Stud. Univ. Babeş-Bolyai Math. 68(2023), No. 2, 387–398
DOI: 10.24193/subbmath.2023.2.13

A coupled system of fractional difference equations with anti-periodic boundary conditions

Jagan Mohan Jonnalagadda

Abstract. In this article, we give sufficient conditions for the existence, uniqueness and Ulam–Hyers stability of solutions for a coupled system of two-point nabla fractional difference boundary value problems subject to anti-periodic boundary conditions, using the vector approach of Precup [4, 14, 19, 21]. Some examples are included to illustrate the theory.

Mathematics Subject Classification (2010): 39A12, 39A70.

Keywords: Nabla fractional difference, boundary value problem, anti-periodic boundary conditions, existence, uniqueness, Ulam–Hyers stability.

1. Introduction

In [21], Precup described the advantage of vector-valued norms in the study of the semilinear operator system

$$\begin{cases} N_1(u_1, u_2) = u_1, \\ N_2(u_1, u_2) = u_2, \end{cases} \tag{1.1}$$

in a Banach space X with norm $|\cdot|$, by some methods of nonlinear analysis. Here N_1, $N_2 : X^2 \to X$ are given nonlinear operators. Obviously, this system can be viewed as a fixed point problem:

$$Nu = u, \tag{1.2}$$

Received 07 March 2020; Accepted 13 April 2020.

in the space X^2, where $u = (u_1, u_2)$ and $N = (N_1, N_2)$. Precup [21] proposed the applications of a few fixed point theorems to the system 1.1 in X^2, by using the vector-valued norm

$$\|u\| = \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix},$$

for $u = (u_1, u_2) \in X^2$. Also, Precup [21] demonstrated that the results obtained by using the vector-valued norm are better than those established by means of any scalar norm in X^2.

Theorem 1.1. [21] *Assume that*

(H1) *for each $i \in \{1, 2\}$, there exist nonnegative numbers a_i and b_i such that*

$$|N_i(u_1, u_2) - N_i(v_1, v_2)| \leq a_i|u_1 - v_1| + b_i|u_2 - v_2|, \tag{1.3}$$

for all (u_1, u_2), $(v_1, v_2) \in X^2$;

(H2) *The spectral radius of $M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ is less than one.*

Then, (1.1) *has a unique solution $(u_1, u_2) \in X^2$.*

Theorem 1.2. [21] *Assume that*

(H3) *for each $i \in \{1, 2\}$, the operator N_i is completely continuous and, there exist nonnegative numbers a_i, b_i and c_i such that*

$$|N_i(u_1, u_2)| \leq a_i|u_1| + b_i|u_2| + c_i, \tag{1.4}$$

for all $(u_1, u_2) \in X^2$.

In addition, assume that condition (H2) is satisfied. Then, (1.1) *has at least one solution $(u_1, u_2) \in X^2$ satisfying*

$$\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \leq (I - M)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{1.5}$$

Further, in [25], the author used the following theorem to establish Ulam–Hyers stability of solutions of (1.1):

Theorem 1.3. [25] *Assume that the hypothesis of Theorem 1.1 holds. Then, the system* (1.1) *is Ulam–Hyers stable.*

Motivated by these results, in this article, we consider the following coupled system of nabla fractional difference equations with anti-periodic boundary conditions

$$\begin{cases} \Big(\nabla_0^{\alpha_1 - 1}\big(\nabla u_1\big)\Big)(t) + f_1(u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}_2^T, \\ \Big(\nabla_0^{\alpha_2 - 1}\big(\nabla u_2\big)\Big)(t) + f_2(u_1(t), u_2(t)) = 0, \quad t \in \mathbb{N}_2^T, \\ u_1(0) + u_1(T) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(T) = 0, \\ u_2(0) + u_2(T) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(T) = 0, \end{cases} \tag{1.6}$$

and apply Theorems 1.1 - 1.3 to establish sufficient conditions on existence, uniqueness, and Ulam–Hyers stability [5, 6, 7, 17, 11, 13, 15, 22, 23, 24] of its solutions. For this purpose, we convert the system (1.6) in the form of (1.1). But the results may not be straightforward because the computation of nonnegative numbers in each theorem for the system (1.6) is complicated due to the presence of nabla fractional difference operators in it.

