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HIGHER ORDER EINSTEIN-SCHRÖDINGER SPACES

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Dedicated to Professor Pavel Enghiş at his 70th anniversary

Abstract. In 1945 A.Einstein [6] and E.Schrödinger [10] started form a generalized Riemann space, thas is, a space M associated with a nonsymmetric tensor $G_{ij}(x)$ and desired to find the set of all linear connections $\Gamma^i_{jk}(x)$ compatible with such a metric : $G_{ij/k} = 0$ (see also [1] and [2]).The geometry of this space $(M.G_{ij})$ is called **the Einsten - Schrödinger's geometry** [3], [4].

The purpose of this paper is to discuss a nonsymmetric tensor field $G_{ij}(x, y^{(1)}, ..., y^{(k)})$,where $(x, y^{(1)}, ..., y^{(k)})$ is a point of the k-osculator bundle $(Osc^k M, \pi, M)$ and to obtain the results for the Einstein - Schrödinger's geometry of the higher order in a natural case.

The fundamental notions and notations concerning the osculator bundle of the higher order are given in the papers [8] [9] and in the recent Miron's book [7] and we suppose them to be known.

For a nonsymmetric tensor field $G_{ij}(x, y^{(1)}, ..., y^{(k)})$ on $Osc^k M$,we have a symmetric tensor field $g_{ij}(x, y^{(1)}, ..., y^{(k)})$ and a skew-symmetric one $a_{ij}(x, y^{(1)}, ..., y^{(k)})$ from the spliting

$$G_{ij} = g_{ij} + a_{ij}, \tag{1}$$

where we suppose that

$$det \parallel g_{ij}(x, y^{(1)}, ..., y^{(k)})\parallel \cdot \parallel a_{ij}(x, y^{(1)}, ..., y^{(k)}) \parallel \neq 0 \tag{2}$$

and dim M = n = 2n'.

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We denote

$$||g_{ij}(x,y^{(1)},...,y^{(k)})||^{-1} = ||g^{ij}(x,y^{(1)},...,y^{(k)})||,$$
$$||a_{ij}(x,y^{(1)},...,y^{(k)})||^{-1} = ||a^{ij}(x,y^{(1)},...,y^{(k)})||$$

We have from $G_{ij|k} = 0$, $G_{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1,...,k)$ the following equations :

$$g_{ij|k} = 0\ ,\ g_{ij} \overset{(\alpha)}{|}_k = 0\ ,\ a_{ij|k} = 0\ ,\ a_{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1,...,k)\ , \tag{3}$$

which is equivalent to

$$g^{ij}{}_{|k} = 0\ ,\ g^{ij} \overset{(\alpha)}{|}_k = 0\ ,\ a^{ij}{}_{|k} = 0\ ,\ a^{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1,...,k). \tag{4}$$

We investigate the set of all N-linear connections $D\Gamma(N) = (L^i{}_{jk},\ \underset{(\alpha)}{C^i{}_{jk}})$ $(\alpha = 1,...,k)$ for which we have (3) in the form

$$L^i{}_{jk} = \overset{\circ}{L^i}{}_{jk} + A^i_{jk}\ ,\quad \underset{(\alpha)}{C^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}} + \underset{(\alpha)}{B^i_{jk}} \quad (\alpha = 1,...,k)\ ,$$

where $D\overset{\circ}{\Gamma}(N) = (\overset{\circ}{L^i}{}_{jk},\ \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}})$ $(\alpha = 1,...,k)$ is a fixed N-linear connection on $Osc^k M$ and A^i_{jk} , $\underset{(\alpha)}{B^i_{jk}}$ are arbitrary tensor fields of type (1,2).

We obtain for A and for $\underset{(\alpha)}{B}$ the equations

$$A^r_{ik}g_{rj} + A^r_{jk}g_{ir} = g_{ij\overset{\circ}{|}k}\ ,\ A^r_{ik}a_{rj} + A^r_{jk}a_{ir} = a_{ij\overset{\circ}{|}k}, \tag{5}$$

$$\text{(6)} \quad \begin{cases} \underset{(\alpha)}{B^r_{ik}}\, g_{rj} + \underset{(\alpha)}{B^r_{jk}}\, g_{ir} = g_{ij} \overset{(\alpha)}{\underset{o}{|}}_{k}, \\ \underset{(\alpha)}{B^r_{ik}}\, a_{rj} + \underset{(\alpha)}{B^r_{jk}}\, a_{ir} = a_{ij} \overset{(\alpha)}{\underset{o}{|}}_{k} \quad (\alpha = 1, ..., k) \end{cases}$$

We do not know the general solution of the equation system (5) and (6)

We give a solution for these equations in the following special case.

