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Unsteady flow of Bingham fluid in a thin layer with mixed boundary conditions

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Abstract. In this paper we consider the dynamic system for Bingham fluid in a three-dimensional thin domain with Fourier and Tresca boundary condition. We study the existence and uniqueness results for the weak solution, then we establish its asymptotic behavior, when the depth of the thin domain tends to zero. This study yields a mechanical laws that give a new description of the behavior this system.

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1. Introduction

This work gives an extension to describe the flow of fluids in a dynamic system to some of the results obtained in a series of papers [1, 2, 4, 5, 9], in which the authors considered a stationary case only of the general equations describing the motion of some fluid flows in bounded thin domain, with slip and mixed boundary conditions. The aim of this paper is to study the asymptotic analysis of an incompressible Bingham fluid in a dynamic regime in a three dimensional thin domain mixed boundary and subject to slip phenomenon on a part of the boundary. We are interested here in the existence and uniqueness for this problem and also its behavior when the thickness of the thin domain tends to zero.

This fluid enters the category of non-Newtonian fluids, and there are many milieus in nature and industry exhibiting the behavior of the Bingham fluid. For example, heavy crude oils, colloid solutions...See also historical ref [3]. More specifically, the model under study is mainly related for lubrication problems in a lot of mechanical papers [10, 11, 13] when the gap between the solid surfaces is very weak. In this dynamic system, the non-slip condition is caused by the chemical structure between the

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lubricants and the surrounding surfaces. On the contrary, tangential stresses, when they reach a certain threshold, destroy the chemical structure and induce a slip phenomena. This phenomenon is implicitly expressed by the Reynolds equation, which was mathematically posed during 1985 in [12].

Thus, following the same ideas as in [5]. The departure point is the laws of conservation, which includes here the effect of the acceleration-dependent inertia forces. A friction law of Tresca and the Fourier boundary condition are assumed on the boundary, so fall into the scope of the work of [8]. Then we will compare our results to stationary problem in [1, 2, 4, 5].

This work is also devoted to prove our results, with suitable conditions on the initial data, contrary to what was assumed in [7, p. 289-290] where the initial conditions for the data were null. The main difficulty here is to estimate the solutions of the problem, due to the fractional term for the Bingham constitutive law and the assumption coming from the initial velocity. The proofs presented in this paper are based on regularization methods and classical results for elliptic variational derived from [6, 7]. The plan of this paper is as follow, we present in section 2, some notation and the weak formulation of problem. In section 3, we give the main results on existence results by the regularization methods. In section 4, we introduce a scaling as in [5, 8], we give some needed estimates on the velocity and pressure, also the convergence results. In sections 5 we present the limit problem and we give the mechanical interpretation of the results.

2. Preliminaries and variational formulation

Let ω be fixed region in the surface $x' = (x_1, x_2) \in \mathbb{R}^2$, and let $h \in C^2(\omega)$ be a smooth positive function such that $0 < \underline{h} \leq h(x') \leq \overline{h}$ for all $(x', 0) \in \omega$. Consider an incompressible Bingham fluid occupying the domain

$$\begin{aligned} &\Omega^\varepsilon = \left\{ x = (x', x_3) \in \mathbb{R}^3 : \, (x', 0) \in \omega, \quad 0 < x_3 < \varepsilon h(x') \right\}, \\ &Q^\varepsilon = \Omega^\varepsilon \times \left]0, T\right[. \end{aligned}$$

where $\varepsilon \in \left]0, 1\right[$ and $T > 0$. Noting Γ^ε the boundary of Ω^ε, we have $\Gamma^\varepsilon = \bar{\omega} \cup \bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$, and Γ_1^ε the upper boundary of equation $x_3 = \varepsilon h(x')$, Γ_L^ε is the lateral boundary. We denote by $\mathbb{S}_n$ $(n = 2, 3)$ the space of symmetric tensors, while '.' and $|.|$ will represent the inner product and the Euclidean norm on $\mathbb{S}_n$ or $\mathbb{R}^n$. We consider the rate of deformation operator defined for every $u^\varepsilon \in H^1(\Omega^\varepsilon)^3$ by $D(u^\varepsilon) = \dfrac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$. Let ν denote the unit outer normal on Γ^ε, and we write u^ε for its trace on Γ^ε, also