Here $T \in \mathbb{N}_2$; $1 < \alpha_1, \alpha_2 < 2$; f_1, $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are continuous, ∇_0^ν denotes the ν^{th}-th order Riemann–Liouville type backward (nabla) difference operator where $\nu \in \{\alpha_1 - 1, \alpha_2 - 1\}$ and ∇ denotes the first order nabla difference operator.

The present article is organized as follows: Section 2 contains preliminaries. In Section 3, we establish sufficient conditions on existence, uniqueness, and Ulam–Hyers stability of solutions of the system (1.6). We provide two examples in Section 4 to illustrate the applicability of established results.

2. Preliminaries

For our convenience, in this section, we present a few useful definitions and fundamental facts of nabla fractional calculus, which can be found in [1, 2, 3, 8, 9, 10, 16, 18, 20].

Denote by $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for any a, $b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by $\rho(t) = \max\{a, t-1\}$, for all $t \in \mathbb{N}_a$. Define the μ^{th}-order nabla fractional Taylor monomial by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-a+\mu)}{\Gamma(t-a)\Gamma(\mu+1)}, \quad t \in \mathbb{N}_a, \quad \mu \in \mathbb{R} \setminus \{\dots, -2, -1\}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Observe that $H_\mu(a,a) = 0$ and $H_\mu(t,a) = 0$ for all $\mu \in \{\dots, -2, -1\}$ and $t \in \mathbb{N}_a$. The first order backward (nabla) difference of $u : \mathbb{N}_a \to \mathbb{R}$ is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$.

Definition 2.1 (See [9]**).** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_a,$$

where by convention $\big(\nabla_a^{-\nu} u\big)(a) = 0$.

Definition 2.2 (See [9]**).** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $0 < \nu \leq 1$. The ν^{th}-order nabla difference of u is given by

$$\big(\nabla_a^\nu u\big)(t) = \Big(\nabla\big(\nabla_a^{-(1-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Lemma 2.3 (See [9]**).** *We have the following properties of nabla fractional Taylor monomials.*

1. $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$, $t \in \mathbb{N}_a$.
2. $\sum_{s=a+1}^{t} H_\mu(s,a) = H_{\mu+1}(t,a)$, $t \in \mathbb{N}_a$.
3. $\sum_{s=a+1}^{t} H_\mu(t,\rho(s)) = H_{\mu+1}(t,a)$, $t \in \mathbb{N}_a$.

Proposition 2.4 (See [12]**).** *Let $s \in \mathbb{N}_a$ and $-1 < \mu$. The following properties hold:*

(a) $H_\mu(t,\rho(s)) \geq 0$ *for* $t \in \mathbb{N}_{\rho(s)}$ *and,* $H_\mu(t,\rho(s)) > 0$ *for* $t \in \mathbb{N}_s$.

(b) $H_\mu(t, \rho(s))$ *is a decreasing function with respect to* s *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in (0, \infty)$.
(c) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (-1, 0)$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* s.
(d) $H_\mu(t, \rho(s))$ *is a non-decreasing function with respect to* t *for* $t \in \mathbb{N}_{\rho(s)}$ *and* $\mu \in [0, \infty)$.
(e) *If* $t \in \mathbb{N}_s$ *and* $\mu \in (0, \infty)$, *then* $H_\mu(t, \rho(s))$ *is an increasing function of* t.
(f) $H_\mu(t, \rho(s))$ *is a decreasing function with respect to* t *for* $t \in \mathbb{N}_{s+1}$ *and* $\mu \in (-1, 0)$.

Proposition 2.5 (See [12]**).** *Let* u *and* v *be two nonnegative real-valued functions defined on a set* S. *Further, assume* u *and* v *achieve their maximum values in* S. *Then,*

$$|u(t) - v(t)| \leq \max\{u(t), v(t)\} \leq \max\big\{\max_{t \in S} u(t), \max_{t \in S} v(t)\big\},$$

for every fixed t *in* S.