Definition 1. *An assymmetric metric (1) is called* **natural** *if we have*

$$\Lambda^{rk}_{is}\Phi^{hs}_{rj} = \Phi^{rk}_{is}\Lambda^{hs}_{rj} \tag{7}$$

where

$$\Lambda^{kh}_{ij} = \frac{1}{2}(\delta^k_i\delta^h_j - g_{ij}g^{kh}), \quad \Phi^{kh}_{ij} = \frac{1}{2}(\delta^k_i\delta^h_j - a_{ij}a^{kh}). \tag{8}$$

Theorem 1. *An assymmetric metric $G_{ij}(x, y^{(1)}, ..., y^{(k)})$ on $Osc^k M$ is natural if and only if there exist a function $\mu(x, y^{(1)}, ..., y^{(k)})$ on $Osc^k M$ such that*

$$g_{ir}g_{js}a^{rs} = \mu g_{ij}. \tag{9}$$

Examples.

1. Let $f^i_j(x, y^{(1)}, ..., y^{(k)})$ be a tensor field of type (1,1) which gives an almost complex d-structure on $Osc^k M$: $f^2 = -\delta$. If we put:

$$a_{ij} = f^r_i g_{rj}, \tag{10}$$

then $a_{ij}(x, y^{(1)}, ..., y^{(k)})$ is alternating and $G_{ij} = g_{ij} + a_{ij}$ is an asymmetric metric on $Osc^k M$. In this case $\mu = -1$.

2. Let $q^i_j(x, y^{(1)}, ..., y^{(k)})$ be a tensor field of type (1,1) which gives an almost product d-structure on $Osc^k M$: $q^2 = +\delta$. If we put:

$$a_{ij} = q^r_i g_{rj} \tag{11}$$

then $a_{ij}(x, y^{(1)}, ..., y^{(k)})$ is alternate and $G_{ij} = g_{ij} + a_{ij}$ is an asymmetric metric on $Osc^k M$. In this case $\mu = +1$.

Theorem 2. *If there exist a N-linear connection on $Osc^k M$ compatible with a natural asymmetric metric $G_{ij}(x, y^{(1)}, ..., y^{(k)})$, then the function μ is constant.*

Definition 2. *A natural asymmetric metric (1) is called* **elliptic** *if $\mu = -c^2$ and* **hyperbolic** *if $\mu = c^2$, where c is a positive constant.*

The converse of Theorem 2 holds as follows:

Theorem 3. *If a natural asymmetric metric (1) is elliptic or hyperbolic, then there exist N-linear connections $D\tilde{\Gamma}(N) = (\tilde{L}^i{}_{jk}, \underset{(\alpha)}{\tilde{C}^i{}_{jk}})$ compatible with $G_{ij}(x, y^{(1)}, ..., y^{(k)})$.*

Let $D\overset{\circ}{\Gamma}(N) = (\overset{\circ}{L^i}{}_{jk}, \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}})$ be a given N-linear connection, then in the elliptic case we have

$$(12)\quad \begin{cases} \tilde{L}^i{}_{jk} = \overset{\circ}{L^i}{}_{jk} + \frac{1}{4}\{g^{ir} g_{rj\overset{\circ}{|}k} + a^{ir} a_{rj\overset{\circ}{|}k} + f^r_j f^i_{r\overset{\circ}{|}k}\} \\ \underset{(\alpha)}{\tilde{C}^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}} + \frac{1}{4}\{g^{ir} g_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k + a^{ir} a_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k + f^r_j f^i_r \overset{(\alpha)}{\overset{\circ}{|}}_k\} \end{cases}$$

