

# Transmission problem between two Herschel-Bulkley fluids in thin layer

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**Abstract.** The paper is devoted to the study of steady-state transmission problem between two Herschel-Bulkley fluids in thin layer.

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**Keywords:** Herschel-Bulkley fluid, transmission, thin layer, asymptotic behaviour.


## 1. Introduction

The rigid viscoplastic and incompressible fluid of Herschel-Bulkley has been studied and used by many mathematicians, physicists and engineers, to model the flow of metals, plastic solids and a variety of polymers. Due to the existence of the yield limit, the model can capture phenomena connected with the development of discontinuous stresses. A particularity of Herschel-Bulkley fluid lies in the presence of rigid zones located in the interior of the flow and as yield limit increases, the rigid zones become larger and may completely block the flow, this phenomenon is known as the blockage property. The literature concerning this topic is extensive; see e.g. [7, 8, 9, 11]. The purpose of this paper is to study the asymptotic behavior of the steady flow of Herschel-Bulkley fluid in a two-dimensional thin layer. The paper is organized as follows. In section 2 we present the mechanical problem of the steady flow of Herschel-Bulkley fluid in a two-dimensional thin layer. We introduce some notations and preliminaries. Moreover, we define some function spaces and we recall the variational formulation. In Section 3, we are interested in the asymptotic behavior, to this aim we prove some convergence results concerning the velocity and pressure when the thickness tends to zero. Besides, the uniqueness of a limit solution has been also established.

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### 2. Problem statement

Denoting by  $I$  the open interval  $I = ]0, 1[$ . Introducing the functions  $h_i : I \rightarrow \mathbb{R}_+^*$  such that  $h_i \in C^1(I)$ ,  $i = 1, 2$ .

Considering the following domains

$$\begin{aligned} \Omega_1 &= \{(x, y) \in \mathbb{R}^2 / x \in I \text{ and } 0 < y < h_1(x)\}, \\ \Omega_1^\varepsilon &= \{(x_1, x_2) \in \mathbb{R}^2 / x_1 \in I \text{ and } 0 < x_2 < \varepsilon h_1(x_1)\}, \\ \Omega_2 &= \{(x, y) \in \mathbb{R}^2 / x \in I \text{ and } h_1(x) < y < h_2(x)\}, \\ \Omega_2^\varepsilon &= \{(x_1, x_2) \in \mathbb{R}^2 / x_1 \in I \text{ and } \varepsilon h_1(x_1) < x_2 < \varepsilon h_2(x_1)\}, \end{aligned}$$

where  $\varepsilon > 0$ . Remark that if  $(x_1, x_2) \in \Omega_i^\varepsilon$  then  $(x, y) = (x_1, \frac{x_2}{\varepsilon}) \in \Omega_i$ . This permits us to define, for every function  $\varphi_i^\varepsilon : \Omega_i^\varepsilon \rightarrow \mathbb{R}$ , the function  $\widehat{\varphi}_i^\varepsilon : \Omega_i \rightarrow \mathbb{R}$  given by  $\widehat{\varphi}_i^\varepsilon(x, y) = \varphi_i^\varepsilon(x_1, x_2)$ ,  $i = 1, 2$ . Let  $1 < p \leq 2$ ,  $p'$  the conjugate  $p$ ,  $(\frac{1}{p} + \frac{1}{p'} = 1)$  and  $f_i = (f_{i1}, f_{i2}) \in L^{p'}(\Omega_i)^2$  a given functions. We define the functions  $f_i^\varepsilon \in L^{p'}(\Omega_i^\varepsilon)^2$  such that  $\widehat{f}_i^\varepsilon = f_i$ ,  $i = 1, 2$ . We consider a mathematical problem modeling the steady flow of a rigid viscoplastic and incompressible Herschel-Bulkley fluid. We suppose that the consistency and yield limit of the fluid are respectively  $\mu_i \varepsilon^p$ ,  $g_i \varepsilon$  where  $\mu_i, g_i > 0$ ,  $i = 1, 2$  and  $p$  represents the power-law index. The first fluid occupies a bounded domain  $\Omega_1^\varepsilon \subset R^2$  with the boundary  $\partial\Omega_1^\varepsilon$  of class  $C^1$ . The second one occupies a bounded domain  $\Omega_2^\varepsilon \subset \mathbb{R}^2$  with the boundary  $\partial\Omega_2^\varepsilon$  of class  $C^1$ . We denote by  $\Omega^\varepsilon$  the domain  $\Omega_1^\varepsilon \cup \Omega_2^\varepsilon$  and we suppose that  $\partial\Omega_1^\varepsilon = \Gamma_0 \cup \Gamma_1$  and  $\partial\Omega_2^\varepsilon = \Gamma_0 \cup \Gamma_2$  the velocity is known and equal to zero, where  $\Gamma_0, \Gamma_1, \Gamma_2$  are measurable domains and  $meas(\Gamma_1), meas(\Gamma_2) > 0$ . The fluids are acted upon by given volume forces of densities  $f_1, f_2$  respectively. We denote by  $S_2$  the space of symmetric tensors on  $\mathbb{R}^2$ . We define the inner product and the Euclidean norm on  $\mathbb{R}^2$  and  $S_2$ , respectively, by

$$\begin{aligned} u.v &= u_l v_l \quad \forall u, v \in \mathbb{R}^2 \quad \text{and} \quad \sigma.\tau = \sigma_{lm} \tau_{lm} \quad \forall \sigma, \tau \in S_2. \\ |u| &= (u.u)^{\frac{1}{2}} \quad \forall u \in \mathbb{R}^2 \quad \text{and} \quad |\sigma| = (\sigma.\sigma)^{\frac{1}{2}} \quad \forall \sigma \in S_2. \end{aligned}$$

Here and below, the indices  $l$  and  $m$  run from 1 to 2 and the summation convention over repeated indices is used. We denote by  $\widetilde{\sigma}_i^\varepsilon$  the deviator of  $\sigma_i^\varepsilon$  given by

$$\sigma_i^\varepsilon = -p_i^\varepsilon I_2 + \widetilde{\sigma}_i^\varepsilon,$$

where  $p_i^\varepsilon$ ,  $i = 1, 2$  represents the hydrostatic pressure and  $I_2$  denotes the identity matrix of size 2. We consider the rate of deformation operator defined for every  $v_i^\varepsilon \in W^{1,p}(\Omega_i^\varepsilon)^2$  by

$$D(v_i^\varepsilon) = (D_{lm}(v_i^\varepsilon)), \quad D_{lm}(v_i^\varepsilon) = \frac{1}{2}((v_i^\varepsilon)_{l,m} + (v_i^\varepsilon)_{m,l}), \quad i = 1, 2.$$

We denote by  $n$  the unit outward normal vector on the boundary  $\Gamma_0$  oriented to the exterior of  $\Omega_1^\varepsilon$  and to the interior of  $\Omega_2^\varepsilon$ , see the figure below. For every vector field  $v_i^\varepsilon \in W^{1,p}(\Omega_i^\varepsilon)^2$  we also write  $v_i^\varepsilon$  for its trace on  $\partial\Omega_i^\varepsilon$ ,  $i = 1, 2$ .

The steady-state transmission problem for the Herschel-Bulkley fluids in thin layer is given by the following mechanical problem.