3. Green's function and its property

Assume $T \in \mathbb{N}_2$, $1 < \alpha < 2$, and $h : \mathbb{N}_2^T \to \mathbb{R}$. Consider the boundary value problem

$$\begin{cases} \Big(\nabla_0^{\alpha-1}\big(\nabla u\big)\Big)(t) + h(t) = 0, & t \in \mathbb{N}_2^T, \\ u(0) + u(T) = 0, & \big(\nabla u\big)(1) + \big(\nabla u\big)(T) = 0. \end{cases} \tag{3.1}$$

First, we construct the Green's function, $G(t, s)$ corresponding to (3.1), and obtain an expression for its unique solution. Denote by

$$D_1 = \{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \geq s\}, \quad D_2 = \{(t, s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T : t \leq \rho(s)\},$$

and

$$\xi_\alpha = 2\left[1 + H_{\alpha-2}(T, 0)\right]. \tag{3.2}$$

Theorem 3.1. *The unique solution of the nabla fractional boundary value problem* (3.1) *is given by*

$$u(t) = \sum_{s=2}^{T} G_\alpha(t, s) h(s), \quad t \in \mathbb{N}_0^T, \tag{3.3}$$

where

$$G_\alpha(t, s) = \begin{cases} K_\alpha(t, s) - H_{\alpha-1}(t, \rho(s)), & (t, s) \in D_1, \\ K_\alpha(t, s), & (t, s) \in D_2. \end{cases} \tag{3.4}$$

Here

$$K_\alpha(t, s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T, \rho(s)) + 2H_{\alpha-1}(t, 0)H_{\alpha-2}(T, \rho(s)) \\ + H_{\alpha-1}(T, \rho(s))H_{\alpha-2}(T, 0) - H_{\alpha-1}(T, 0)H_{\alpha-2}(T, \rho(s))\Big].$$

Proof. Denote by

$$\big(\nabla u\big)(t) = v(t), \quad t \in \mathbb{N}_1^T.$$

Subsequently, the difference equation in (3.1) takes the form

$$\big(\nabla_0^{\alpha-1} v\big)(t) + h(t) = 0, \quad t \in \mathbb{N}_2^T. \tag{3.5}$$

Let $v(1) = c_2$. Then, by Lemma 5.1 of [1], the unique solution of (3.5) is given by

$$v(t) = H_{\alpha-2}(t,0)c_2 - \big(\nabla_1^{-(\alpha-1)}h\big)(t), \quad t \in \mathbb{N}_1^T.$$

That is,

$$\big(\nabla u\big)(t) = H_{\alpha-2}(t,0)c_2 - \big(\nabla_1^{-(\alpha-1)}h\big)(t), \quad t \in \mathbb{N}_1^T. \tag{3.6}$$

Applying the first order nabla sum operator, ∇^{-1} on both sides of (3.6), we obtain

$$u(t) = c_1 + H_{\alpha-1}(t,0)c_2 - \big(\nabla_1^{-\alpha}h\big)(t), \quad t \in \mathbb{N}_0^T, \tag{3.7}$$

where $c_1 = u(0)$. We use the pair of anti-periodic boundary conditions considered in (3.1) to eliminate the constants c_1 and c_2 in (3.7). It follows from the first boundary condition $u(0) + u(T) = 0$ that

$$2c_1 + H_{\alpha-1}(T,0)c_2 = \big(\nabla_1^{-\alpha}h\big)(T). \tag{3.8}$$

The second boundary condition $\big(\nabla u\big)(1) + \big(\nabla u\big)(T) = 0$ yields

$$\left[1 + H_{\alpha-2}(T,0)\right] c_2 = \big(\nabla_1^{-(\alpha-1)}h\big)(T). \tag{3.9}$$