$$u_\nu^\varepsilon = u^\varepsilon.\nu, \quad u_\tau^\varepsilon = u^\varepsilon - u_\nu^\varepsilon.\nu, \quad \sigma_\nu^\varepsilon = (\sigma^\varepsilon.\nu).\nu \text{ and } \sigma_\tau^\varepsilon = \sigma^\varepsilon.\nu - (\sigma_\nu^\varepsilon).\nu$$

be, respectively, the components of the normal, the tangential of u^ε on Γ^ε, the normal and the tangential of σ^ε on Γ^ε.

The unstable flow of Bingham fluid that will be studied in this paper is given by the following mechanical problem.

Problem P. Find the velocity fields $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ and the scalar pressure p^ε such that

$$\frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(\sigma^\varepsilon) = -\nabla p^\varepsilon + f^\varepsilon \quad \text{in } \Omega^\varepsilon \times [0,T], \tag{2.1}$$

$$\operatorname{div}(u^\varepsilon) = 0 \qquad \text{in } \Omega^\varepsilon \times [0,T], \tag{2.2}$$

$$\begin{cases} \sigma_{ij}^\varepsilon = \varepsilon^{-1}\alpha \dfrac{D_{ij}(u^\varepsilon)}{|D(u^\varepsilon)|} + 2\mu D_{ij}(u^\varepsilon) & \text{if } |D(u^\varepsilon)| \neq 0 \\ |\sigma^\varepsilon| \leq \varepsilon^{-1}\alpha \quad \text{if } |D(u^\varepsilon)| = 0 \end{cases} \quad \text{in } \Omega^\varepsilon \times [0,T], \tag{2.3}$$

$$u^\varepsilon = 0 \text{ on } \Gamma_L^\varepsilon \times]0,T[, \tag{2.4}$$

$$u^\varepsilon \cdot \nu = 0 \quad \text{on } (\omega \cup \Gamma_1^\varepsilon) \times]0,T[, \tag{2.5}$$

$$\sigma_\tau(u^\varepsilon) = -l^\varepsilon u^\varepsilon \text{ on } \Gamma_1^\varepsilon \times]0,T[, \tag{2.6}$$

$$\begin{cases} |\sigma_\tau^\varepsilon| < \varepsilon^{-1}k \Rightarrow u_\tau^\varepsilon(t) = 0 \\ |\sigma_\tau^\varepsilon| = \varepsilon^{-1}k \Rightarrow \exists\lambda \geq 0 \; u_\tau^\varepsilon(t) = -\lambda\sigma_\tau^\varepsilon \end{cases} \quad \text{on } \omega \times [0,T], \tag{2.7}$$

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \;\; \forall x \in \Omega^\varepsilon. \tag{2.8}$$

Here, the flow is given by equation (2.1), where $f^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, f_3^\varepsilon)$ denote the volume force of density. The equation (2.2) represent the incompressibility condition. Relation (2.3) represents the constitutive law of Bingham fluid of viscosity μ and plasticity threshold α, where μ, $\alpha > 0$ are constants independent of ε. The condition (2.4) is the Dirichlet boundary. (2.5) give the non-slip condition of velocity on Γ_1^ε and ω. (2.4) represent the Fourier condition on Γ_1^ε, where $l^\varepsilon > 0$ is a given constant. Condition (2.7) represents a Tresca's friction law on ω, where k is a coefficient independent of ε, finally, the initial velocity is a given by (2.8), with $u_0^\varepsilon \neq 0$ is a given function.

