u) < c - \varepsilon'$, then from b) we get

$$f(\eta(u, t)) \leq f(u) < c - \varepsilon'.$$

The part e) of the theorem is proved in same way as d).

Now we consider the general case. For this let $C^{**} = \{(u,\xi) \in epi(f) | \; u \in C\}$. The set $epi(f)$ is closed in $X \times \mathbb{R}$ and it follow that $epi(f)$ is a complete metric space. In the next we prove that for every $(u,\xi) \in epi(f)$ with $(u,\xi) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$, we have $|d\mathcal{G}_f|(u,\xi) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.
We distinguish two cases:

I) Let $f(u) = \xi$. In this case we have two subcases.

a) $|df|(u) < \infty$. If $(u, f(u)) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$, then we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon])$ and $d_{ep}((u,f(u)), C^{**}) \leq 2\delta$. Because $d(u,C) \leq d_{ep}((u,f(u)), C^{**}) \leq 2\delta$ we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}$ and using the hypothesis of theorem follow $|df|(u) > \varepsilon$.

Because $|df|(u) < \infty$ from Proposition 1.3 we have $|\mathcal{G}_f|(u, f(u)) = \frac{|df|(u)}{\sqrt{1+|df|^2(u)}}$ and using the fact that the function $x \mapsto \frac{x}{\sqrt{1+x^2}}$ is increasing we have $|d\mathcal{G}_f|(u, f(u)) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

b) If $|df|(u) = \infty$ using Proposition 1.3 we get $|d\mathcal{G}_f|(u, f(u)) = 1 > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

II) If $f(u) < \xi$, then from Proposition 1.3 we have $|d\mathcal{G}_f|(u, f(u)) = 1 > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

From these we get that for every $(u,\xi) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$ implies $|d\mathcal{G}_f|(u,\xi) > \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}$.

The set $A := \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**} \cap epi(f) \neq \emptyset$, because if $u \in C$, then $(u, f(u)) \in A$. We apply the previous step for $X := epi(f)$, $f := \mathcal{G}_f$ and $C := C^{**}$. Then there exists two positive numbers ε', $\lambda > 0$ and a continuous mapping $\overline{\eta} := (\overline{\eta_1}, \overline{\eta_2}) : epi(f) \times [0,1] \to epi(f)$ such that the following holds:

2.6) $d_{ep}((\overline{\eta}(u,\xi), t), (u,\xi)) \leq \lambda t, \quad \forall \; (u,\xi) \in epi(f), \forall \; t \in [0,1]$;

2.7) $\mathcal{G}_f(\overline{\eta}(u,\xi), t) = \overline{\eta}_2((u,\xi), t) \leq \xi = \mathcal{G}_f(u,\xi)$, for all$(u,\xi) \in epi(f)$, and $\forall \; t \in [0,1]$;

2.8) $\overline{\eta}((u,\xi), t) = (u,\xi)$ for every $(u,\xi) \in epi(f)$ with$(u,\xi) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$;

2.9) $\overline{\eta}(\mathcal{G}_f^{c+\varepsilon'} \cap C^{**}, 1) \subset \mathcal{G}_f^{c-\varepsilon'}$;

2.10) $f(\overline{\eta}((u,\xi), t) < c$ for every $t \in]0,1]$ and $\forall \; (u,\xi) \in \mathcal{G}_f^c \cap C^{**}$.

We define the function $\eta : X \times [0,1] \to X$ by

2.11) $\eta(u,t) = \overline{\eta}_1((u, f(u)), t)$.

Because $\overline{\eta}$ takes its values in $epi(f)$, we have

2.12) $f(\overline{\eta}_1((u, f(u)), t) \le \overline{\eta}_2((u, f(u)), t)$

From 2.6) we have:

$$d(\eta(u,t), u) = d((\overline{\eta}_1(u, f(u), t), u) \le$$

$$\le [d^2((\overline{\eta}_1(u, f(u)), t), u) + (\overline{\eta}_2((u, f(u)), t) - f(u))^2]^{\frac{1}{2}} =$$

$$= d_{ep}((\overline{\eta}(u, f(u)), t), (u, f(u)) \le \lambda t.$$

From the relations 2.7) and 2.12) we get

$$f(\eta(u,t)) = f(\overline{\eta}_1(u, f(u)), t) \le \overline{\eta}_2((u, f(u)), t) \le f(u).$$

If $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}$ then

2.13) $(u, f(u)) \notin \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$.

