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The norm of pre-Schwarzian derivatives of certain analytic functions with bounded positive real part

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Abstract. For real numbers $0 \leq \alpha < 1$ and $\beta > 1$ we define the univalent function in the unit disk Δ which maps Δ on to the strip domain ω with $\alpha < Re\,\omega < \beta$. In this paper we give the best estimates for the norm of the pre-Schwarzian derivative $T_f(z) = \frac{f''(z)}{f'(z)}$ where $\|T_f\| = \sup_{|z|<1} \left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right|$.

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1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of A, consisting of all univalent functions f in Δ is denoted by S. In [5] the authors introduced a new class for certain analytic functions, and they denote by $S(\alpha, \beta)$ the class of functions $f \in A$ which satisfy the inequality

$$\alpha < Re\,\frac{zf'(z)}{f(z)} < \beta, \ (z \in \Delta). \tag{1.2}$$

for some real number $0 \leq \alpha < 1$ and some real number $\beta > 1$. Also, the authors introduced the class $\nu(\alpha, \beta)$ of functions $f \in A$ which satisfy the inequality

$$\alpha < Re\left\{\left(\frac{z}{f(z)}\right)^2 f'(z)\right\} < \beta, \ (z \in \Delta). \tag{1.3}$$

where $0 \leq \alpha < 1$ and $\beta > 1$.
Let f and g be analytic in Δ. The function f is called to be *subordinate* to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function ω such that $\omega(0) = 0$, $|\omega(z)| < 1$, and $f(z) = g(\omega(z))$ on Δ. The pre-Schwarzian derivative of f is denoted by

$$T_f(z) = \frac{f''(z)}{f'(z)},$$

and we define the norm of T_f by

$$\|T_f\| = \sup_{|z|<1} \left(1 - |z|^2\right) \left|\frac{f''(z)}{f'(z)}\right|.$$

This norm have a significant meaning in the theory of Teichmuller spaces. For a univalent function f, it is well known that $\|T_f\| < 6$, and this estimate is the best possible [3,6]. On the other hand the following result is important to be noted:

Theorem 1.1. *Let f be analytic and locally univalent in Δ. Then,*
(i) if $\|T_f\| \leq 1$ then f is univalent, and
(ii) if $f \in S^*(\alpha)$, then $\|T_f\| \leq 6 - 4\alpha$.

The part (i) is due to Becker [1], and the sharpness of the constants is due to Becker and Pommerenke [2]. The part (ii) is due to Yamashita [8]. The norm estimates for typical subclasses of univalent functions are investigated by many authors like [4,7,8].
In this paper we shall give the best estimate for the norm of pre-Schwarzian derivatives of the class $S(\alpha, \beta)$ and $\nu(\alpha, \beta)$.

2. Main Results

To prove our main results we shall need the Schwartz' lemma.
Now, we define an analytic function $P : \Delta \to \mathbb{C}$ by

$$P(z) = 1 + \frac{\beta - \alpha}{\pi} \, i \, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right),$$

due to Kuroki and Owa [5]. They proved that p maps conformally Δ onto a convex domain ω with $\alpha < Re\,\omega < \beta$. Using this fact and the definition of subordination, we can directly obtain the following lemmas:

Lemma 2.1. *Let $f \in A$ and $0 \leq \alpha < \alpha < 1 < \beta$. Then, $f \in S(\alpha, \beta)$ if and only if*

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{\beta - \alpha}{\pi} \, i \, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1 - z}\right),$$

Lemma 2.2. *Let $f \in A$ and $0 \le \alpha < 1 < \beta$. Then, $f \in \nu(\alpha,\beta)$ if and only if*

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec 1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right).$$

In this work, first we find norm estimate of the pre-Schwarzian derivative for $f \in S(\alpha,\beta)$, and then we find the norm estimate of the pre-Schwarzian derivative for $f \in \nu(\alpha,\beta)$.

Theorem 2.3. *For $0 \le \alpha < 1 < \beta$, if $f \in S(\alpha,\beta)$, then*

$$\|T_f\| \le \frac{2(\beta-\alpha)}{\pi}\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right).$$

Proof. For an arbitrary function $f \in S(\alpha,\beta)$, set $g(z) = \dfrac{zf'(z)}{f(z)}$. Then, g is a holomorphic function on Δ satisfying $g(0) = 1$ and

$$g(\Delta) \subset \{\omega \in \mathbb{C} : \alpha < Re\,\omega < \beta\} := H(\alpha,\beta).$$

The univalent map $P(z) = 1+\dfrac{\beta-\alpha}{\pi}\, i\, \log\left(\dfrac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right)$ on Δ satisfies $P(0) = 1$ and $P(z) = H(\alpha,\beta)$, therefore g is subordinate to P. Thus, there exists a holomorphic function $\omega = \omega_f : \Delta \to \Delta$ with $\omega(0) = 0$ such that,

$$g(z) = (P\circ\omega)(z) = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}\right). \tag{2.1}$$

By the logarithmic differentation of (2.1), we have

$$\log\frac{zf'(z)}{f(z)} = \log\left\{1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}\right)\right\},$$

and consequently

$$\log z + \log f'(z) - \log f(z) = \log\left\{1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}\right)\right\}.$$