Problem  $P_\varepsilon$ . Find the velocity field  $u_i^\varepsilon = (u_{i1}^\varepsilon, u_{i2}^\varepsilon) : \Omega_i^\varepsilon \rightarrow \mathbb{R}^2$ , the stress field  $\sigma_i^\varepsilon = (\sigma_{i1}^\varepsilon, \sigma_{i2}^\varepsilon) : \Omega_i^\varepsilon \rightarrow S_2$  and the pressure  $p_i^\varepsilon : \Omega_i^\varepsilon \rightarrow \mathbb{R}$ ,  $i = 1, 2$  such that

$$\operatorname{div} \sigma_1^\varepsilon + f_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon. \quad (2.1)$$

$$\operatorname{div} \sigma_2^\varepsilon + f_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon. \quad (2.2)$$

$$\left. \begin{aligned} \widetilde{\sigma}_1^\varepsilon &= \mu_1 \varepsilon^p |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon) + g_1 \varepsilon \frac{D(u_1^\varepsilon)}{|D(u_1^\varepsilon)|} \text{ if } |D(u_1^\varepsilon)| \neq 0 \\ \left| \widetilde{\sigma}_1^\varepsilon \right| &\leq g_1 \varepsilon \text{ if } |D(u_1^\varepsilon)| = 0 \end{aligned} \right\} \text{ in } \Omega_1^\varepsilon, \quad (2.3)$$

$$\left. \begin{aligned} \widetilde{\sigma}_2^\varepsilon &= \mu_2 \varepsilon^p |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon) + g_2 \varepsilon \frac{D(u_2^\varepsilon)}{|D(u_2^\varepsilon)|} \text{ if } |D(u_2^\varepsilon)| \neq 0 \\ \left| \widetilde{\sigma}_2^\varepsilon \right| &\leq g_2 \varepsilon \text{ if } |D(u_2^\varepsilon)| = 0 \end{aligned} \right\} \text{ in } \Omega_2^\varepsilon, \quad (2.4)$$

$$\operatorname{div} u_1^\varepsilon = 0 \text{ in } \Omega_1^\varepsilon, \quad (2.5)$$

$$\operatorname{div} u_2^\varepsilon = 0 \text{ in } \Omega_2^\varepsilon, \quad (2.6)$$

$$u_1^\varepsilon = 0 \text{ on } \Gamma_1, \quad (2.7)$$

$$u_2^\varepsilon = 0 \text{ on } \Gamma_2, \quad (2.8)$$

$$u_1^\varepsilon - u_2^\varepsilon = 0 \text{ on } \Gamma_0, \quad (2.9)$$

$$\sigma_1^\varepsilon \cdot \mathbf{n} - \sigma_2^\varepsilon \cdot \mathbf{n} = 0 \text{ on } \Gamma_0. \quad (2.10)$$

Here, the flow is given by the equations (2.1) and (2.2). Equations (2.3) and (2.4) represent the constitutive law of Herschel-Bulkley fluid. Equations (2.5) and (2.6) represents the incompressibility condition. Equality (2.7), (2.8) give the velocities on the boundaries  $\Gamma_1$  and  $\Gamma_2$ , respectively. Finally, on the boundary part  $\Gamma_0$ , equations (2.9) and (2.10) represent the transmission condition for liquid-liquid interface. Let us define now the following Banach spaces

$$W_{\Gamma_i}^{1,p}(\Omega_i^\varepsilon) = \{v_i \in W^{1,p}(\Omega_i^\varepsilon)^2 : v_i = 0 \text{ on } \Gamma_i, i = 1, 2\}, \quad (2.11)$$

$$W_{\operatorname{div}}^{p,\varepsilon}(\Omega_i^\varepsilon) = \{v_i \in W^{1,p}(\Omega_i^\varepsilon)^2 : \operatorname{div}(v_i) = 0 \text{ in } \Omega_i^\varepsilon, i = 1, 2\}, \quad (2.12)$$

$$W_{\operatorname{div}}^p(\Omega_i) = \{v_i \in W^{1,p}(\Omega_i)^2 : \operatorname{div}(v_i) = 0 \text{ in } \Omega_i, i = 1, 2\}, \quad (2.13)$$

$$W_p = \left\{ \begin{aligned} (\varphi_1, \varphi_2) &\in L^p(\Omega_1) \times L^p(\Omega_2) : \\ \frac{\partial \varphi_1}{\partial y} &\in L^p(\Omega_1), \frac{\partial \varphi_2}{\partial y} \in L^p(\Omega_2) \end{aligned} \right\}, \quad (2.14)$$

$$L_0^p(\Omega_i^\varepsilon) = \left\{ \begin{aligned} \varphi_i^\varepsilon &\in L^p(\Omega_i^\varepsilon) : \\ \int_{\Omega_i^\varepsilon} \varphi_i^\varepsilon(x_1, x_2) dx_1 dx_2 &= 0, i = 1, 2 \end{aligned} \right\}, \quad (2.15)$$

$$L_0^p(\Omega_i) = \left\{ \varphi_i \in L^p(\Omega_i) : \int_{\Omega_i} \varphi_i(x, y) dx dy = 0, i = 1, 2 \right\}, \quad (2.16)$$

$$W^\varepsilon = \left\{ \begin{aligned} (v_1, v_2) &\in W_{\operatorname{div}}^{p,\varepsilon}(\Omega_1^\varepsilon) \times W_{\operatorname{div}}^{p,\varepsilon}(\Omega_2^\varepsilon) : v_1 = v_2 \\ &\text{on } \Gamma_0, v_1 = 0 \text{ on } \Gamma_1, v_2 = 0 \text{ on } \Gamma_2 \end{aligned} \right\}. \quad (2.17)$$

For the rest of this article, we will denote by  $c$  possibly different positive constants depending only on the data of the problem.

The use of Green's formula permits us to derive the following variational formulation of the mechanical problem  $(P_\varepsilon)$ , see [10, 11].

Problem  $PV_\varepsilon$  . For prescribed data  $(f_1^\varepsilon, f_2^\varepsilon) \in L^{p'}(\Omega_1^\varepsilon)^2 \times L^{p'}(\Omega_2^\varepsilon)^2$ . Find  $(u_1^\varepsilon, u_2^\varepsilon) \in W^\varepsilon$  and  $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$  satisfying the variational inequality

$$\begin{aligned} & \mu_1 \varepsilon^p \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon) D(v_1 - u_1^\varepsilon) dx_1 dx_2 + g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1)| dx_1 dx_2 \\ & - g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)| dx_1 dx_2 + \mu_2 \varepsilon^p \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon) D(v_2 - u_2^\varepsilon) dx_1 dx_2 \\ & + g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2)| dx_1 dx_2 - g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)| dx_1 dx_2 \\ & \geq \int_{\Omega_1^\varepsilon} f_1^\varepsilon \cdot (v_1 - u_1^\varepsilon) dx_1 dx_2 + \int_{\Omega_1^\varepsilon} p_1^\varepsilon \operatorname{div}(v_1 - u_1^\varepsilon) dx_1 dx_2 \\ & + \int_{\Omega_2^\varepsilon} f_2^\varepsilon \cdot (v_2 - u_2^\varepsilon) dx_1 dx_2 + \int_{\Omega_2^\varepsilon} p_2^\varepsilon \operatorname{div}(v_2 - u_2^\varepsilon) dx_1 dx_2, \quad \forall (v_1, v_2) \in W^\varepsilon. \end{aligned} \tag{2.18}$$

It is known that this variational problem has a unique solution  $(u_1^\varepsilon, u_2^\varepsilon) \in W^\varepsilon$  and  $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$ , see for more details [7, 10, 11].