Now we assume that $(u, f(u)) \in \mathcal{G}_f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}^{**}$. From this follow that

2.14) $f(u) \in [c-2\varepsilon, c+2\varepsilon]$

and $(u, f(u)) \in C_{2\delta}^{**}$, which is equivalent with $d_{ep}((u, f(u)), C^{**}) \le 2\delta$. But we have $d(u, C) = \inf\{d(u,v) | v \in C\} \le \inf\{d_{ep}(u, f(u)), (v, \xi)) \mid (v,\xi) \in C^{**}\} = d_{ep}((u, f(u)), C^{**})$. From this and from 2.14) we get $u \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}$ which is a contradiction with assumption. If $u \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta}$, then from 2.8) we get $\eta(u,t) = \overline{\eta}_1((u, f(u)), t) = u$.

If $f(u) \le c + \varepsilon'$ then from 2.9) and 2.12) we get

$$f(\eta(u,1)) = f(\overline{\eta}_1(u, f(u)), 1) \le \overline{\eta}_2((u, f(u)), 1) \le c - \varepsilon'.$$

From 2.10) and 2.12) we get the relation e).

In the following we use the next form of Willem deformation theorem.

Corollary 2.2. *Let (X,d) be a complet metric space, $f : X \to \mathbb{R}$ a continuous function, C a closed subset of X and $c \in \mathbb{R}$ a real number. Let $\varepsilon > 0$ be a number such:*
$\forall\ x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$: *we have* $|df|(x) > \varepsilon$.
Then there exists two real numbers $\varepsilon' \in (0,\varepsilon)$ and $\lambda > 0$ and a continuous map $\eta : X \times [0,1] \to X$ such that:

a') $d(\eta(u,t),u) \leq \lambda t$, for every $t \in [0,1]$.
b') $f(\eta(u,t)) \leq f(u)$, for every $t \in [0,1]$ and $x \in X$.
c') if $x \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon} : \eta(x,t) = x, \quad \forall\ t \in [0,1]$.
d') $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$ with $\varepsilon' = \dfrac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$.
e') $\forall\ t \in]0,1]$ and $\forall\ x \in f^c \cap C$ we have $f(\eta(t,x)) < c$.

Proof. In the proof of Willem deformation lemma we take $\delta := \varepsilon$, $t_1 = \frac{\varepsilon}{2}$ and $\varepsilon' = \dfrac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$.

3. A minmax result

Definition 3.1. Let B a closed subset of M. We shall say that the class $\mathcal{F}$ of subsets of M is homotopy stable with boundary B if:

(a) Every set in $\mathcal{F}$ contains B;
(b) For any set $A \in \mathcal{F}$ and any continuous function $\eta \in \mathcal{C}([0,1] \times M, M)$ verifying $\eta(t,x) = x$ for all $(t,x) \in (\{0\} \times M) \cup ([0,1] \times B)$ we have $\eta(1,A) \in \mathcal{F}$.

Definition 3.2. We say that a set F is dual $\mathcal{F}$ if F verifies the following conditions:

1°) $\operatorname{dist}(F,B) > 0$;
2°) $F \cap A \neq \emptyset$ for all $A \in \mathcal{F}$.

Denote by $\mathcal{F}^\star$ a family of subsets which are dual to $\mathcal{F}$ and we say that $\mathcal{F}^\star$ is dual family to $\mathcal{F}$. We have the following relation

$$c^\star := \sup_{F \in \mathcal{F}^\star} \inf_{x \in F} f(x) \leq \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) =: c.$$

Examples:

3.1) Let K be a compact metric space, $K_0 \subset K$ a closed set, X a Banach space, $\chi \in \mathcal{C}(K,X)$. Then the set $\mathcal{F} = \{\, A = g(K) \,|\, g \in \mathcal{C}(K,X) \text{ with } g(K_0) = \chi(K_0) \,\}$ is a homotopy stable family wits boundary $B = \chi(K_0)$.

3.2) For each $n \in \mathbf{N}$ the families

$$\mathcal{F}_n = \{ A \mid A \subset X \text{with} \operatorname{cat}_X(A) \geq n \}$$

is a homotopy stable family, where $\operatorname{cat}_X(A)$ denote the Ljusternik-Schnirelmann category.

3.3) For each $n \in \mathbf{N}$ the families

$$\mathcal{F}_n = \{ A \mid Y \subset A \text{and} \operatorname{cat}_{(X,Y)}(A) \geq n \}$$

is a homotopy stable family with boundary Y, where $\operatorname{cat}_{(X,Y)}(A)$ denote the relative category, see [25].