Now, we us consider the following function spaces

$$\begin{aligned} K^\varepsilon &= \left\{\phi \in H^1(\Omega^\varepsilon)^3 : \phi = 0 \text{ on } \Gamma_L^\varepsilon,\; \phi.\nu = 0 \text{ on } \omega \cup \; \Gamma_1^\varepsilon\right\}, \\ K_{\text{div}}^\varepsilon &= \{\phi \in K^\varepsilon : \operatorname{div}(\phi) = 0 \text{ in } \Omega^\varepsilon \}, \\ L_0^2(\Omega^\varepsilon) &= \left\{q \in L^2(\Omega^\varepsilon) : \int_{\Omega^\varepsilon} q \; dx = 0\right\}. \end{aligned}$$

Let us introduce the bilinear forms a, $\breve{a}$ and functional J^ε defined by

$$a(u^\varepsilon, \phi - u^\varepsilon) = 2\mu \int_{\Omega^\varepsilon} D_{ij}(u^\varepsilon) D_{ij}(\phi - u^\varepsilon)\, dx,$$

$$\breve{a}(u^\varepsilon, \phi - u^\varepsilon) = a(u^\varepsilon, \phi - u^\varepsilon) + l^\varepsilon \int_{\Gamma_1^\varepsilon} u^\varepsilon.(\phi - u^\varepsilon)\, d\tau,$$

$$J^\varepsilon(\phi) = \varepsilon^{-1} \int_\omega k\, |\phi_\tau| \; dx' + \sqrt{2}\alpha\varepsilon^{-1} \int_{\Omega^\varepsilon} |D(\phi)|\, dx.$$

J^ε is convex and continuous but non differentiable in K^ε.

Following [5, 8], the variational inequality of the problem (2.1)-(2.8) is given by

Problem Pv. Find $\{u^\varepsilon, p^\varepsilon\}$ where $u^\varepsilon(t) \in K^\varepsilon_{\text{div}}$, $\dfrac{\partial u^\varepsilon}{\partial t}(t) \in K^\varepsilon$ and $p^\varepsilon(t) \in L^2_0(\Omega^\varepsilon)$ such that

$$\begin{aligned} &\int\limits_{\Omega^\varepsilon} \frac{\partial u^\varepsilon}{\partial t}(t).\left(\phi - u^\varepsilon(t)\right) dx + \check{a}(u^\varepsilon(t), \phi - u^\varepsilon(t)) - \int\limits_{\Omega^\varepsilon} p^\varepsilon \, div\ (\phi)\, dx + \\ &J^\varepsilon(\phi) - J^\varepsilon(u^\varepsilon(t)) \geq \int\limits_{\Omega^\varepsilon} f^\varepsilon.\left(\phi - u^\varepsilon(t)\right) dx \ \forall t\ \in]0,T[\ \ \forall \phi \in K^\varepsilon \end{aligned} \tag{2.9}$$

with

$$u^\varepsilon(0) = u^\varepsilon_0\,(\neq 0)\,. \tag{2.10}$$

Notation. To simplify the writing, we will denote the norm in $L^2(\Omega^\varepsilon)^3$ by$\|.\|_{0,\Omega^\varepsilon}$ and the norm in $H^s(\Omega^\varepsilon)^3$ by $\|.\|_{s,\Omega^\varepsilon}$, the inner products on the space $L^2(\Omega^\varepsilon)^3$ designed by $(.,.)$ and le $\langle .,. \rangle$ denote the duality pairing between $(K^\varepsilon_{\text{div}})'$ and $K^\varepsilon_{\text{div}}$.

3. Existence and uniqueness results

We establish here a theorem of existence of weak solutions for Pv.