### 3. Asymptotic behavior

In this section, we establish some results concerning the asymptotic behavior of the solution when  $\varepsilon$  tends to zero. We begin by recalling the following lemmas (see [12, 1, 3, 6])

**Lemma 3.1.** 1. *Poincaré’s inequality.* For every  $v_i \in W_{\Gamma_i}^{1,p}(\Omega_i^\varepsilon)$  we have

$$\|v_i^\varepsilon\|_{L^p(\Omega_i^\varepsilon)^2} \leq \varepsilon \left\| \frac{\partial v_i^\varepsilon}{\partial x_2} \right\|_{L^p(\Omega_i^\varepsilon)^2}, \quad i = 1, 2. \tag{3.1}$$

2. *Korn’s inequality.* For every  $v_i \in W_{\Gamma_i}^{1,p}(\Omega_i^\varepsilon)$  there exists a positive constant  $C_0$  independent on  $\varepsilon$ , such that

$$\|\nabla v_i^\varepsilon\|_{L^p(\Omega_i^\varepsilon)^4} \leq C_0 \|D(v_i^\varepsilon)\|_{L^p(\Omega_i^\varepsilon)^4}, \quad i = 1, 2. \tag{3.2}$$

**Lemma 3.2.** Let  $E$  be a Banach space,  $A : E \rightarrow E'$  a monotone and hemi-continuous operator,  $J : E \rightarrow ]-\infty, +\infty]$  a proper and convex functional. Let  $u \in E$  and  $f \in E'$ . The following assertions are equivalent:

1.  $\langle Au; v - u \rangle_{E' \times E} + J(v) - J(u) \geq \langle f; v - u \rangle_{E' \times E} \quad \forall v \in E.$
2.  $\langle Av; v - u \rangle_{E' \times E} + J(v) - J(u) \geq \langle f; v - u \rangle_{E' \times E} \quad \forall v \in E.$

The main results of this section are stated by the following proposition.

**Proposition 3.3.** Let  $(u_1^\varepsilon, u_2^\varepsilon) \in W^\varepsilon$  and  $(p_1^\varepsilon, p_2^\varepsilon) \in L_0^{p'}(\Omega_1^\varepsilon) \times L_0^{p'}(\Omega_2^\varepsilon)$  be the solution of variational problem  $(PV_\varepsilon)$ . Then, there exists  $(\widehat{u}_1, \widehat{u}_2) \in W_p^2$  and  $(\widehat{p}_1, \widehat{p}_2) \in L_0^{p'}(\Omega_1) \times$

$L_0^{p'}(\Omega_2)$  such that

$$(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon) \rightarrow (\widehat{u}_1, \widehat{u}_2) \text{ in } W_p^2 \text{ weakly,} \tag{3.3}$$

$$\left( \frac{\partial \widehat{u}_{12}^\varepsilon}{\partial y}, \frac{\partial \widehat{u}_{22}^\varepsilon}{\partial y} \right) \rightarrow (0, 0) \text{ in } L^p(\Omega_1) \times L^p(\Omega_2) \text{ weakly,} \tag{3.4}$$

$$(\widehat{p}_1^\varepsilon, \widehat{p}_2^\varepsilon) \rightarrow (\widehat{p}_1, \widehat{p}_2) \text{ in } L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2) \text{ weakly.} \tag{3.5}$$

*Proof.* Choosing  $(v_1, v_2) = (0, 0)$  as test function in inequality (2.18), we deduce that

$$\begin{aligned} & \mu_1 \varepsilon^p \|D(u_1^\varepsilon)\|_{L^p(\Omega_1^\varepsilon)}^p + \mu_2 \varepsilon^p \|D(u_2^\varepsilon)\|_{L^p(\Omega_2^\varepsilon)}^p \\ & \leq \int_{\Omega_1^\varepsilon} f_1^\varepsilon \cdot u_1^\varepsilon dx_1 dx_2 + \int_{\Omega_2^\varepsilon} f_2^\varepsilon \cdot u_2^\varepsilon dx_1 dx_2. \end{aligned}$$

this permits us to obtain, making use of Poincaré’s and Korn’s inequalities and by passage to variables  $x$  and  $y$

$$\left\| \widehat{u}_1^\varepsilon \right\|_{L^p(\Omega_1)^2} + \left\| \widehat{u}_2^\varepsilon \right\|_{L^p(\Omega_2)^2} \leq c, \tag{3.6}$$

$$\left\| \frac{\partial \widehat{u}_1^\varepsilon}{\partial y} \right\|_{L^p(\Omega_1)^2} + \left\| \frac{\partial \widehat{u}_2^\varepsilon}{\partial y} \right\|_{L^p(\Omega_2)^2} \leq c, \tag{3.7}$$

$$\left\| \frac{\partial \widehat{u}_1^\varepsilon}{\partial x} \right\|_{L^p(\Omega_1)^2} + \left\| \frac{\partial \widehat{u}_2^\varepsilon}{\partial x} \right\|_{L^p(\Omega_2)^2} \leq \frac{c}{\varepsilon}. \tag{3.8}$$

Moreover, we get using the incompressibility condition (2.5), (2.6) and Green’s formula, for any function  $(\varphi_1^\varepsilon, \varphi_2^\varepsilon) \in W_{\Gamma_1}^{1,p}(\Omega_1^\varepsilon) \times W_{\Gamma_2}^{1,p}(\Omega_2^\varepsilon)$

$$\begin{aligned} & \int_{\Omega_1} \frac{\partial \widehat{u}_{12}^\varepsilon}{\partial y} \widehat{\varphi}_1^\varepsilon dx dy + \int_{\Omega_2} \frac{\partial \widehat{u}_{22}^\varepsilon}{\partial y} \widehat{\varphi}_2^\varepsilon dx dy \\ & = \varepsilon \int_{\Omega_1} \widehat{u}_{11}^\varepsilon \frac{\partial \widehat{\varphi}_1^\varepsilon}{\partial x} dx dy + \varepsilon \int_{\Omega_2} \widehat{u}_{21}^\varepsilon \frac{\partial \widehat{\varphi}_2^\varepsilon}{\partial x} dx dy. \end{aligned}$$

Which gives, making use of (2.14)

$$\left\| \frac{\partial \widehat{u}_{12}^\varepsilon}{\partial y} \right\|_{W^{-1,p'}(\Omega_1)} + \left\| \frac{\partial \widehat{u}_{22}^\varepsilon}{\partial y} \right\|_{W^{-1,p'}(\Omega_2)} \leq c\varepsilon. \tag{3.9}$$

We can then extract a subsequences still denoted by  $(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon)$  such that

$$(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon) \rightarrow (\widehat{u}_1, \widehat{u}_2) \text{ in } L^p(\Omega_1)^2 \times L^p(\Omega_2)^2 \text{ weakly,} \tag{3.10}$$

$$\left( \frac{\partial \widehat{u}_1^\varepsilon}{\partial y}, \frac{\partial \widehat{u}_2^\varepsilon}{\partial y} \right) \rightarrow \left( \frac{\partial \widehat{u}_1}{\partial y}, \frac{\partial \widehat{u}_2}{\partial y} \right) \text{ in } L^p(\Omega_1)^2 \times L^p(\Omega_2)^2 \text{ weakly,} \tag{3.11}$$

$$\left( \frac{\partial \widehat{u}_{12}^\varepsilon}{\partial y}, \frac{\partial \widehat{u}_{22}^\varepsilon}{\partial y} \right) \rightarrow (0, 0) \text{ in } L^p(\Omega_1) \times L^p(\Omega_2) \text{ weakly.} \tag{3.12}$$