3.4) We recall the definition of the P-ideal valued cohomological index. Let E be a paracompact space and $(X, A) \in \mathcal{E}_E$ whwre $\mathcal{E}_E$ is the category of paracompact pair (X, A) ion E for a fixed closed subset A of E. Let $H^*(,)$ be the Alexander-Spanier cohomology theory with a field coefficient K, see [34]. The cup product defines a multiplication on $H^*(X, A)$ as flows:

$$H^*(X, A) \otimes H^*(E) \stackrel{1 \otimes i^*}{\rightarrow} H^*(X, A) \otimes H^*(X) \rightarrow H^*(X, A),$$

where 1 is the identity on $H^*(X, A)$ and i is the inclusion map $X \stackrel{i}{\rightarrow} E$. Therefore , $H^*(X, A)$ is an $H^*(E)$ module. In particular, $H^*(A)$ is also an $H^*(E)$-module. We introduce the following notation: $\Lambda = H^*(E)$. For an $H^*(E)$-submodule P of $H^*(A)$ the P-ideal value cohomological index of (X, A) over K is an ideal denoted by

$$P - Index_E(X, A) = \{\lambda \in \Lambda \mid u \cdot \lambda = 0, \forall u \in M^*(X, A) \},$$

where $M^q(X, A) = \delta^q(P)$ for $q \geq -1$, $M^o(X, A) = \mathcal{E}(K)$, δ^* is the coboundary operator for the pair (X, A) and $\mathcal{E}$ is the augmentation. In the next we consider A and B two disjoint closed subsets of X. We say that A is P-ideal linking to B if and only if

$$P - Index_E(E \setminus B, A) \supset P - Index_E(E, A).$$

Let E be a connected paracompact space. We suppose that the following conditions holds:

1') There are two disjoint sets A and B such that A is P-ideal linking to $B, P \subset H^*(A)$;

2') There exists a closed set $\tilde{X} \supset A$ in E such that $\tilde{X} \setminus A$ is precompact and

$$P - Index_E(\tilde{X}, A) = P - Index_E(E, A).$$

We denote by $\alpha = P - Index_E(E, A)$ and $\beta = P - Index_E(E \setminus B, A)$. Since A is P-Ideal linking to B, we have $\beta \supset \alpha$ and $\beta \neq \alpha$. We define the set

$$\Sigma_\alpha = \{ (X, A) \in \mathcal{E}_E : P - Index_E(X, A) = \alpha \},$$

where $\mathcal{E}_E$ is the class of all paracompact pair (X, A) in$E$. Note that $\Sigma_\alpha \neq \emptyset$ since $(\tilde{X}, A) \in \Sigma_\alpha$. We prove that Σ_α is homotopy stable with boundary A. Let $(X, A) \in \Sigma_\alpha$ be a paracompact pair and $\eta \in \mathcal{C}([0, 1] \times E, E)$ a deformation such that $\eta(t, x) = x$ for all $(t, x) \in (\{0\} \times E) \cup ([0, 1] \times A)$. From the invariance property of the P-ideal valued index we get $\eta(1, X) \in \Sigma_\alpha$. Since A is P-Ideal linking to B, then for every $X \in \Sigma_\alpha$ we have $X \cap B \neq \emptyset$.

In the next we prove the main result of this section which generalize the main results from [21], [6] and [9].

Theorem 3.3. *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a continuous function. Consider a homotopy stable family $\mathcal{F}$ of subsets of X with boundary B and a dual family $\mathcal{F}^\star$ of $\mathcal{F}$. Let $F \in \mathcal{F}^\star$ be a fixed, element which verifies the following condition*

$$\inf_{x \in F} f(x) \geq c, \tag{3.1}$$

where $c := \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x)$.
Let $\varepsilon \in (0, \frac{dist(B,F)}{2})$ and $\delta > 0$ be arbitrarly fixed numbers. Then for any $A \in \mathcal{F}$

which verifies the relation

$$\sup_{x \in A \cap F_\delta} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}, \tag{3.2}$$

then there exists $x_\varepsilon \in X$ such that the following holds:

(i) $c - 2\varepsilon \le f(x_\varepsilon) \le c + 2\varepsilon$*;*

(ii) $|df|(x_\varepsilon) \le \varepsilon$*;*

(iii) $dist(x_\varepsilon, F) \le 2\varepsilon$*;*

(iv) $dist(x_\varepsilon, A) \le 2\varepsilon$*.*

Proof. From the definition of the number c, there exists a subset $A \subset X$ such that $\sup_{x \in A} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$. From this we get that for every $\delta > 0$ we have the following relation:

$$\sup_{x \in A \cap F_\delta} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}.$$

