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ON BIRECURRENT WEYL SPACES

ELİF ÖZKARA CANFES AND ABDÜLKADİR ÖZDEĞER

Dedicated to Professor Pavel Enghiş at his 70^{th} anniversary

Abstract. In this paper, birecurrent Weyl spaces are defined and it is proved that the birecurrence tensor of a birecurrent Weyl space is symmetric if and only if the space is Riemannian. Moreover, some results concerning birecurrent hypersurfaces of a birecurrent Weyl space are obtained.

1. Introduction.

An n-dimensional manifold W_n is said to be a Weyl space if it has a conformal metric tensor g_{ij} and a symmetric connection ∇ satisfying the compatibility condition given by the equation

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \tag{1.1}$$

where T_k denotes a covariant vector field and $\nabla_k g_{ij}$ denotes the usual covariant derivative.

Under a renormalization of the fundamental tensor of the form

$$\breve{g}_{ij} = \lambda^2 g_{ij} \tag{1.2}$$

the complementary vector T_k is transformed by the law

$$\breve{T}_k = T_k + \partial_k \ln \lambda \tag{1.3}$$

where λ is a function defined on W_n.

A quantity A is called a satellite with weight $\{p\}$ of the tensor g_{ij}, if it admits a transformation of the form

$$\breve{A} = \lambda^p A \tag{1.4}$$

under the renormalization (1.2) of the metric tensor g_{ij} ([1], [2]).

The prolonged covariant derivative of a satellite A of the tensor g_{ij} with weight $\{p\}$ is defined by ([1], [2])

$$\dot{\nabla}_k A = \nabla_k A - pT_k A. \tag{1.5}$$

We note that the prolonged covariant derivative preserves the weight.

According to Norden [3], we have

$$x^a_{,ijk} - x^a_{,ikj} = R^h_{ijk} x^a_h \tag{1.6}$$

where R^h_{ijk} is the curvature tensor of the Weyl space defined by

$$R^h_{ijk} = \frac{\partial}{\partial x^j}\Gamma^h_{ik} - \frac{\partial}{\partial x^k}\Gamma^h_{ij} + \Gamma^h_{lj}\Gamma^l_{ik} - \Gamma^h_{lk}\Gamma^l_{ij}. \tag{1.7}$$

The first and the second Bianchi identities for Weyl spaces are, by [4],

$$R^h_{ijk} + R^h_{kij} + R^h_{jki} = 0 \tag{1.8}$$

$$\dot{\nabla}_r R^i_{jkh} + \dot{\nabla}_k R^i_{jhr} + \dot{\nabla}_h R^i_{jrk} = 0. \tag{1.9}$$

2. Birecurrent Weyl Spaces

A Weyl space $W_n(g_{ij}, T_k)$ is called **recurrent**,[4], if the curvature tensor satisfies the following condition for some non-zero covariant vector field ϕ_s ($\neq T_s$):

$$\dot{\nabla}_s R^h_{ijl} = \phi_s R^h_{ijl}. \tag{2.1}$$

We call a non-flat Weyl space $W_n(g_{ij}, T_k)$ **birecurrent** if the curvature tensor satisfies the condition

$$\dot{\nabla}_r \dot{\nabla}_s R^h_{ijk} = \phi_{sr} R^h_{ijk} \tag{2.2}$$

for some non-zero covariant tensor field ϕ_{sr}. Transvecting (2.2) by g_{hl} and remembering that the prolonged covariant differentiation preserves the metric, we obtain the equivalent form of (2.2) as

$$\dot{\nabla}_r \dot{\nabla}_s R_{lijk} = \phi_{sr} R_{lijk} \ , \ R_{lijk} = g_{lh} R^h_{ijk}. \tag{2.2)'$$

It is easy to see that a recurrent Weyl space is birecurrent. In fact, by taking the prolonged covariant derivative of (2.1) with respect to u^r, we get

$$\dot{\nabla}_r \dot{\nabla}_s R^h_{ijk} = (\phi_{s,r} + \phi_s \phi_r) R^h_{ijk}. \tag{2.3}$$

with $\phi_{sr} = \phi_{s,r} + \phi_s \phi_r$.