Solving (3.8) and (3.9) for c_1 and c_2, we obtain

$$c_1 = \frac{1}{2}\left[\sum_{s=2}^{T} H_{\alpha-1}(T,\rho(s))h(s) - \frac{2H_{\alpha-1}(T,0)}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s)\right], \tag{3.10}$$

$$c_2 = \frac{2}{\xi_\alpha}\sum_{s=2}^{T} H_{\alpha-2}(T,\rho(s))h(s). \tag{3.11}$$

Substituting these expressions in (3.7), we achieve (3.4). $\square$

Lemma 3.2. *Observe that*

$$|K_\alpha(t,s)| \le \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big], \tag{3.12}$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$.

Proof. Denote by

$$K'_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(t,0)H_{\alpha-2}(T,\rho(s)) + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big], \tag{3.13}$$

and

$$K''_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s))\Big], \tag{3.14}$$

so that

$$K_\alpha(t,s) = K'_\alpha(t,s) - K''_\alpha(t,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

Clearly, from Proposition 2.4,

$$K'_\alpha(t,s) \ge 0, \quad K''_\alpha(t,s) > 0, \text{ for all } (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T.$$

From Proposition 2.5, it is obvious that

$$|K_\alpha(t,s)| \le \left\{ \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s) \right\}. \tag{3.15}$$

First, we evaluate the first backward difference of $K'_\alpha(t,s)$ with respect to t for a fixed s. Consider

$$\nabla K'_\alpha(t,s) = \frac{1}{\xi_\alpha}\Big[2H_{\alpha-2}(t,0)H_{\alpha-2}(T,\rho(s))\Big] > 0,$$

for all $(t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T$, implying that $K'_\alpha(t,s)$ is an increasing function of t for a fixed s. Thus, we have

$$K'_\alpha(t,s) \le K'_\alpha(T,s), \quad (t,s) \in \mathbb{N}_0^T \times \mathbb{N}_2^T. \tag{3.16}$$

It follows from (3.13) - (3.16) that

$$\begin{aligned}
&|K_\alpha(t,s)| \\
&\le \left\{ \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K'_\alpha(t,s), \max_{(t,s)\in\mathbb{N}_0^T\times\mathbb{N}_2^T} K''_\alpha(t,s) \right\} \\
&\le \left\{ \max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s), \max_{s\in\mathbb{N}_2^T} K''_\alpha(t,s) \right\} \\
&= \max_{s\in\mathbb{N}_2^T} K'_\alpha(T,s) \\
&= \frac{1}{\xi_\alpha} \max_{s\in\mathbb{N}_2^T} \Big[H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(s)) \\
&\quad + H_{\alpha-1}(T,\rho(s))H_{\alpha-2}(T,0)\Big] \\
&\le \frac{1}{\xi_\alpha}\Big[\max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s)) + 2H_{\alpha-1}(T,0)\max_{s\in\mathbb{N}_2^T} H_{\alpha-2}(T,\rho(s)) \\
&\quad + H_{\alpha-2}(T,0)\max_{s\in\mathbb{N}_2^T} H_{\alpha-1}(T,\rho(s))\Big] \\
&= \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,\rho(2)) + 2H_{\alpha-1}(T,0)H_{\alpha-2}(T,\rho(T)) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,\rho(2))\Big] \\
&= \frac{1}{\xi_\alpha}\Big[H_{\alpha-1}(T,1) + 2H_{\alpha-1}(T,0) + H_{\alpha-2}(T,0)H_{\alpha-1}(T,1)\Big].
\end{aligned}$$