Using the fact that $dist(x, A) = dist(x, \overline{A})$, the assertions i)-iv) from theorem is equivalent with:

$$\text{exists an } \; x_\varepsilon \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{F}_{2\varepsilon} \cap \overline{A}_{2\varepsilon} \; \text{ such that } \; |df|(x_\varepsilon) \le \varepsilon.$$

We suppose the contrary, i.e.

$$\forall \;\; x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap \overline{F}_{2\varepsilon} \cap \overline{A}_{2\varepsilon} \quad \text{we have } \; |df|(x) > \varepsilon. \tag{3.3}$$

We consider the set $C := \overline{A \cap F}$ then we have $C_{2\varepsilon} = (\overline{F \cap A})_{2\varepsilon} \subset (\overline{F} \cap \overline{A})_{2\varepsilon} \subset \overline{F}_{2\varepsilon} \cap \overline{A}_{2\varepsilon}$. From the relation (3.3) we have the following implication;

$$\text{if } \; x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_\varepsilon \; \text{ then } \; |df|(x) > \varepsilon.$$

From Corollary 2.2 we have a continuous function $\eta : X \times [0,1] \to X$ and $\lambda > 0$ which satisfies the assertions a')-e') with $\lambda \le \min\{\varepsilon, \delta\}$, see the proof of Theorem 2.1. Let $A_1 = \eta(A, 1)$. If $x \in C_X C_{2\varepsilon}$, where $C_X C_{2\varepsilon}$ denote the complementary of the set $C_{2\varepsilon}$ in X, then from the propertie c') we have $\eta(x,t) = x$ for every $t \in [0,1]$. Using the fact that $dist(F, B) > 2\varepsilon$, we get $B \subset C_X C_{2\varepsilon}$, thus we have $B = \eta(B, 1)$. Because $\mathcal{F}$ is a homotopy stable family with boundary B, result that $B = \eta(B,1) \subset \eta(A,1) = A_1$, thus $A_1 \in \mathcal{F}$. We have the following relation $\eta(A,1) \cap F \subset \eta(A \cap F_\lambda, 1)$.

Indeed, if $x \in \eta(A,1) \cap F$, then there exists an $y \in A$ such that $x = \eta(y,1) \in F$. But $d(y, \eta(y,1)) \leq \lambda$, thus $y \in F_\lambda$. From the relation $y \in A \cap F_\lambda$ it follows that $x = \eta(y,1) \in \eta(A \cap F_\lambda, 1)$. From (3.2) we have

$$A \cap F_\lambda \subset f^{c+\varepsilon'}, \quad \text{with } \varepsilon' = \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}.$$

Indeed, since $\lambda \leq \delta$, we have $A \cap F_\lambda \subset A \cap F_\delta$. But $\sup_{x \in A \cap F_\delta} f(x) \leq c + \varepsilon'$ we get $A \cap F_\lambda \subset f^{c+\varepsilon'}$. From the properties d') we have

$$\eta(A \cap F_\lambda, 1) \subset \eta(f^{c+\frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}}, 1) \subset f^{c-\frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}}. \tag{3.5}$$

From the relations (3.4) and (3.5) we get $A_1 \cap F \subset f^{c-\frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}}$, which is equivalent to $f(x) \leq c - \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$ for every $x \in A_1 \cap F$. From this relation we get

$$\inf_{x \in F} f(x) \leq \inf_{x \in F \cap A_1} f(x) \leq c - \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}.$$

From the relation (3.1) we have $c \leq \inf_{x \in F} f(x) \leq c - \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$, which is a contradiction.

In the following we give a simplified proof for Theorem 1.5 from [6] without using the Ekeland's variational principle.

Corollary 3.4. *(Theorem 1.10,*[6]*) Let $f : X \to \mathbb{R}$ be a continuous functional on a complete metric space (X, d). We consider a homotopy stable family $\mathcal{F}$ of compact subsets of X with closed boundary B and a dual family $\mathcal{F}^*$ of $\mathcal{F}$. Assume that*

$$\sup_{F \in \mathcal{F}^*} \inf_{x \in F} f(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} f(x) = c$$

and suppose that the number c is finite.

Let $\varepsilon > 0$ and F a subset of X dual to the family $\mathcal{F}$ and satisfying the relation

$$\inf_{x \in F} f(x) \geq c - \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}.$$

Suppose that $0 < \varepsilon < \frac{dist(B,F)}{2}$, the for any set $A \in \mathcal{F}$ satisfying

$$\sup_{x \in A} f(x) \leq c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}},$$

then there exists an $x_\varepsilon \in X$ such that:

i) $c - 2\varepsilon \le f(x_\varepsilon) \le c + 2\varepsilon$;

ii) $|df|(x_\varepsilon) \le \varepsilon$;

iii) $dist(x_\varepsilon, F) \le 2\varepsilon$;

iv) $dist(x_\varepsilon, A) \le 2\varepsilon$.