We examine Weyl spaces which satisfy (2.2), but not (2.1).

We remark that the definition of a birecurrent Weyl space agrees with that of a birecurrent Riemannian space if we take the complementary vector field of $W_n(g_{ij}, T_k)$ as zero.

Theorem 2.1. *The birecurrency tensor of a birecurrent Weyl space with a non-vanishing scalar curvature is symmetric if and only if the space is locally Riemannian.*

Proof. Assume ϕ_{sr} is a symmetric tensor. Transvecting (2.2)′ by $g^{hj} g^{ik}$ and remembering that the Ricci tensor R_{ij} and the scalar curvature R of the Weyl space are respectively defined by $R_{ij} = R^h_{ihj}$, $R = R_{ij} g^{ij}$, we get

$$\dot{\nabla}_r \dot{\nabla}_s R = \phi_{sr} R. \tag{2.4}$$

Changing the order of the indices r and s in (2.4) and subtracting the expression so obtained from (2.4), we have

$$\dot{\nabla}_{[r} \dot{\nabla}_{s]} R = \phi_{[sr]} R.$$

where the bracket indicates antisymmetrization.

Since, by assumption, ϕ_{sr} is a symmetric tensor, we get

$$\dot{\nabla}_{[r} \dot{\nabla}_{s]} R = 0. \tag{2.4$'$}$$

Expanding $\dot{\nabla}_{[r} \dot{\nabla}_{s]} R$ and remembering that R is a satellite of g_{ij} with weight $\{-2\}$, we find that

$$\dot{\nabla}_{[r} \dot{\nabla}_{s]} R = \nabla_{[r} \nabla_{s]} R + 2 \nabla_{[r} T_{s]} R = 0 \text{ with } \nabla_{[r} \nabla_{s]} \mathrm{R} = 0.$$

Since $R \neq 0$, we have

$$\nabla_{[s} T_{r]} = 0$$

which means that W_n is locally Riemannian.

The sufficiency of the condition is a well-known fact from the Riemannian Geometry [5]. □

Corollary 2.1. *If $\dot{\nabla}_r \dot{\nabla}_s R_{ijkh} = 0$, then the Weyl space is locally Riemannian.*

Corollary 2.2. *If $\dot{\nabla}_{[r} \dot{\nabla}_{s]} R_{ijkh} = 0$, then the Weyl space is locally Riemannian.*

Theorem 2.2. *The birecurency tensor ϕ_{is} of a birecurrent Weyl space is the solution of the equation*

$$\phi_{is}(R^{i}_{\cdot\cdot h} - R^{i}_{\cdot h} + \delta^{i}_{h} R) = 0$$

where $R^{i}_{\cdot\cdot h} = R^{i}_{jkh} g^{jk}$, $R^{i}_{\cdot h} = R_{kh} g^{ik}$ and $R = R_{jk} g^{jk}$.

Proof. By taking the prolonged covariant derivative of (1.9) with respect to u^s, we get

$$\dot{\nabla}_s \dot{\nabla}_r R^{i}_{jkh} + \dot{\nabla}_s \dot{\nabla}_k R^{i}_{jhr} + \dot{\nabla}_s \dot{\nabla}_h R^{i}_{jrk} = 0 \tag{2.5}$$

from which, by (2.2), it follows that

$$\phi_{rs} R^{i}_{jkh} + \phi_{ks} R^{i}_{jhr} + \phi_{hs} R^{i}_{jrk} = 0. \tag{2.6}$$

Contracting (2.6) with respect to i and r and remembering that

$$R_{ij} = R^{h}_{ihj} \text{ and } R^{h}_{ilh} = -R^{h}_{ihl}$$

we get

$$\phi_{is} R^{i}_{jkh} - \phi_{ks} R_{jh} + \phi_{hs} R_{jk} = 0. \tag{2.7}$$

or,

$$\phi_{is}(R^{i}_{jkh} - \delta^{i}_{k} R_{jh} + \delta^{i}_{h} R_{jk}) = 0. \tag{2.8}$$