The proof is complete. □

4. Main results

Let $X = \mathbb{R}^{T+1}$ be the Banach space of all real $(T+1)$-tuples equipped with the maximum norm

$$|u| = \max_{t\in\mathbb{N}_0^T} |u(t)|.$$

Obviously, the product space X^2 is also a Banach space with the vector-norm

$$\|u\| = \begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix},$$

for $u = (u_1, u_2) \in X^2$.
For our convenience, denote by

$$\Lambda_i = \frac{1}{\xi_{\alpha_i}} \Big[H_{\alpha_i - 1}(T, 1) + 2H_{\alpha_i - 1}(T, 0) + H_{\alpha_i - 2}(T, 0) H_{\alpha_i - 1}(T, 1) \Big], \tag{4.1}$$

$$a_i = l_i \left[\Lambda_i (T - 1) + H_{\alpha_i}(T, 1) \right], \tag{4.2}$$

$$b_i = m_i \left[\Lambda_i (T - 1) + H_{\alpha_i}(T, 1) \right], \tag{4.3}$$

$$c_i = n_i \left[\Lambda_i (T - 1) + H_{\alpha_i}(T, 1) \right], \tag{4.4}$$

for $i = 1, 2$.
Define the operator $T : X^2 \to X^2$ by

$$T(u_1, u_2)(t) = \begin{pmatrix} T_1(u_1, u_2)(t) \\ T_2(u_1, u_2)(t) \end{pmatrix}, \quad t \in \mathbb{N}_0^T, \tag{4.5}$$

where

$$\begin{aligned} & T_1(u_1, u_2)(t) \\ & = \sum_{s=2}^{T} G_{\alpha_1}(t, s) f_1(u_1(s), u_2(s)) \\ & = \sum_{s=2}^{T} K_{\alpha_1}(t, s) f_1(u_1(s), u_2(s)) - \sum_{s=2}^{t} H_{\alpha_1 - 1}(t, s) f_1(u_1(s), u_2(s)), \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} & T_2(u_1, u_2)(t) \\ & = \sum_{s=2}^{T} G_{\alpha_2}(t, s) f_2(u_1(s), u_2(s)) \\ & = \sum_{s=2}^{T} K_{\alpha_2}(t, s) f_2(u_1(s), u_2(s)) - \sum_{s=2}^{t} H_{\alpha_2 - 1}(t, s) f_2(u_1(s), u_2(s)). \end{aligned} \tag{4.7}$$

Theorem 4.1. *A couple $(u_1, u_2) \in X^2$ is a solution of* (1.6) *if, and only if,*

$$\begin{cases} T_1(u_1, u_2) = u_1, \\ T_2(u_1, u_2) = u_2. \end{cases} \tag{4.8}$$

In view of Theorem 4.1 it is enough to apply Theorems 1.1 - 1.3 to the system (4.8).

Theorem 4.2. *Assume that*

(I) *for each $i \in \{1, 2\}$, there exist nonnegative numbers l_i and m_i such that*

$$|f_i(u_1, u_2) - f_i(v_1, v_2)| \le l_i |u_1 - v_1| + m_i |u_2 - v_2|, \tag{4.9}$$

for all (u_1, u_2), $(v_1, v_2) \in X^2$;

In addition, assume that condition (H2) is satisfied. Then, (4.8) *has a unique solution $(u_1, u_2) \in X^2$.*

Proof. For each $i \in \{1,2\}$ and for all (u_1,u_2), $(v_1,v_2) \in X^2$, consider

$$\begin{aligned}
&|T_i(u_1,u_2) - T_i(v_1,v_2)| \\
&\leq \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(u_1(s),u_2(s)) - f_i(v_1(s),v_2(s))| \\
&\quad + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(u_1(s),u_2(s)) - f_i(v_1(s),v_2(s))| \\
&\leq [l_i|u_1-v_1| + m_i|u_2-v_2|] \left[\sum_{s=2}^{T} |K_{\alpha_i}(t,s)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\right] \\
&\leq [l_i|u_1-v_1| + m_i|u_2-v_2|]\,[\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
&\leq [l_i|u_1-v_1| + m_i|u_2-v_2|]\,[\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
&\leq a_i|u_1-v_1| + b_i|u_2-v_2|,
\end{aligned}$$

implying that (H1) holds. Thus, by Theorem 1.1, the system (4.8) has a unique solution $(u_1,u_2) \in X^2$. □

Theorem 4.3. *Assume that*

(II) *for each $i \in \{1,2\}$, there exist nonnegative numbers a_i, b_i and c_i such that*

$$|f_i(u_1,u_2)| \leq l_i|u_1| + m_i|u_2| + n_i, \tag{4.10}$$

for all $(u_1,u_2) \in X^2$.