Let now  $(v_1^\varepsilon, v_2^\varepsilon) \in W_{\Gamma_1}^{1,p}(\Omega_1^\varepsilon) \times W_{\Gamma_2}^{1,p}(\Omega_2^\varepsilon)$ , we obtain by setting  $(u_1^\varepsilon - v_1^\varepsilon, u_2^\varepsilon - v_2^\varepsilon)$  as test function in inequality (2.18), using the incompressibility conditions (2.5) and (2.6) as well as the Green formula and Hölder’s inequality

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} \nabla p_1^\varepsilon v_1^\varepsilon dx_1 dx_2 + \int_{\Omega_2^\varepsilon} \nabla p_2^\varepsilon v_2^\varepsilon dx_1 dx_2 \\ & \leq \mu_1 \varepsilon^p \left( \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p'}} \left( \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \\ & \quad + g_1 \varepsilon^{\frac{1}{p'}+1} \text{meas}(\Omega_1^\varepsilon)^{\frac{1}{p'}} \left( \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \\ & + \varepsilon \left\| \widehat{f}_1^\varepsilon \right\|_{L^{p'}(\Omega_1^\varepsilon)^2} \left\| \widehat{v}_1^\varepsilon \right\|_{W_{\Gamma_1}^{1,p}(\Omega_1)} + \varepsilon \left\| \widehat{f}_2^\varepsilon \right\|_{L^{p'}(\Omega_2^\varepsilon)^2} \left\| \widehat{v}_2^\varepsilon \right\|_{W_{\Gamma_2}^{1,p}(\Omega_2)} \\ & + \mu_2 \varepsilon^p \left( \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p'}} \left( \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \\ & \quad + g_2 \varepsilon^{\frac{1}{p'}+1} \text{meas}(\Omega_2^\varepsilon)^{\frac{1}{p'}} \left( \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}}. \end{aligned} \tag{3.13}$$

On the other hand, it is easy to check that, after some algebraic manipulations, we find

$$\left( \int_{\Omega_i^\varepsilon} |D(v_i^\varepsilon)|^p dx_1 dx_2 \right)^{\frac{1}{p}} \leq \varepsilon^{\frac{1}{p}-1} \left\| \widehat{v}_i^\varepsilon \right\|_{W_{\Gamma_i}^{1,p}(\Omega_i)}, \quad i = 1, 2. \tag{3.14}$$

Hence, from (3.7), (3.8), (3.13) and (3.14) it follows that

$$\begin{aligned} & \int_{\Omega_1^\varepsilon} \nabla p_1^\varepsilon v_1^\varepsilon dx_1 dx_2 + \int_{\Omega_2^\varepsilon} \nabla p_2^\varepsilon v_2^\varepsilon dx_1 dx_2 \\ & \leq c\varepsilon \left( \left\| \widehat{v}_1^\varepsilon \right\|_{W_{\Gamma_1}^{1,p}(\Omega_1)} + \left\| \widehat{v}_2^\varepsilon \right\|_{W_{\Gamma_2}^{1,p}(\Omega_2)} \right). \end{aligned} \tag{3.15}$$

Passing to the variables  $x$  and  $y$  in the left hand side of (3.15) we find the following estimates

$$\left\| \widehat{p}_1^\varepsilon \right\|_{L_0^{p'}(\Omega_1)} + \left\| \widehat{p}_2^\varepsilon \right\|_{L_0^{p'}(\Omega_2)} \leq c, \tag{3.16}$$

$$\left\| \frac{\partial \widehat{p}_1^\varepsilon}{\partial x} \right\|_{W^{-1,p'}(\Omega_1)} + \left\| \frac{\partial \widehat{p}_2^\varepsilon}{\partial x} \right\|_{W^{-1,p'}(\Omega_2)} \leq c, \tag{3.17}$$

$$\left\| \frac{\partial \widehat{p}_1^\varepsilon}{\partial y} \right\|_{W^{-1,p'}(\Omega_1)} + \left\| \frac{\partial \widehat{p}_2^\varepsilon}{\partial y} \right\|_{W^{-1,p'}(\Omega_2)} \leq \varepsilon c. \tag{3.18}$$

Consequently, we can extract a subsequence still denoted by  $(\widehat{p}_1^\varepsilon, \widehat{p}_2^\varepsilon)$  such that

$$(\widehat{p}_1^\varepsilon, \widehat{p}_2^\varepsilon) \rightarrow (\widehat{p}_1, \widehat{p}_2) \text{ in } L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2) \text{ weakly,} \tag{3.19}$$

which achieves the proof. This proof permits also to deduce that the limit pressure verify  $(\widehat{p}_1(x, y), \widehat{p}_2(x, y)) = (\widehat{p}_1(x), \widehat{p}_2(x))$ .  $\square$

**Proposition 3.4.** *The velocity limit given by (3.3) verifies*

$$\int_0^{h_1(x)} \widehat{u}_{11}(x, y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u}_{21}(x, y)dy = 0 \quad \forall x \in I. \tag{3.20}$$

*Proof.* We know from incompressibility conditions (2.5) and (2.6) that

$$\begin{aligned} \int_{\Omega_1^\varepsilon} \operatorname{div} u_1^\varepsilon(x_1, x_2)\varphi_1(x_1)dx_1dx_2 + \int_{\Omega_2^\varepsilon} \operatorname{div} u_2^\varepsilon(x_1, x_2)\varphi_2(x_1)dx_1dx_2 \\ = 0 \text{ for all } (\varphi_1, \varphi_2) \in D(I)^2. \end{aligned}$$

This implies, using Green’s formula

$$\begin{aligned} \int_{\Omega_1^\varepsilon} u_{11}^\varepsilon(x_1, x_2) \frac{d\varphi_1}{dx_1}(x_1)dx_1dx_2 + \int_{\Omega_2^\varepsilon} u_{21}^\varepsilon(x_1, x_2) \frac{d\varphi_2}{dx_1}(x_1)dx_1dx_2 \\ = \int_{\Omega_1^\varepsilon} \frac{\partial u_{12}^\varepsilon}{\partial x_2}(x_1, x_2)\varphi_1(x_1)dx_1dx_2 + \int_{\Omega_2^\varepsilon} \frac{\partial u_{22}^\varepsilon}{\partial x_2}(x_1, x_2)\varphi_2(x_1)dx_1dx_2. \end{aligned}$$

Hence, by passage to the variables  $x$  and  $y$  using Fubini’s theorem and Green’s formula, we can infer

$$\begin{aligned} \int_0^1 \varphi_1(x) \left( \frac{d}{dx} \int_0^{h_1(x)} \widehat{u}_{11}^\varepsilon(x, y)dy \right) dx + \int_0^1 \varphi_2(x) \left( \frac{d}{dx} \int_{h_1(x)}^{h_2(x)} \widehat{u}_{21}^\varepsilon(x, y)dy \right) dx \\ = 0 \quad \forall (\varphi_1, \varphi_2) \in D(I)^2. \end{aligned}$$

Then,

$$\int_0^1 \varphi(x) \left( \frac{d}{dx} \left( \int_0^{h_1(x)} \widehat{u}_{11}^\varepsilon(x, y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u}_{21}^\varepsilon(x, y)dy \right) \right) dx = 0, \quad \forall \varphi \in D(I).$$

Then,

$$\frac{d}{dx} \left( \int_0^{h_1(x)} \widehat{u}_{11}^\varepsilon(x, y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u}_{21}^\varepsilon(x, y)dy \right) = 0.$$

Moreover, the fact that  $(\widehat{u}_{11}^\varepsilon, \widehat{u}_{21}^\varepsilon) \in L^p(\Omega_1) \times L^p(\Omega_2)$  and  $(h_1, h_2) \in C^1(I)^2$  gives, using the Sobolev embedding  $W^{1,p}(I) \subset C^0(\bar{I})$

$$\int_0^{h_1(x)} \widehat{u}_{11}^\varepsilon(x, y)dy + \int_{h_1(x)}^{h_2(x)} \widehat{u}_{21}^\varepsilon(x, y)dy \in C^0(\bar{I}).$$

Thus, by passage to the limit when  $\varepsilon$  tends to zero, taking into account the boundaries conditions (2.7), (2.8) and (2.9), the assertion (3.20) can be deduced.  $\square$

We derive in the proposition below the strong equation verified by the limit solution  $(\widehat{u}_1, \widehat{u}_2) \in W_p^2$  and  $(\widehat{p}_1, \widehat{p}_2) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ .

**Proposition 3.5.** *If  $(\frac{\partial \widehat{u}_{11}}{\partial y}, \frac{\partial \widehat{u}_{21}}{\partial y}) \neq (0, 0)$  then the limit point  $(\widehat{u}_{11}, \widehat{u}_{21})$  and  $(\widehat{p}_1, \widehat{p}_2)$  given by (3.3) and (3.5) verify the limit problem*

$$\begin{aligned}
 -\frac{\partial}{\partial y} \left( \frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} + \frac{\sqrt{2}}{2} g_1 \operatorname{sign} \left( \frac{\partial \widehat{u}_{11}}{\partial y} \right) + \frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right. \\
 \left. + \frac{\sqrt{2}}{2} g_2 \operatorname{sign} \left( \frac{\partial \widehat{u}_{21}}{\partial y} \right) \right) \\
 = \widehat{f}_{11} - \frac{d\widehat{p}_1}{dx} + \widehat{f}_{21} - \frac{d\widehat{p}_2}{dx} \text{ in } W^{-1,p'}(\Omega). \tag{3.21}
 \end{aligned}$$

*Proof.* Introducing the operator  $\Phi$  defined as follows

$$\Phi : W^\varepsilon \rightarrow W^{\varepsilon'}$$

$$\begin{aligned}
 \langle \Phi(u_1^\varepsilon, u_2^\varepsilon), (v_1^\varepsilon, v_2^\varepsilon) \rangle_{W^{\varepsilon'} \times W^\varepsilon} &= \mu_1 \varepsilon^p \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)|^{p-2} D(u_1^\varepsilon) D(v_1^\varepsilon) dx_1 dx_2 \\
 &+ \mu_2 \varepsilon^p \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)|^{p-2} D(u_2^\varepsilon) D(v_2^\varepsilon) dx_1 dx_2.
 \end{aligned}$$

It is easy to verify that  $\Phi$  is monotone and hemi-continuous (see for more details the reference [11, 4, 2]). Moreover, we know that the functional

$$(v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon \rightarrow g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)| dx_1 dx_2 + g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)| dx_1 dx_2$$

is proper and convex. Then, the use of Minty’s lemma permits us to affirm that (2.18) is equivalent to the following inequality

$$\begin{aligned}
 &\mu_1 \varepsilon^p \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)|^{p-2} D(v_1^\varepsilon) D(v_1^\varepsilon - u_1^\varepsilon) dx_1 dx_2 + g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)| dx_1 dx_2 \\
 &- g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(u_1^\varepsilon)| dx_1 dx_2 + \mu_2 \varepsilon^p \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)|^{p-2} D(v_2^\varepsilon) D(v_2^\varepsilon - u_2^\varepsilon) dx_1 dx_2 \\
 &+ g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)| dx_1 dx_2 - g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(u_2^\varepsilon)| dx_1 dx_2 \\
 &\geq \int_{\Omega_1^\varepsilon} f_1^\varepsilon \cdot (v_1^\varepsilon - u_1^\varepsilon) dx_1 dx_2 + \int_{\Omega_1^\varepsilon} p_1^\varepsilon \operatorname{div}(v_1^\varepsilon - u_1^\varepsilon) dx_1 dx_2 \\
 &+ \int_{\Omega_2^\varepsilon} f_2^\varepsilon \cdot (v_2^\varepsilon - u_2^\varepsilon) dx_1 dx_2 + \int_{\Omega_2^\varepsilon} p_2^\varepsilon \operatorname{div}(v_2^\varepsilon - u_2^\varepsilon) dx_1 dx_2 \quad \forall (v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon.
 \end{aligned}$$

Our object now is to pass to the limit when  $\varepsilon$  tends to zero. To this aim, we use Proposition (3.3) and the weak lower semi-continuity of the convex and continuous functional

$$(v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon \rightarrow g_1 \varepsilon \int_{\Omega_1^\varepsilon} |D(v_1^\varepsilon)| dx_1 dx_2 + g_2 \varepsilon \int_{\Omega_2^\varepsilon} |D(v_2^\varepsilon)| dx_1 dx_2.$$



We find the following limit inequality

$$\begin{aligned}
 & \mu_1 \int_{\Omega_1} \frac{1}{2^{\frac{p-2}{2}}} \left[ \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v}_{12}}{\partial y} \right|^2 \right]^{\frac{p-2}{2}} \times \frac{1}{2} \left[ \frac{\frac{\partial \widehat{v}_{11}}{\partial y} \frac{\partial(\widehat{v}_{11} - \widehat{u}_{11})}{\partial y}}{\frac{\partial \widehat{v}_{12}}{\partial y} \frac{\partial(\widehat{v}_{12} - \widehat{u}_{12})}{\partial y}} \right] dx dy \\
 & + g_1 \int_{\Omega_1} \frac{1}{\sqrt{2}} \left[ \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v}_{12}}{\partial y} \right|^2 \right]^{\frac{1}{2}} dx dy - g_1 \int_{\Omega_1} \frac{1}{\sqrt{2}} \left[ \frac{\left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^2}{\left| \frac{\partial \widehat{u}_{12}}{\partial y} \right|^2} \right]^{\frac{1}{2}} dx dy \\
 & + \mu_2 \int_{\Omega_2} \frac{1}{2^{\frac{p-2}{2}}} \left[ \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v}_{22}}{\partial y} \right|^2 \right]^{\frac{p-2}{2}} \times \frac{1}{2} \left[ \frac{\frac{\partial \widehat{v}_{21}}{\partial y} \frac{\partial(\widehat{v}_{21} - \widehat{u}_{21})}{\partial y}}{\frac{\partial \widehat{v}_{22}}{\partial y} \frac{\partial(\widehat{v}_{22} - \widehat{u}_{22})}{\partial y}} \right] dx dy \\
 & + g_2 \int_{\Omega_2} \frac{1}{\sqrt{2}} \left[ \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right|^2 + \left| \frac{\partial \widehat{v}_{22}}{\partial y} \right|^2 \right]^{\frac{1}{2}} dx dy - g_2 \int_{\Omega_2} \frac{1}{\sqrt{2}} \left[ \frac{\left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^2}{\left| \frac{\partial \widehat{u}_{22}}{\partial y} \right|^2} \right]^{\frac{1}{2}} dx dy \\
 & \geq \int_{\Omega_1} \widehat{f}_1 \cdot (\widehat{v}_1 - \widehat{u}_1) dx dy + \int_{\Omega_1} \widehat{p}_1 \operatorname{div}(\widehat{v}_1 - \widehat{u}_1) dx dy + \int_{\Omega_2} \widehat{f}_2 \cdot (\widehat{v}_2 - \widehat{u}_2) dx dy \\
 & \quad + \int_{\Omega_2} \widehat{p}_2 \operatorname{div}(\widehat{v}_2 - \widehat{u}_2) dx dy \quad \forall (v_1^\varepsilon, v_2^\varepsilon) \in W^\varepsilon. \tag{3.22}
 \end{aligned}$$