Proof. From the assumption of theorem we see that

$$\sup_{F\in\mathcal{F}^*} \inf_{x\in F} f(x) = \inf_{A\in\mathcal{F}} \max_{x\in A} f(x) = c,$$

and exists $A \in \mathcal{F}$ and $F \in \mathcal{F}^*$ such that:

$$\sup_{x\in A} f(x) \le c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}},$$

$$\inf_{x\in F} f(x) \ge c - \frac{\varepsilon}{3\sqrt{1+\varepsilon^2}}.$$

The proof is same to proof of Theorem 3.1, if we choose $C := A \cap F$. We have that C is closed, because A is compact set and $A_1 = \eta(A, 1)$ is compact, because the function $x \mapsto \eta(x, 1)$ is continuous. Therefore we have $A_1 \in \mathcal{F}$.

Remark 3.5 If we choose different homotopy stable family we give different min-max results. For example if we choose the homotopy stable family $\mathcal{F} = \{\, A = g(K) \,|\, g \in \mathcal{C}(K, X) \text{ with } g(K_0) = \chi(K_0) \,\}$, where K is a compact metric space and $K_0 \subset K$ we obtain a generalization for continuous functionals of the Theorem 4.3, see [26].

4. Equivariant version of min-max result

In this section we give a generalization of some min-max and multiplicity results of Ghoussoub [21] for continuous functionals which represent an another proof of some min-max and multiplicity results of some results of Corvellec [5]. First we recall some definition and results from [3] and [5].

In this section (X, d) will denote a metric space and G a group of isometries of X, i.e.

$$G = \{g : X \to X \mid \ d(g(x), g(y)) = d(x, y), \ \text{ for all } \ x, y \in X \ \text{ and } \ g \in G\}.$$

As usual, we say that

$$A \subset X \ \text{ is G-invariant if } g(A) = A \ \text{ for all} g \in G;$$

$$h : X \to \mathbb{R} \ \text{ is G-invariant if } \ h \circ g = h \ \text{ for all } \ g \in G;$$

$$h : X \to X \ \text{ is G-equivariant if } \ h \circ g = g \circ h \ \text{ for all} g \in G.$$

Definition 4.1. Let (X, d) be a complete G- metric space and let $f :\to \mathbb{R}$ be a continuous, G-ivariant function and $u \in X$ a fixed element. We denote by $|df|_G(u)$ the supremum of the $\sigma \in [0, \infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B(Gu, \delta) \times [0, \delta] \to \mathbb{R}$$

such that $\forall v \in B(Gu, \delta)$ *for all* $t \in [0, \delta]$ we have

a) $\eta(., t)$ is G-invariant for each $t \in [0, \delta]$

b) $d(\mathcal{H}(v, t), v) \leq t$

c) $f(\mathcal{H}(v, t)) \leq f(v) - \sigma t$

The extended real number $|df|_G(u)$ is called *the G-weak slope* of f at u.

The epigraph function $\mathcal{G}_f$ is Lipschitz continuous of constant 1 and is G-invariant, because the function f is G-invariant.

Proposition 4.2. *Let* $f : X \to \mathbb{R}$ *be a continuous function, G-inariant and let* $(u, \xi) \in epi(f)$. *Then*

$$|\mathcal{G}_f|_G(u, \xi) = \begin{cases} \dfrac{|df|_G(u)}{\sqrt{1 + |df|_G(u)^2}}, & \text{if } f(u) = \xi \text{and } \ |df|_G(u) < \infty, \\ 1 & \text{if } f(u) < \xi \text{or } \ |df|_G(u) = \infty. \end{cases}$$

We recall a result from Corvellec [5].

Proposition 4.3. *Let* (X, d) *be a complete G- metric space and let* $f : X \to \mathbb{R}$ *be a continuous G-invariant function,* C *a closed G-invariant subset of* X *and* $\delta, \sigma > 0$ *such that*

$$d(u, C) \leq \delta \Longrightarrow |df|_G(u) > \sigma.$$

Then there exists a continuous G-equivariant map $\eta : X \times [0, \delta] \to X$ *such that*

1) $d(\eta(u,t),u)) \leq t$,

2) $f(\eta(u,t)) \leq f(u)$,

3) $d(u,C) \geq \delta \Longrightarrow \eta(u,t) = u$,

4) $u \in C \Longrightarrow f(\eta(u,t)) \leq f(u) - \sigma t$.