In addition, assume that condition (H2) is satisfied. Then, (4.8) *has at least one solution $(u_1,u_2) \in X^2$ satisfying* (1.5).

Proof. Since T_i, $i = 1,2$, is a summation operator on a discrete finite set, it is trivially completely continuous on X^2. For each $i \in \{1,2\}$ and for all $(u_1,u_2) \in X^2$, consider

$$\begin{aligned}
|T_i(u_1,u_2)| &\leq \sum_{s=2}^{T} |K_{\alpha_i}(t,s)|\,|f_i(u_1(s),u_2(s))| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\,|f_i(u_1(s),u_2(s))| \\
&\leq [l_i|u_1| + m_i|u_2| + n_i] \left[\sum_{s=2}^{T} |K_{\alpha_i}(t,s)| + \sum_{s=2}^{t} H_{\alpha_i-1}(t,s)\right] \\
&\leq [l_i|u_1| + m_i|u_2| + n_i]\,[\Lambda_i(T-1) + H_{\alpha_i}(t,1)] \\
&\leq [l_i|u_1| + m_i|u_2| + n_i]\,[\Lambda_i(T-1) + H_{\alpha_i}(T,1)] \\
&\leq a_i|u_1| + b_i|u_2| + c_i,
\end{aligned}$$

implying that (H3) holds. Thus, by Theorem 1.2, the system (4.8) has at least one solution $(u_1,u_2) \in X^2$ satisfying (1.5). □

Definition 4.4. [25] Let X be a Banach space and T_1, $T_2 : X \times X \to X$ be two operators. Then, the system (4.8) is said to be Ulam–Hyers stable if there exist C_1,

C_2, C_3, $C_4 > 0$ such that for each ε_1, $\varepsilon_2 > 0$ and each solution-pair $(u_1^*, u_2^*) \in X \times X$ of the in-equations:

$$\begin{cases} \|u_1 - T_1(u_1, u_2)\|_X \leq \varepsilon_1, \\ \|u_2 - T_2(u_1, u_2)\|_X \leq \varepsilon_2, \end{cases} \tag{4.11}$$

there exists a solution $(v_1^*, v_2^*) \in X \times X$ of (4.8) such that

$$\begin{cases} \|u_1^* - v_1^*\|_X \leq C_1\varepsilon_1 + C_2\varepsilon_2, \\ \|u_2^* - v_2^*\|_X \leq C_3\varepsilon_1 + C_4\varepsilon_2. \end{cases} \tag{4.12}$$

Theorem 4.5. *Assume that the hypothesis of Theorem 4.2 holds. Then, the system* (4.8) *is Ulam–Hyers stable.*

5. Examples

In this section, we provide two examples to illustrate the applicability of Theorem 4.2, Theorem 4.3, and Theorem 4.5.

Example 5.1. Consider the following boundary value problem for a coupled system of fractional difference equations

$$\begin{cases} \Big(\nabla_0^{0.5}\big(\nabla u_1\big)\Big)(t) + (0.001)\left[1 + \tan^{-1} u_1(t) + \tan^{-1} u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ \Big(\nabla_0^{0.5}\big(\nabla u_2\big)\Big)(t) + (0.002)\left[1 + \sin u_1(t) + \sin u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^9, \\ u_1(0) + u_1(9) = 0, \quad \big(\nabla u_1\big)(1) + \big(\nabla u_1\big)(9) = 0, \\ u_2(0) + u_2(9) = 0, \quad \big(\nabla u_2\big)(1) + \big(\nabla u_2\big)(9) = 0. \end{cases} \tag{5.1}$$