Furthermore, from (3.3) and (3.4) we find

$$\left( \frac{\partial \widehat{u}_{12}}{\partial y}, \frac{\partial \widehat{u}_{22}}{\partial y} \right) = (0, 0) \text{ in } \Omega_1 \times \Omega_2.$$

It follows, keeping in mind (3.20), that  $\widehat{u}_1(x, y) = (\widehat{u}_{11}(x, y), 0)$  and,  $\widehat{u}_2(x, y) = (\widehat{u}_{21}(x, y), 0)$ , this permits also to choose  $(\widehat{v}_{12}, \widehat{v}_{22}) = (0, 0)$  in (3.22). Considering now the operator  $\Phi$  such that

$$\begin{aligned}
 & \Phi : W_p \rightarrow W'_p, \\
 & \langle \Phi(\widehat{u}_{11}, \widehat{u}_{21}), (\widehat{v}_{11}, \widehat{v}_{21}) \rangle_{W'_p \times W_p} \\
 & = \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \frac{\partial \widehat{v}_{11}}{\partial y} dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \frac{\partial \widehat{v}_{21}}{\partial y} dx dy.
 \end{aligned}$$

It is clear that the operator  $\Phi$  is monotone and hemi-continuous and the functional

$$(\widehat{v}_{11}, \widehat{v}_{21}) \in W_p \rightarrow \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right| dx dy + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right| dx dy$$

is proper and convex. Hence, we deduce using again Minty's lemma

$$\begin{aligned}
 & \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \frac{\partial(\widehat{v}_{11} - \partial \widehat{u}_{11})}{\partial y} dx dy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right| dx dy \\
 & - \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \frac{\partial(\widehat{v}_{21} - \partial \widehat{u}_{21})}{\partial y} dx dy \\
 & \quad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_1} \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right| dx dy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| dx dy
 \end{aligned}$$

$$\begin{aligned}
 &\geq \int_{\Omega_1} \widehat{f}_{11} \cdot (\widehat{v}_{11} - \widehat{u}_{11}) dx dy - \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} (\widehat{v}_{11} - \widehat{u}_{11}) dx dy \\
 &+ \int_{\Omega_2} \widehat{f}_{21} \cdot (\widehat{v}_{21} - \widehat{u}_{21}) dx dy - \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} (\widehat{v}_{21} - \widehat{u}_{21}) dx dy \quad \forall (\widehat{v}_{11}, \widehat{v}_{21}) \in W_p. \quad (3.23)
 \end{aligned}$$

This yields, via Green’s formula

$$\begin{aligned}
 &-\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) (\widehat{v}_{11} - \widehat{u}_{11}) dx dy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right| dx dy \\
 &-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| dx dy - \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) (\widehat{v}_{21} - \widehat{u}_{21}) dx dy \\
 &\quad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right| dx dy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| dx dy \\
 &\geq \int_{\Omega_1} \widehat{f}_{11} (\widehat{v}_{11} - \widehat{u}_{11}) dx dy - \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} (\widehat{v}_{11} - \widehat{u}_{11}) dx dy \\
 &+ \int_{\Omega_2} \widehat{f}_{21} (\widehat{v}_{21} - \widehat{u}_{21}) dx dy - \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} (\widehat{v}_{21} - \widehat{u}_{21}) dx dy \quad \forall (\widehat{v}_{11}, \widehat{v}_{21}) \in W_p. \quad (3.24)
 \end{aligned}$$

Due to the fact that  $W_{\Gamma_i}^{1,p}(\Omega_i)$  is dense in  $W_p(\Omega_i)$ , see [1, 5], we can take  $\widehat{v}_{11} = \widehat{u}_{11} \pm \varphi_1$  and  $\widehat{v}_{21} = \widehat{u}_{21} \pm \varphi_2$  in (3.24), where  $(\varphi_1, \varphi_2) \in W_{\Gamma_1}^{1,p}(\Omega_1) \times W_{\Gamma_2}^{1,p}(\Omega_2)$  to obtain the following inequalities

$$\begin{aligned}
 &-\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) \varphi_1 dx dy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial (\widehat{u}_{11} + \varphi_1)}{\partial y} \right| dx dy \\
 &-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| dx dy - \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) \varphi_2 dx dy \\
 &\quad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial (\widehat{u}_{21} + \varphi_2)}{\partial y} \right| dx dy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| dx dy \\
 &\geq \int_{\Omega_1} \widehat{f}_{11} \varphi_1 dx dy - \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} \varphi_1 dx dy + \int_{\Omega_2} \widehat{f}_{21} \varphi_2 dx dy \\
 &\quad - \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} \varphi_2 dx dy \quad \forall (\varphi_1, \varphi_2) \in W_{\Gamma_1}^{1,p}(\Omega_1) \times W_{\Gamma_2}^{1,p}(\Omega_2).
 \end{aligned}$$

and

$$\begin{aligned}
 &\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) \varphi_1 dx dy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial (\widehat{u}_{11} - \varphi_1)}{\partial y} \right| dx dy \\
 &-\frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) \varphi_2 dx dy \\
 &\quad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial (\widehat{u}_{21} - \varphi_2)}{\partial y} \right| dx dy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| dx dy
 \end{aligned}$$

$$\begin{aligned} &\geq - \int_{\Omega_1} \widehat{f}_{11} \varphi_1 dx dy + \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} \varphi_1 dx dy - \int_{\Omega_2} \widehat{f}_{21} \varphi_2 dx dy \\ &+ \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} \varphi_2 dx dy \quad \forall (\varphi_1, \varphi_2) \in W_{\Gamma_1}^{1,p}(\Omega_1) \times W_{\Gamma_2}^{1,p}(\Omega_2). \end{aligned}$$

Replacing in these two inequalities the test function  $(\varphi_1, \varphi_2)$  by  $(\lambda\varphi_1, \lambda\varphi_2)$ ,  $\lambda > 0$ , dividing the obtained inequalities by  $\lambda$ . The passage to the limit when  $\lambda$  tends to 0 implies, under the hypothesis  $\left(\frac{\partial \widehat{u}_{11}}{\partial y}, \frac{\partial \widehat{u}_{21}}{\partial y}\right) \neq (0, 0)$ , that

$$\begin{aligned} &-\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) \varphi_1 dx dy \\ &+ \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \text{sign} \left( \frac{\partial \widehat{u}_{11}}{\partial y} \right) \left( \frac{\partial \varphi_1}{\partial y} \right) dx dy \\ &-\frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) \varphi_2 dx dy \\ &+ \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \text{sign} \left( \frac{\partial \widehat{u}_{21}}{\partial y} \right) \left( \frac{\partial \varphi_2}{\partial y} \right) dx dy \\ &\geq \int_{\Omega_1} \widehat{f}_{11} \varphi_1 dx dy - \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} \varphi_1 dx dy + \int_{\Omega_2} \widehat{f}_{21} \varphi_2 dx dy \\ &- \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} \varphi_2 dx dy \quad \forall (\varphi_1, \varphi_2) \in W_{\Gamma_1}^{1,p}(\Omega_1) \times W_{\Gamma_2}^{1,p}(\Omega_2), \end{aligned}$$

and

$$\begin{aligned} &\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) \varphi_1 dx dy \\ &- \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \text{sign} \left( \frac{\partial \widehat{u}_{11}}{\partial y} \right) \left( \frac{\partial \varphi_1}{\partial y} \right) dx dy \\ &+ \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\partial}{\partial y} \left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) \varphi_2 dx dy \\ &- \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \text{sign} \left( \frac{\partial \widehat{u}_{21}}{\partial y} \right) \left( \frac{\partial \varphi_2}{\partial y} \right) dx dy \\ &\geq - \int_{\Omega_1} \widehat{f}_{11} \varphi_1 dx dy + \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} \varphi_1 dx dy - \int_{\Omega_2} \widehat{f}_{21} \varphi_2 dx dy \\ &+ \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} \varphi_2 dx dy \quad \forall (\varphi_1, \varphi_2) \in W_{\Gamma_1}^{1,p}(\Omega_1) \times W_{\Gamma_2}^{1,p}(\Omega_2). \end{aligned}$$