We have the following equivariant version of Willem deformation lemma.

Theorem 4.4. *Let* (X,d) *be a complet* G*-metric space,* $f : X \to \mathbb{R}$ *a continuous* G*-invariant function,* C *a closed* G*-invariant subset of* X *and* $c \in \mathbb{R}$ *a real number. Let* $\varepsilon > 0$ *number such:*

$\forall\ x \in f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\varepsilon}$: *we have* $|df|_G(x) > \varepsilon$.

Then there exists two real numbers $\varepsilon' \in (0,\varepsilon)$ *and* $\lambda > 0$ *and a continuous* G*-equivariant map* $\eta : X \times [0,1] \to X$ *such that:*

a') $d(\eta(u,t),u) \leq \lambda t$, *for every* $t \in [0,1]$.

b') $f(\eta(u,t)) \leq f(u)$, *for every* $t \in [0,1]$ *and* $x \in X$.

c') *if* $x \notin f^{-1}([c-2\varepsilon, c+2\varepsilon]) \cap C_{2\delta} : \eta(x,t) = x, \quad \forall\ t \in [0,1]$.

d') $\eta(f^{c+\varepsilon'} \cap C, 1) \subset f^{c-\varepsilon'}$ *with* $\varepsilon' = \dfrac{\varepsilon}{2\sqrt{1+\varepsilon^2}}$.

e') $\forall\ t \in]0,1]$ *and* $\forall\ x \in f^c \cap C$ *we have* $f(\eta(t,x)) < c$.

Definition 4.5. Let X be a paracompact space on wich act a compact Lie G. We denote by $\mathcal{P}_G(X) = \{\, A \subset X \mid A$ closed invariant subset of $X\,\}$. A **topological index** Ind_G associated to a compact Lie group G is a function $Ind_G : \mathcal{P}_G(X) \to \mathbf{N} \cup \{\infty\}$ verifying the following properties:

(I1) $Ind_G(A) = 0$ if and only if $A = \emptyset$;

(I2) If $f : A_1 \to A_2$ is a G-equivariant continuous map the $Ind_G(A_1) \leq Ind_G(A_2)$;

(I3) If K is a compact invariant, then there exists a closed invariant neighborhood U of K, such that $Ind_G(U) = Ind_G(K)$.

(I4) $Ind_G(A_1 \cup A_2) \leq Ind_G(A_1) + Ind_G(A_2)$

(I5) If K is compact invariant set with $K \cap I(G) = \emptyset$, then K contains at least n orbits provided $ind_G(K) \geq n$, where $I(G) = \{\, x \in X \mid \exists g \in G \setminus \{e\} \text{ with } gx = x\}$.

(I6) If K is a compact invariant set with $K \cap I(G) = \emptyset$, then $Ind_G(K) < +\infty$.

Definition 4.6. Let X be a G-paracompact space. We introduce the following notation:

$$\widehat{\mathcal{P}}_G(X) = \{ (A, B) \,|\, B \subset A \subset X \text{ and } A, B \text{are closed and invariant}\}.$$

A **relative index** is a function $Ind_G(,) : \widehat{\mathcal{P}} \to \mathbf{N} \cup \{+\infty\}$ such that we have:

(R1) $Ind_G(, \emptyset)$ verifies the properties (I1)-(I6) of the index and will be denoted $Ind_G()$.

(R2) If $f : (A_1, B) \to (A_2, B)$ is equivariant and $f_{|B}$ is a homeomrphism, then $Ind_G(A_1, B) \leq Ind_G(A_2, B)$.

(R3) $Ind_G(A_1 \cup A_2) \leq Ind_G(A_1, B) + Ind_G(A_2)$.

Examples:

4.1) If we consider the cat_G-category introduced by [19], [14], [27] or $\mathcal{A}$- category or relative $\mathcal{A}$-category or the $\mathcal{A}$-genus, see [14], [15],[12] we get another class of G-index and relative index.

4.2) The relative cohomological index introduced by Fadell and Husseini see [20], the eqivariant cup-length see [13] and the ideal valued index see [33], is another relative index.

Definition 4.7. Let B a closed subset of M. We shall say that the class $\mathcal{F}$ of subsets of M is G-**homotopy stable with boundary** B if:

(a) Every set in $\mathcal{F}$ is G-invariant;

(b) Every set in $\mathcal{F}$ contains B;

(c) For any set $A \in \mathcal{F}$ and any G-equivariant $\eta \in \mathcal{C}([0,1] \times M, M)$ verifying $\eta(t, x) = x$ for all $(t, x) \in (\{0\} \times M) \cup ([0,1] \times B)$ we have $\eta(1, A) \in \mathcal{F}$.