Transvection of (2.8) by g^{jk} yields

$$\phi_{is}(R^{i}_{\cdot\cdot h} - R^{i}_{\cdot h} + \delta^{i}_{h} R) = 0$$

where $R^{i}_{\cdot\cdot h} = R^{i}_{jkh} g^{jk}$, $R^{i}_{\cdot h} = R_{kh} g^{ik}$ and $R = R_{jk} g^{jk}$. □

Corollary 2.3. *If $\det A^{i}_{h} \neq 0$, then W_n is Riemannian, where $A^{i}_{h} = R^{i}_{\cdot\cdot h} - R^{i}_{\cdot h} + \delta^{i}_{h} R$.*

3. Hypersurfaces of Birecurrent Weyl Spaces.

Let $W_n(g_{ij}, T_k)$ be a hypersurface, with coordinates $u^i (i = 1, 2, \cdots, n)$ of a Weyl space $W_{n+1}(g_{ab}, T_c)$ with coordinates $x^a (a = 1, 2, \cdots, n+1)$. The metrics of W_n and W_{n+1} are connected by the relations

$$g_{ij} = g_{ab} x_i^a x_j^b \ (i, j = 1, 2, \cdots, n \ ; \ a, b = 1, 2, \cdots, n+1) \tag{3.1}$$

where x_i^a denotes the covariant derivative of x^a with respect to u^i.

It is easy to see that the prolonged covariant derivative of a satellite A, relative to W_n, and W_{n+1}, are related by

$$\dot{\nabla}_k A = x_k^c \dot{\nabla}_c A \ (k = 1, 2, \cdots, n \ ; \ c = 1, 2, \cdots, n+1) \tag{3.2}$$

Let n^a be the contravariant components of the vector field of W_{n+1} normal to W_n which is normalized by the condition

$$g_{ab} n^a n^b = 1. \tag{3.3}$$

The moving frame $\{x_a^i, n_a\}$ in W_n, reciprocal to the moving frame $\{x_i^a, n^a\}$ is defined by the relations [3]

$$n_a x_i^a = 0 \ n^a x_a^i = 0 \ x_i^a x_a^j = \delta_i^j. \tag{3.4}$$

Remembering that the weight of x_i^a is $\{0\}$, the prolonged covariant derivative of x_i^a with respect to u^k is found as

$$\dot{\nabla}_k x_i^a = \nabla_k x_i^a = \omega_{ik} n^a \tag{3.5}$$

where ω_{ik} is the second fundamental form. It can be shown that ω_{ik} is a satellite of g_{ij} with weight $\{1\}$.

The generalized Gauss and Mainardi-Codazzi equations are obtained in [4], respectively as

$$R_{pijk} = \Omega_{pijk} + \overline{R}_{dbce} x_p^d x_i^b x_j^c x_k^e \tag{3.6}$$

$$\dot{\nabla}_k \omega_{ij} - \dot{\nabla}_j \omega_{ik} + \overline{R}_{dbce} x_i^b x_j^c x_k^e n^d = 0, \tag{3.7}$$

where $\overline{R}_{dbce}$ is the covariant curvature tensor of W_{n+1} and Ω_{pijk} is the Sylvesterian of ω_{ij} defined by $\Omega_{pijk} = \omega_{pj}\omega_{ik} - \omega_{pk}\omega_{ij}$.

In the following we will use the notation

$$B^{ab\cdots cd}_{ij\cdots kl} = x^a_i x^b_j \cdots x^c_k x^d_l \tag{3.8}$$

the same as in [6].