Consequently, we get by combining these two inequalities and using a simple integration by parts

$$- \int_{\Omega_1} \frac{\partial}{\partial y} \left[ \left( \frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_1 \text{sign} \left( \frac{\partial \widehat{u}_{11}}{\partial y} \right) \right] \varphi_1 dx dy$$

$$\begin{aligned}
 & - \int_{\Omega_2} \frac{\partial}{\partial y} \left[ \left( \frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_2 \operatorname{sign} \left( \frac{\partial \widehat{u}_{21}}{\partial y} \right) \right] \varphi_2 dx dy \\
 & = \int_{\Omega_1} \left( \widehat{f}_{11} - \frac{d\widehat{p}_1}{dx} \right) \varphi_1 dx dy + \int_{\Omega_2} \left( \widehat{f}_{21} - \frac{d\widehat{p}_2}{dx} \right) \varphi_2 dx dy \\
 & \quad \forall (\varphi_1, \varphi_2) \in W_{\Gamma_1}^{1,p}(\Omega_1) \times W_{\Gamma_2}^{1,p}(\Omega_2).
 \end{aligned}$$

Let us consider

$$\varphi \in W_0^{1,p}(\Omega) : \varphi = \begin{cases} \varphi_1 & \text{in } \Omega_1 \\ \varphi_2 & \text{in } \Omega_2 \end{cases},$$

and

$$\begin{aligned}
 \widetilde{a}_1 & = \begin{cases} -\frac{\partial}{\partial y} \left[ \left( \frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_1 \operatorname{sign} \left( \frac{\partial \widehat{u}_{11}}{\partial y} \right) \right] & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2, \end{cases} \\
 \widetilde{a}_2 & = \begin{cases} 0 & \text{in } \Omega_1, \\ -\frac{\partial}{\partial y} \left[ \left( \frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_2 \operatorname{sign} \left( \frac{\partial \widehat{u}_{21}}{\partial y} \right) \right] & \text{in } \Omega_2, \end{cases} \\
 \widetilde{b}_1 & = \begin{cases} \widehat{f}_{11} - \frac{d\widehat{p}_1}{dx} & \text{in } \Omega_1, \\ 0 & \text{in } \Omega_2, \end{cases} \\
 \widetilde{b}_2 & = \begin{cases} 0 & \text{in } \Omega_1, \\ \widehat{f}_{21} - \frac{d\widehat{p}_2}{dx} & \text{in } \Omega_2. \end{cases}
 \end{aligned}$$

Then,

$$\begin{aligned}
 \int_{\Omega} (\widetilde{a}_1 + \widetilde{a}_2) \varphi dx dy & = \int_{\Omega_1} (\widetilde{a}_1 + \widetilde{a}_2) \varphi_1 dx dy + \int_{\Omega_2} (\widetilde{a}_1 + \widetilde{a}_2) \varphi_2 dx dy \\
 & = \int_{\Omega_1} \widetilde{a}_1 \varphi_1 dx dy + \int_{\Omega_2} \widetilde{a}_2 \varphi_2 dx dy \\
 & = \int_{\Omega_1} -\frac{\partial}{\partial y} \left[ \left( \frac{\mu_1}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_1 \operatorname{sign} \left( \frac{\partial \widehat{u}_{11}}{\partial y} \right) \right] \varphi_1 dx dy \\
 & + \int_{\Omega_2} -\frac{\partial}{\partial y} \left[ \left( \frac{\mu_2}{2^{\frac{p}{2}}} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) + \frac{\sqrt{2}}{2} g_2 \operatorname{sign} \left( \frac{\partial \widehat{u}_{21}}{\partial y} \right) \right] \varphi_2 dx dy \\
 & = \int_{\Omega_1} \left( \widehat{f}_{11} - \frac{d\widehat{p}_1}{dx} \right) \varphi_1 dx dy + \int_{\Omega_2} \left( \widehat{f}_{21} - \frac{d\widehat{p}_2}{dx} \right) \varphi_2 dx dy \\
 & = \int_{\Omega_1} \widetilde{b}_1 \varphi_1 dx dy + \int_{\Omega_2} \widetilde{b}_2 \varphi_2 dx dy \\
 & = \int_{\Omega} (\widetilde{b}_1 + \widetilde{b}_2) \varphi dx dy \quad \forall \varphi \in W_0^{1,p}(\Omega).
 \end{aligned}$$

Which eventually gives (3.21).

From now on we will denote by  $(\widehat{u}_{11}, \widehat{u}_{21}) \in W_p$  and  $(\widehat{p}_1, \widehat{p}_2) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$  the solution of the limit problem (3.21). □

The following proposition shows the uniqueness of the limit solution  $(\widehat{u}_{11}, \widehat{p}_1)$  and  $(\widehat{u}_{21}, \widehat{p}_2)$ .

**Proposition 3.6.** *The limit strong problem (3.21) has a unique, solution  $(\widehat{u}_{11}, \widehat{u}_{21}) \in W_p$  and  $(\widehat{p}_1, \widehat{p}_2) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$  with the condition (3.20) .*

*Proof.* Suppose that the limit problem (3.21) has at least two solutions  $(\widehat{u}_{11}, \widehat{u}_{21}) \in W_p$  ,  $(\widehat{p}_1, \widehat{p}_2) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$  and  $(\overline{\widehat{u}}_{11}, \overline{\widehat{u}}_{21}) \in W_p$ ,  $(\overline{\widehat{p}}_1, \overline{\widehat{p}}_2) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_2)$ . In particular,  $(\widehat{u}_{11}, \widehat{u}_{21})$ ,  $(\widehat{p}_1, \widehat{p}_2)$  and  $(\overline{\widehat{u}}_{11}, \overline{\widehat{u}}_{21})$ ,  $(\overline{\widehat{p}}_1, \overline{\widehat{p}}_2)$  are solutions of the weak formulation (3.23). Then