Examples:

4.3) If we consider the Cat_G-category or $\mathcal{A}$- category or relative $\mathcal{A}$-category or the $\mathcal{A}$-genus, see [14], [15],[12] we get another class of G-homotpy stable family.

4.4) In general, if we consider an index or a relative index we get different G-homotopy stable families.

In the next we generalize the result of Ghoussoub in the equivariant case see [21], for continuous and G-ivariant functionals.

Theorem 4.8. *Let (X, d) be a complete G-metric space and let $f : X \to \mathbb{R}$ be a continuous G-invariant function. Consider a G-homotopy stable family $\mathcal{F}$ of subsets of X with boundary B and a dual family $\mathcal{F}^\star$ of $\mathcal{F}$. Let $F \in \mathcal{F}^\star$ be a fixed which element which verifies the following condition*

$$\inf_{x\in F} f(x) \geq c, \tag{4.1}$$

where $c := \inf_{A\in\mathcal{F}} \sup_{x\in A} f(x)$.
Let $\varepsilon \in (0, \frac{dist(B,F)}{2})$ and $\delta > 0$ be arbitrarly fixed numbers. Then for any $A \in \mathcal{F}$ which verifies the relation

$$\sup_{x\in A\cap F_\delta} f(x) \leq c + \frac{\varepsilon}{2\sqrt{1+\varepsilon^2}}, \tag{4.2}$$

then there exists $x_\varepsilon \in X$ such that the following holds:

(i) $c - 2\varepsilon \leq f(x_\varepsilon) \leq c + 2\varepsilon$;

(ii) $|df|_G(x_\varepsilon) \leq \varepsilon$;

(iii) $dist(x_\varepsilon, F) \leq 2\varepsilon$;

(iv) $dist(x_\varepsilon, A) \leq 2\varepsilon$.

With the aid of Theorem 4.8 it is easy to prove a result which is very useful in state differant multiplicity results. For this we need the following definition.

Definition 4.9. We say that the continuous and G-invariant function $f : M \to \mathbf{R}$ verifies the G- Palais-Smale condition at the level c and around the set F (shortly $G - (PS)_{F,c}$) along the sequence $(A_n)_n \subset \mathcal{F}$ if any sequence $(x_n)_n \subset M$ verifying $f(x_n) \to c$, $||df|_G(x_n) \to 0$, $\text{dist}(x_n, F) \to 0$ and $\text{dist}(x_n, A_n) \to 0$ has a convergent subsequence.

We shall denote by $A_\infty = \{x \in M \mid \lim_{n\to\infty} \text{dist}(x, A_n) = 0\}$. Under the hypothesis of Theorem 4.8 and assuming that $f : M \to \mathbf{R}$ verifies $(PS)_{F,c}$ along a minimaxing sequence $(A_n)_n$, the set $F \cap K_c \cap A_\infty$ is non empty.

We have the following two general multiplicity results of Ghoussoub [21] for continuous G-invariant functions. Because the proof it is same way we omit.

Theorem 4.10. *(Ghoussoub-Corvellec) Let G ,M and c as in Theorem 4.8 and $f : M \to \mathbb{R}$ a G-invariant continuous function veryfying the condition G-(PS). Let $(\mathcal{F}_j)_{j=1}^{N}$ be an decreasing sequence of G-homotopy stable family with boundaries $(B_j)_{j=1}^{N}$ and verifying the following excision property with respect to an index Ind_G:*

(E) For every $1 \le j \le j+p \le N$ any $A \in \mathcal{F}_{j+p}$ and any U open and invariant such that $\overline{U} \cap B_j = \emptyset$ and $Ind_G(\overline{U}) \ge p$ we have $A \setminus U \in \mathcal{F}_j$.
Let F be a closed invariant set such that for each $1 \le j \le N$, F verifies (F1) and $\sup f(B) \le \inf f(F)$ *with respect to $\mathcal{F}_j$. Set $c_j = \inf_{A\in\mathcal{F}_j} \sup_{x\in A} f(x)$, $d = \inf f(F)$ and let $M = \sup\{k : c_k = d\} \vee 0$. Then we have:*

(a) $Ind_G(K_{c_M} \cap F \cap A_\infty) \ge M$ for every minimaxing sequence $(A_n)_n$ in $\mathcal{F}_M$.

(b) For every $M < i \le j+p \le N$ such that $c_j = c_{j+p}$ we have $Ind_G(K_{c_j} \cap A_\infty \ge p+1$ for every minimaxing sequence $(A_n)_n$ in $\mathcal{F}_{j+p}$. In particular if $I(G) \subset (M \setminus F) \cap (f \le d)$ then:

(c) f has at least N distinct critical orbits.