Hence,

$$\frac{1}{z}+\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}=$$

$$=\frac{\beta-\alpha}{\pi}\, i\, \frac{-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega'(z)\left(1-\omega(z)\right)+\omega'(z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)}{\left(1-\omega(z)\right)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)},$$

Then,

$$\frac{f''(z)}{f'(z)}=\frac{\beta-\alpha}{\pi}\, i\left(\frac{1}{z}\log\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}+\frac{-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega'(z)}{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}+\frac{\omega'(z)}{1-\omega(z)}\right),$$

and therefore,

$$T_f(z)=\frac{f''(z)}{f'(z)}=$$

$$=\frac{\beta-\alpha}{\pi}\, i\left(\frac{1}{z}\log\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)}{1-\omega(z)}+\frac{\omega'(z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right)}{\left(1-\omega(z)\right)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)}\right).$$

Setting $\omega=id_{\Delta}$, we also have

$$T_{f_{\alpha,\beta}}(z)=\frac{\beta-\alpha}{\pi}\, i\left(\frac{1}{z}\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right)+\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}\right),$$

and we conclude by using of Schwartz' lemma that,

$$\left(1-|z|^2\right)|T_f(z)|\le\left(1-|z|^2\right)\left|T_{f_{\alpha,\beta}}(z)\right|. \tag{2.2}$$

Thus, we can estimate as follows

$$\left(1-|z|^2\right)|T_f(z)| \leq \frac{\beta-\alpha}{\pi}\left(\frac{1-|z|^2}{|z|}\left|\log\left(\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}\right)\right|\right.$$
$$\left.+\left(1-|z|^2\right)\left|\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}\right|\right).$$

By using of maximum principle we can obtain upper bound of $\|T_f\|$, therefore

$$\lim_{z\to 0}\left(1-|z|^2\right)\left|\frac{\log\dfrac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z}{1-z}}{z}\right|$$
$$=\lim_{z\to 0}\left(1-|z|^2\right)\cdot\lim_{z\to 0}\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}$$
$$=1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}} \tag{2.3}$$

Also, we have

$$\lim_{z\to 0}\left(1-|z|^2\right)\left|\frac{1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}z\right)}\right| = 1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}, \tag{2.4}$$

hence, by (2.2) and (2.3) combined with (2.4), we conclude

$$\sup\left(1-|z|^2\right)|T_f(z)| \leq \frac{2(\beta-\alpha)}{\pi}\left(1-e^{2\pi i\frac{1-\alpha}{\beta-\alpha}}\right),$$

and this completes our proof. □

Theorem 2.4. *For $0 \leq \alpha < 1 < \beta$, if $f \in \nu(\alpha, \beta)$, then*

$$\|T_f\| \leq \frac{3(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right).$$

Proof. Let $f \in \nu(\alpha, \beta)$, and set $g(z) = \left(\frac{z}{f(z)}\right)^2 f'(z)$. Then, the functiong is a holomorphic function on Δ satisfying $g(0) = 1$ and

$$g(\Delta) \subset \{\omega \in \mathbb{C} : \alpha < Re\,\omega < \beta\} := H(\alpha, \beta).$$

The univalent map $P(z) = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log\left(\frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z}\right)$ on Δ satisfies $P(0) = 1$ and $P(z) = H(\alpha, \beta)$, hence g is subordinate to P. So, there exists a holomorphic function $\omega = \omega_f : \Delta \to \Delta$ with $\omega(0) = 0$ such that

$$g(z) = (P \circ \omega)(z) = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1 - \omega(z)}. \tag{2.5}$$

By the logarithmic differentiation of (2.5) and using the same method as proof of Theorem 2.3, we have

$$2\left(\frac{1}{z} - \frac{f'(z)}{f(z)}\right) + \frac{f''(z)}{f'(z)} =$$

$$= \frac{\beta-\alpha}{\pi}\, i\, \frac{-e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega'(z)\,(1-\omega(z)) + \omega'(z)\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)}{(1-\omega(z))\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\omega(z)\right)}. \tag{2.6}$$

With (2.1) we have,

$$\frac{zf'(z)}{f(z)} = 1 + \frac{\beta-\alpha}{\pi}\, i\, \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1 - \omega(z)},$$

therefore

$$T_f(z) = \frac{f''(z)}{f'(z)} =$$

$$= \frac{\beta-\alpha}{\pi}\, i \left(\frac{2}{z} \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)}{1-\omega(z)} + \frac{\omega'(z)\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right)}{(1-\omega(z))\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} \omega(z)\right)} \right)$$

Setting $\omega = id_{\Delta}$, we also have

$$T_{f_{\alpha,\beta}}(z) = \frac{\beta-\alpha}{\pi}\, i \left(\frac{2}{z} \log \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z}{1-z} + \frac{1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}}{(1-z)\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}} z\right)} \right).$$

Therefore,

$$\left(1-|z|^2\right)|T_f(z)| \leq \left(1-|z|^2\right)\left|T_{f_{\alpha,\beta}}(z)\right|,$$

hence we have

$$\sup \left(1-|z|^2\right)|T_f(z)| \leq \frac{3(\beta-\alpha)}{\pi}\left(1 - e^{2\pi i \frac{1-\alpha}{\beta-\alpha}}\right).$$

This completes the proof of our theorem. □

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