Comparing (1.6) and (5.1), we have $T = 9$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(u_1, u_2) = (0.001)\left[1 + \tan^{-1} u_1 + \tan^{-1} u_2\right],$$

and

$$f_2(u_1, u_2) = (0.002)\left[1 + \sin u_1 + \sin u_2\right],$$

for all $(u_1, u_2) \in \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (I) with $l_1 = 0.001$, $m_1 = 0.001$, $l_2 = 0.002$ and $m_2 = 0.002$. We have,

$$\begin{aligned} a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.0527, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.1053, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T,1)\right] = 0.0527, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T,1)\right] = 0.1053. \end{aligned}$$

Further,

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.0527 & 0.0527 \\ 0.1053 & 0.1053 \end{pmatrix}.$$

The spectral radius of M is 0.158, which is less than one, implying that M converges to zero. Hence, by Theorem 4.2, the system (5.1) has a unique solution $(u_1, u_2) \in X^2$. Also, by Theorem 4.5, the unique solution of (5.1) is Ulam–Hyers stable.

Example 5.2. Consider the following boundary value problem for a coupled system of fractional difference equations

$$\begin{cases} \left(\nabla_0^{0.5}\left(\nabla u_1\right)\right)(t) + (0.01)\left[1 + \frac{1}{\sqrt{1+u_1^2(t)}} + u_2(t)\right] = 0, \quad t \in \mathbb{N}_2^4, \\ \left(\nabla_0^{0.5}\left(\nabla u_2\right)\right)(t) + (0.02)\left[1 + u_1(t) + \frac{1}{\sqrt{1+u_2^2(t)}}\right] = 0, \quad t \in \mathbb{N}_2^4, \\ u_1(0) + u_1(4) = 0, \quad \left(\nabla u_1\right)(1) + \left(\nabla u_1\right)(4) = 0, \\ u_2(0) + u_2(4) = 0, \quad \left(\nabla u_2\right)(1) + \left(\nabla u_2\right)(4) = 0. \end{cases} \tag{5.2}$$

Comparing (1.6) and (5.2), we have $T = 4$, $\alpha_1 = \alpha_2 = 1.5$,

$$f_1(u_1, u_2) = (0.01)\left[1 + \frac{1}{\sqrt{1 + u_1^2(t)}} + u_2(t)\right],$$

and

$$f_2(u_1, u_2) = (0.02)\left[1 + u_1(t) + \frac{1}{\sqrt{1 + u_2^2(t)}}\right],$$

for all $(u_1, u_2) \in \mathbb{R}^2$. Clearly, f_1 and f_2 are continuous on $\mathbb{R}^2$. Next, f_1 and f_2 satisfy assumption (II) with $l_1 = 0.01$, $m_1 = 0.01$, $l_2 = 0.02$, $m_2 = 0.02$, $n_1 = 0.01$ and $n_2 = 0.02$. We have,

$$\begin{aligned} a_1 &= l_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.1219, \\ a_2 &= l_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.2438, \\ b_1 &= m_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.1219, \\ b_2 &= m_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.2438, \\ c_1 &= n_1\left[\Lambda_1(T-1) + H_{\alpha_1}(T, 1)\right] = 0.1219, \\ c_2 &= n_2\left[\Lambda_2(T-1) + H_{\alpha_2}(T, 1)\right] = 0.2438. \end{aligned}$$

Further,

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} 0.1219 & 0.1219 \\ 0.2438 & 0.2438 \end{pmatrix}.$$

The spectral radius of M is 0.3657, which is less than one, implying that M converges to zero. Hence, by Theorem 4.3, the system (5.2) has at least one solution $(u_1, u_2) \in X^2$ satisfying

$$\begin{pmatrix} |u_1| \\ |u_2| \end{pmatrix} \leq (I - M)^{-1}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0.1757 \\ 0.2658 \end{pmatrix}.$$

Conclusion

In this article, we obtained sufficient conditions on existence, uniqueness and Ulam–Hyers stability of solutions of the system (1.6) using the approaches of Precup and Urs. We also provided two examples to demonstrate the applicability of established results. Observe that Theorem 4.2 is not applicable to the system (5.1).