(d) If $N \to \infty$ then f has an unbounded critical value.

If in the Theorem 4.8 we take $Ind_G = cat_G$ and $F = M$ we get the following multiplicity result.

Corollary 4.11. *Let $f : M \to \mathbf{R}$ a G-invariant continuous function, which satisfied the G-(PS) condition and is bounded below, then f has at least $cat_G(M)$ distinct critical orbit.*

This corollary is a generalization for continuous G-invariant function of the Fadell multiplicity result for cat_G, see [19].

As a consequence of Corollary 4.11 is the main result from [32] and the main result from section 1 of [23] and Theorem 4.12 see [26].

Corollary 4.12. *Let G be a discrete subgroup of the Banach space X and $f : X \to \mathbb{R}$ a G-invariant continuous function wich satisfied the G-(PS) condition and is bounded below. If the dimension n of the space generated by G is finit, then f has at least $n+1$ distinct critical orbit.*

Proof. Using Corollary 4.11 we get f has at least $\mathrm{cat}_G(X)$ distinct critical orbit. But $cat_G(X) \geq cat(X/G) = cat_{T^n}(T^n) = n+1$, where T^n is the n-dimensional torus.

Now we consider the group Lie $G = (S^1)^k \mathrm{or} G = (\mathbf{Z}_p)^k$, $k \geq 1$ and let X be an infinite dimensional ortogonal reprsentation of the group G. Using Corollary 3 from [12] we get $\mathrm{cat}_G(SX) = \infty$ if $X^G = 0$ and $\mathrm{cat}_G(SX) = 2$ if $X^G \neq 0$, where SX denote the unit sphere in X and X^G the fixed point set of the group action G on X.

Corollary 4.13. *Let G, X be as above, and $f : X \to \mathbf{R}$ a G-invariant continuous function. We suppose that the function f is bounded from belov on SX and f satisfies the G-(PS) condition on SX. If $X^G = 0$ then f has an infinitely many distinct critical orbits on SX.*

The Corollary 4.13 is a generalization of the main result from section 3 of [23], where the authors are considered $\mathbf{Z}/\mathbf{p}$-action.

Corollary 4.14. *(Li-Santos) Let E be a complete metric space and $f : X \to \mathbb{R}$ a continuous functional. We suppose that the following conditions holds:*

1. *f satisfied the (PS) conditions;*
2. *There are teo disjoint sets A and B such that A is P-ideal linking to B;*
3. $\sup_{x \in A} f(x) \leq \inf_{x \in B} f(x)$;
4. *There exists a closed subset $\tilde{X} \supset A$ in E such that $\tilde{X} \setminus A$ is precompact and*

$$P - Index_E(\tilde{X}, A) = P - Index_E(E, A).$$

Then f possesses at least one critical value $c \geq \inf_{x \in B} f(x)$.

Proof. We denote by $\alpha = P - Index_E(E, A)$ and $\beta = P - Index_E(E \setminus B, A)$. Since A is P-ideal linking to B, we have $\beta \supset \alpha$ and $\beta \neq \alpha$. We define the set

$$\Sigma_\alpha = \{(X, A) \in \mathcal{E}_E \quad : \quad P - Index_E(X, A) = \alpha \ \},$$

where $\mathcal{E}_E$ is the class of all paracompact pair (X, A) in E. Note that $\Sigma_\alpha \neq \emptyset$ since $(\tilde{X}, A) \in \Sigma_\alpha$ and is homotopy stable with boundary A. Thus the conditions of Theorem 4.10 are satisfied and the conclusion of this corollary is true.

If we consider the relative index we have the following result of Ghoussoub for G-invariant continuous functions.

Theorem 4.15. *(Ghoussoub) Let G and M as in Theorem 4.8 and let $f : M \to \mathbb{R}$ be a continuous G-invariant function satifying the G-(PS) condition. Let B and F be two disjoint closed and invariant subset of M such that:*

(1) $k = Ind_G(M \setminus F) < Ind_G(X, B) = n.$

(2) $\sup f(B) \leq \inf f(F)$

(3) $I(G) \subset B.$

Then f has at least $n - k$ distinct critical orbits. Moreover, if $Ind_G(X, B) = \infty$, then f has an unbounded sequence of critical values.

Remark 4.16 If in the Theorem 4.15 we take for relative index the relativ cohomological index we get a generalization of Theorem 5.6 see [19]. If we take different relative index we obtain different multiplicity results for continuous G-invariant function.

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