Theorem 3.1. *For a hypersurface of a birecurrent Weyl space* W_{n+1} *with birecurence tensor* ϕ_{ef} *we have the identity*

$$\dot{\nabla}_r\dot{\nabla}_s R_{ijkl} - \phi_{rs}R_{ijkl} = \dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} - \phi_{rs}\Omega_{ijkl} + S_{ijkl(sr)} + D_{ijkl}\omega_{sr} + \overline{R}_{abcd}\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} \tag{3.9}$$

where

$$S_{ijklsr} = \dot{\nabla}_e\overline{R}_{abcd}x^e_s\dot{\nabla}_r B^{abcd}_{ijkl}\ ,\ D_{ijkl} = B^{abcd}_{ijkl}n^e\dot{\nabla}_e\overline{R}_{abcd} \text{ and } \phi_{rs} = \phi_{ef}B^{ef}_{rs}$$

and the paranthesis ( ) denotes symmetrization.

Proof. By taking the prolonged covariant derivative of Gauss equation with respect to u^s and u^r successively, we have

$$\begin{aligned} \dot{\nabla}_r\dot{\nabla}_s R_{ijkl} &= \dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} + (\dot{\nabla}_f\dot{\nabla}_e\overline{R}_{abcd})B^{abcdef}_{ijklsr} + (\dot{\nabla}_e\overline{R}_{abcd})(\dot{\nabla}_r B^{abcde}_{ijkls}) \\ &+ (\dot{\nabla}_f\overline{R}_{abcd})x^f_r\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_s B^{abcd}_{ijkl}. \end{aligned}$$

If W_{n+1} is birecurrent Weyl, then by the definition $\dot{\nabla}_f\dot{\nabla}_e\overline{R}_{abcd} = \phi_{ef}\overline{R}_{abcd}$ so that we have

$$\begin{aligned} \dot{\nabla}_r\dot{\nabla}_s R_{ijkl} &= \dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} + \phi_{ef}\overline{R}_{abcd}B^{abcdef}_{ijklsr} + (\dot{\nabla}_e\overline{R}_{abcd})(\dot{\nabla}_r B^{abcde}_{ijkls}) \\ &+ (\dot{\nabla}_f\overline{R}_{abcd})x^f_r\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_s B^{abcd}_{ijkl}. \end{aligned}$$

By using the Gauss equation (3.6), the above equation can be brought into the form

$$\begin{aligned} \dot{\nabla}_r\dot{\nabla}_s R_{ijkl} &= \phi_{sr}R_{ijkl} + \dot{\nabla}_s\dot{\nabla}_r\Omega_{ijkl} - \phi_{sr}\Omega_{ijkl} + \dot{\nabla}_e\overline{R}_{abcd}\dot{\nabla}_r B^{abcde}_{ijkls} \\ &+ x^f_r\dot{\nabla}_f\overline{R}_{abcd}\dot{\nabla}_s B^{abcd}_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl}. \end{aligned}$$

where $\phi_{rs} = \phi_{ef}B^{ef}_{rs}$. Hence, by (3.5), the result follows. □

Theorem 3.2. *If a hypersurface of a Birecurrent Weyl space is birecurrrent then*

$$\dot{\nabla}_r\dot{\nabla}_s\Omega_{ijkl} - \phi_{rs}\Omega_{ijkl} + S_{ijkl(sr)} + \omega_{sr}D_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_r\dot{\nabla}_s B^{abcd}_{ijkl} = 0. \tag{3.10}$$

or, equivalently,

$$\dot{\nabla}_{[r}\dot{\nabla}_{s]}\Omega_{ijkl} - \phi_{[rs]}\Omega_{ijkl} + \overline{R}_{abcd}\dot{\nabla}_{[r}\dot{\nabla}_{s]}B^{abcd}_{ijkl} = 0. \tag{3.11}$$

Proof. It is clear from (2.2) and Theorem 3.1. □

A hypersurface of a Weyl space is called totally geodesic if $\omega_{ij} = 0$.

Theorem 3.3. *Every totally geodesic hypersurface of a birecurrent Weyl space is birecurrent.*

Proof. Since the hypersurface is totally geodesic, by putting $\omega_{ij} = 0$ in (3.9) we get the result. □

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Istanbul Technical University, Faculty of Science and Letters, 80626 Maslak, Istanbul, TURKEY