References

[1] Abdeljawad, T., Atici, F.M., *On the definitions of nabla fractional operators*, Abstr. Appl. Anal., **2012**(2012), Art. ID 406757, 13 pp.

[2] Ahrendt, K. Castle, L., Holm, M., Yochman, K., *Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula*, Commun. Appl. Anal., **16**(2012), no. 3, 317-347.

[3] Atici, Ferhan, M., Eloe, P.W., *Discrete fractional calculus with the nabla operator*, Electron. J. Qual. Theory Differ. Equ., 2009, Special Edition I, no. 3, 12 pp.

[4] Bolojan-Nica, O., Infante, G., Precup, R., *Existence results for systems with coupled nonlocal initial conditions*, Nonlinear Anal., **94**(2014), 231-242.

[5] Chen, C., Bohner, M., Jia, B., *Ulam-Hyers stability of Caputo fractional difference equations*, Math. Meth. Appl. Sci., (2019), 1-10.

[6] Chen, F., Zhou, Y., *Existence and Ulam stability of solutions for discrete fractional boundary value problem*, Discrete Dynamics in Nature and Society, 2013, Art. ID 459161, 7 pp.

[7] Dutta, B.K., Arora, L.K., *Hyers-Ulam stability for a class of nonlinear fractional differential equations*, Rev. Bull. Calcutta Math. Soc., **21**(2013), no. 1, 95-102.

[8] Gholami, Y., Ghanbari, K., *Coupled systems of fractional ∇-difference boundary value problems*, Differ. Equ. Appl., **8**(2016), no. 4, 459-470.

[9] Goodrich, C., Peterson, A.C., *Discrete Fractional Calculus*, Springer, Cham, 2015.

[10] Hein, J., McCarthy, S., Gaswick, N., McKain, B., Speer, K., *Laplace transforms for the nabla difference operator*, Panamer. Math. J., **21**(2011), 79-96.

[11] Hyers, D.H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27**(1941), 222-224.

[12] Ikram, A., *Lyapunov inequalities for nabla Caputo boundary value problems*, J. Difference Equ. Appl., **25**(2019), no. 6, 757-775.

[13] Jagan, M.J., *Hyers-Ulam stability of fractional nabla difference equations*, Int. J. Anal. 2016, Art. ID 7265307, 5 pp.

[14] Jebelean, P., Precup, R., *Solvability of p, q-Laplacian systems with potential boundary conditions*, Appl. Anal., **89**(2010), no. 2, 221-228.

[15] Jung, S.M., *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, 2011.

[16] Kelley, W.G., Peterson, A.C., *Difference Equations. An Introduction with Applications*, Second edition, Harcourt/Academic Press, San Diego, CA, 2001.

[17] Khan, H., Li, Y., Chen, W., Baleanu, D., Khan, A., *Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p-Laplacian operator*, Bound. Value Probl., 2017, Paper No. 157, 16.

[18] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, **204**, Elsevier Science B.V., Amsterdam, 2006.

[19] Nica, O., Precup, R., *On the nonlocal initial value problem for first order differential systems*, Stud. Univ. Babeş-Bolyai Math., **56**(2011), no. 3, 113-125.

[20] Podlubny, I., *Fractional Differential Equations*, Mathematics in Science and Engineering, **198**, Academic Press, Inc., San Diego, CA, 1999.

[21] Precup, R., *The role of matrices that are convergent to zero in the study of semilinear operator systems*, Math. Comput. Modelling, **49**(2009), no. 3-4, 703-708.

[22] Rassias, T.M., *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72**(1978), no. 2, 297-300.

[23] Rus, I.A., *Ulam stability of ordinary differential equations*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), 125-133.

[24] Ulam, S., *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, **8**, New York-London, Interscience Publishers, 1960.

[25] Urs, C., *Coupled fixed point theorems and applications to periodic boundary value problems*, Miskolc Mathematical Notes, **14**(2013), no. 1, 323-333.

Jagan Mohan Jonnalagadda
Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad – 500078, Telangana, India
e-mail: j.jaganmohan@hotmail.com