$(\alpha = 1, ..., k)$ *, and in the hyperbolic case we have*

$$(13)\quad \begin{cases} \tilde{L}^i{}_{jk} = \overset{\circ}{L^i}{}_{jk} + \frac{1}{4}\{g^{ir} g_{rj\overset{\circ}{|}k} + a^{ir} a_{rj\overset{\circ}{|}k} - q^r_j q^i_{r\overset{\circ}{|}k}\} \\ \underset{(\alpha)}{\tilde{C}^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C^i}{}_{jk}} + \frac{1}{4}\{g^{ir} g_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k + a^{ir} a_{rj} \overset{(\alpha)}{\overset{\circ}{|}}_k - q^r_j q^i_r \overset{(\alpha)}{\overset{\circ}{|}}_k\} \end{cases}$$

$(\alpha = 1, ..., k)$

Theorem 4. *The set of all N-linear connections* $D\overset{*}{\Gamma}(N) = (\overset{*}{L}{}^i_{jk}, \underset{(\alpha)}{\overset{*}{C}}{}^i_{jk})$ *compatible with a natural asymmetric metric (1) on* $Osc^k M$ *is given by*

$$(14) \qquad \overset{*}{L}{}^i_{jk} = \tilde{L}^i_{jk} + \Lambda^{pi}_{jq}\Phi^{rq}_{ps}Y^s_{rk}, \quad \underset{(\alpha)}{\overset{*}{C}}{}^i_{jk} = \underset{(\alpha)}{\tilde{C}}{}^i_{jk} + \Lambda^{pi}_{jq}\Phi^{rq}_{ps}\underset{(\alpha)}{Z}{}^s_{rk}$$

where $D\tilde{\Gamma}(N)$ *is the N-linear connection in Theorem 3 and* Y^i_{jk}, $\underset{(\alpha)}{Z}{}^i_{jk}$ $(\alpha = 1, ..., k)$ *are arbitrary tensor fields on* $Osc^k M$.

If we put $D\overset{o}{\Gamma}(N) = D\overset{c}{\Gamma}(N) = (\overset{c}{L}{}^i_{jk}, \underset{(\alpha)}{\overset{c}{C}}{}^i_{jk})$, $(\alpha = 1, ..., k)$ for $g_{ij}(x, y^{(1)}, ..., y^{(k)})$, that is:

$$(15) \qquad \begin{cases} \overset{c}{L}{}^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right), \\ \underset{(\alpha)}{\overset{c}{C}}{}^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta y^{(\alpha)k}} + \frac{\delta g_{sk}}{\delta y^{(\alpha)j}} - \frac{\delta g_{jk}}{\delta y^{(\alpha)s}}\right), \end{cases}$$

the generalized Christoffel symbols we have:

Theorem 5. *The canonical N-linear connection compatible with a natural asymmetric metric* $G_{ij}(x, y^{(1)}, ..., y^{(k)})$ *is given in the elliptic case by:*

$$(16) \qquad \begin{cases} L^i_{jk} = \overset{c}{L}{}^i_{jk} + \frac{1}{4}\{a^{ir}a_{rj\overset{c}{|}k} + f^r_j f^i_{r\overset{c}{|}k}\} \\ \underset{(\alpha)}{C}{}^i_{jk} = \underset{(\alpha)}{\overset{c}{C}}{}^i_{jk} + \frac{1}{4}\{a^{ir}a_{rj}\overset{(\alpha)}{\overset{c}{|}}_k + f^r_j f^i_r\overset{(\alpha)}{\overset{c}{|}}_k\} \; (\alpha = 1, ..., k) \end{cases}$$

and in the hyperbolic case by

$$(17)\quad \begin{cases} L^{i}{}_{jk} = \overset{c}{L^{i}}{}_{jk} + \frac{1}{4}\{a^{ir} a_{rj|k}^{\;\;c} - q_j^r q^i{}_{r|k}^{\;c}\} \\ \underset{(\alpha)}{C^{i}{}_{jk}} = \underset{(\alpha)}{\overset{c}{C^{i}}{}_{jk}} + \frac{1}{4}\{a^{ir} a_{rj} \overset{(\alpha)}{\underset{c}{|}}_k - q_j^r q_r^i \overset{(\alpha)}{\underset{c}{|}}_k\}, \quad (\alpha = 1, ..., k) \end{cases}$$

Now, the Einstein equations, electromagnetic tensors, Maxwell equations for the higher order Einstein-Schrödinger geometry can be studied using these canonical N-linear connections.

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