$$\begin{aligned}
& \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \frac{\partial(\widehat{v}_{11} - \partial \widehat{u}_{11})}{\partial y} dx dy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right| dx dy \\
& - \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \overline{\widehat{u}}_{11}}{\partial y} \right| dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \frac{\partial(\widehat{v}_{21} - \partial \widehat{u}_{21})}{\partial y} dx dy \\
& \quad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right| dx dy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \overline{\widehat{u}}_{21}}{\partial y} \right| dx dy \\
& \geq \int_{\Omega_1} \widehat{f}_{11}(\widehat{v}_{11} - \overline{\widehat{u}}_{11}) dx dy - \int_{\Omega_1} \frac{d\widehat{p}_1}{dx} (\widehat{v}_{11} - \overline{\widehat{u}}_{11}) dx dy \\
& + \int_{\Omega_2} \widehat{f}_{21}(\widehat{v}_{21} - \overline{\widehat{u}}_{21}) dx dy - \int_{\Omega_2} \frac{d\widehat{p}_2}{dx} (\widehat{v}_{21} - \overline{\widehat{u}}_{21}) dx dy \quad \forall (\widehat{v}_{11}, \widehat{v}_{21}) \in W_p, \quad (3.25)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left| \frac{\partial \overline{\widehat{u}}_{11}}{\partial y} \right|^{p-2} \frac{\partial \overline{\widehat{u}}_{11}}{\partial y} \frac{\partial(\widehat{v}_{11} - \partial \overline{\widehat{u}}_{11})}{\partial y} dx dy + \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \widehat{v}_{11}}{\partial y} \right| dx dy \\
& - \frac{\sqrt{2}}{2} g_1 \int_{\Omega_1} \left| \frac{\partial \overline{\widehat{u}}_{11}}{\partial y} \right| dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left| \frac{\partial \overline{\widehat{u}}_{21}}{\partial y} \right|^{p-2} \frac{\partial \overline{\widehat{u}}_{21}}{\partial y} \frac{\partial(\widehat{v}_{21} - \partial \overline{\widehat{u}}_{21})}{\partial y} dx dy \\
& \quad + \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \widehat{v}_{21}}{\partial y} \right| dx dy - \frac{\sqrt{2}}{2} g_2 \int_{\Omega_2} \left| \frac{\partial \overline{\widehat{u}}_{21}}{\partial y} \right| dx dy \\
& \geq \int_{\Omega_1} \widehat{f}_{11}(\widehat{v}_{11} - \overline{\widehat{u}}_{11}) dx dy - \int_{\Omega_1} \frac{d\overline{\widehat{p}}_1}{dx} (\widehat{v}_{11} - \overline{\widehat{u}}_{11}) dx dy \\
& + \int_{\Omega_2} \widehat{f}_{21}(\widehat{v}_{21} - \overline{\widehat{u}}_{21}) dx dy - \int_{\Omega_2} \frac{d\overline{\widehat{p}}_2}{dx} (\widehat{v}_{21} - \overline{\widehat{u}}_{21}) dx dy \quad \forall (\widehat{v}_{11}, \widehat{v}_{21}) \in W_p. \quad (3.26)
\end{aligned}$$

Setting  $\widehat{v}_{11} = \overline{\widehat{u}}_{11}$  ,  $\widehat{v}_{21} = \overline{\widehat{u}}_{21}$  and  $\widehat{v}_{11} = \widehat{u}_{11}$  ,  $\widehat{v}_{21} = \widehat{u}_{21}$  as test functions in (3.25) and (3.26), respectively. Subtracting the two obtained inequalities, we can infer

$$\begin{aligned}
 & \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} - \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{11}}{\partial y} \right) \frac{\partial(\widehat{u}_{11} - \widehat{u}_{11})}{\partial y} dx dy \\
 & + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} - \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right|^{p-2} \frac{\partial \widehat{u}_{21}}{\partial y} \right) \frac{\partial(\widehat{u}_{21} - \widehat{u}_{21})}{\partial y} dx dy \\
 & \leq \int_{\Omega_1} \frac{d(\widehat{p}_1 - \widehat{p}_1)}{dx} (\widehat{u}_{11} - \widehat{u}_{11}) dx dy + \int_{\Omega_2} \frac{d(\widehat{p}_2 - \widehat{p}_2)}{dx} (\widehat{u}_{21} - \widehat{u}_{21}) dx dy. \tag{3.27}
 \end{aligned}$$

Observe that for every  $x, y \in \mathbb{R}^n$ ,

$$\left( |x|^{p-2} x - |y|^{p-2} y \right) (x - y) \geq c \frac{|x - y|^2}{(|x| + |y|)^{2-p}}, \quad 1 < p \leq 2.$$

This leads, making use (3.27), to

$$\begin{aligned}
 & \frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\left| \frac{\partial(\widehat{u}_{11} - \widehat{u}_{11})}{\partial y} \right|^2}{\left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| + \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| \right)^{2-p}} dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\left| \frac{\partial(\widehat{u}_{21} - \widehat{u}_{21})}{\partial y} \right|^2}{\left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| + \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| \right)^{2-p}} dx dy \\
 & \leq \int_{\Omega_1} \frac{d(\widehat{p}_1 - \widehat{p}_1)}{dx} (\widehat{u}_{11} - \widehat{u}_{11}) dx dy + \int_{\Omega_2} \frac{d(\widehat{p}_2 - \widehat{p}_2)}{dx} (\widehat{u}_{21} - \widehat{u}_{21}) dx dy \\
 & = \int_0^1 \left[ \frac{d(\widehat{p}_1 - \widehat{p}_1)}{dx} \int_0^{h_1(x)} (\widehat{u}_{11} - \widehat{u}_{11}) dy \right] dx \\
 & + \int_0^1 \left[ \frac{d(\widehat{p}_2 - \widehat{p}_2)}{dx} \int_{h_1(x)}^{h_2(x)} (\widehat{u}_{21} - \widehat{u}_{21}) dy \right] dx.
 \end{aligned}$$

The use of (3.20) gives

$$\frac{\mu_1}{2^{\frac{p}{2}}} \int_{\Omega_1} \frac{\left| \frac{\partial(\widehat{u}_{11} - \widehat{u}_{11})}{\partial y} \right|^2}{\left( \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| + \left| \frac{\partial \widehat{u}_{11}}{\partial y} \right| \right)^{2-p}} dx dy + \frac{\mu_2}{2^{\frac{p}{2}}} \int_{\Omega_2} \frac{\left| \frac{\partial(\widehat{u}_{21} - \widehat{u}_{21})}{\partial y} \right|^2}{\left( \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| + \left| \frac{\partial \widehat{u}_{21}}{\partial y} \right| \right)^{2-p}} dx dy = 0. \tag{3.28}$$

Which gives, keeping in mind (3.28)

$$\left( \frac{\partial(\widehat{u}_{11} - \widehat{u}_{11})}{\partial y}, \frac{\partial(\widehat{u}_{21} - \widehat{u}_{21})}{\partial y} \right) = (0, 0).$$

Since  $(\widehat{u}_{11}(x, h_1(x)), \widehat{u}_{21}(x, h_2(x))) = (\widehat{u}_{11}(x, h_1(x)), \widehat{u}_{21}(x, h_2(x))) = (0, 0)$ , we deduce that  $(\widehat{u}_{11}, \widehat{u}_{21}) = (\widehat{u}_{11}, \widehat{u}_{21})$  a.e. in  $\Omega_1 \times \Omega_2$ . Finally, to prove the uniqueness of the pressure, we use equation (3.21), with the two pressures  $(\widehat{p}_1, \widehat{p}_1)$  and  $(\widehat{p}_2, \widehat{p}_2)$ . We find

$$\frac{d(\widehat{p}_1 - \widehat{p}_1)}{dx} = 0 \quad \text{and} \quad \frac{d(\widehat{p}_2 - \widehat{p}_2)}{dx} = 0.$$

Then, due to fact that  $(\widehat{p}_1, \overline{\widehat{p}_1}) \in L_0^{p'}(\Omega_1) \times L_0^{p'}(\Omega_1)$ ,  $(\widehat{p}_2, \overline{\widehat{p}_2}) \in L_0^{p'}(\Omega_2) \times L_0^{p'}(\Omega_2)$  the result can be easily deduced.  $\square$

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