Theorem 3.1. *We make the following assumptions :*

$$f^\varepsilon, \frac{\partial f^\varepsilon}{\partial t} \in L^2\left(0, T; L^2(\Omega^\varepsilon)^3\right),\ f^\varepsilon(0) \in L^2(\Omega^\varepsilon)^3 \tag{3.1}$$

$$k \in C^\infty_0(\omega),\ k > 0 \text{ does not depend on } t, \tag{3.2}$$

$$u^\varepsilon_0 \in H^2(\Omega^\varepsilon)^3 \cap H^1_0(\Omega^\varepsilon)^3,\ \left(D\left(u^\varepsilon_0\right)\right)_\tau = 0 \text{ on } \omega \cup \Gamma^\varepsilon_1, \tag{3.3}$$

$$\exists \eta > 0 \quad \left|D\left(u^\varepsilon_0\right)\right| \geq \varepsilon^{-1}\eta \ \text{ a.e. in } \Omega^\varepsilon. \tag{3.4}$$

Under these assumptions, there exist a function u^ε unique solution of (2.9)-(2.10) with

$$u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \in L^\infty\left(0, T; L^2(\Omega^\varepsilon)^3\right) \cap L^2\left(0, T; H^1(\Omega^\varepsilon)^3\right). \tag{3.5}$$

Remark 3.1. The hypothesis $\langle u^\varepsilon_0 \neq 0 \rangle$ leads us to make additional techniques in the resolution of (2.9)-(2.10). First, we introduce two technical lemmas in the following paragraph, which will be used to obtain the needed estimates, then we will give the demonstration of theorem 3.1.

3.1. Regularization

For $\zeta > 0$, we consider the operator ψ_ζ and Ψ_ζ defined by

$$\psi_\zeta \ : L^2(\omega)^2 \to L^2(\omega)^2,\quad v \to \psi_\zeta(v) = |v|^{\zeta - 1} v$$

$$\Psi_\zeta : H^1(\Omega^\varepsilon)^{3\times 3} \to H^1(\Omega^\varepsilon)^{3\times 3},\quad \sigma \to \Psi_\zeta(\sigma) = |\sigma|^{\zeta - 1}\sigma$$

From [7], we approach J^ε by differentiable family;

$$J^\varepsilon_\zeta(v) = \varepsilon^{-1}\int_\omega k(x')\frac{|v_\tau|^{(1+\zeta)}}{1+\zeta}dx' + \sqrt{2}\alpha\varepsilon^{-1}\int_{\Omega^\varepsilon}\frac{|D(v)|^{(1+\zeta)}}{1+\zeta}dx,$$

we have

$$\left\langle \left(J^\varepsilon_\zeta\right)'(v), \phi \right\rangle = \varepsilon^{-1}\int_\omega k\psi_\zeta(v_\tau).\phi_\tau dx' + \sqrt{2}\alpha\varepsilon^{-1}\int_{\Omega^\varepsilon}\Psi_\zeta(D(v)).D(\phi)\,dx. \tag{3.6}$$

Then, we can approach the inequality (2.9) by the following equation, for all $\phi \in K^{\varepsilon}_{\text{div}}$:

$$\left(\frac{\partial u^{\varepsilon}_{\zeta}}{\partial t}(t),\phi\right)+\breve{a}\left(u^{\varepsilon}_{\zeta}(t),\phi\right)+\left\langle \left(J^{\varepsilon}_{\zeta}\right)'(v),\phi\right\rangle=\left(f^{\varepsilon}(t),\phi\right) \tag{3.7}$$

with

$$u^{\varepsilon}_{\zeta}(0)=u^{\varepsilon}_{0} \tag{3.8}$$

Lemma 3.1. Let $G:\mathbb{S}^{\star}_{3}\to\mathbb{S}_{3}$ be defined by $G(\tau)=|\tau|^{\zeta-1}\tau$ such that $\zeta\in\,]0,1[$. Let $\sigma\in H^{1}(\Omega^{\varepsilon})^{3\times 3}$, we suppose that there exist a strictly positive constant β such that $|\sigma|\geq\beta$ a. e. in $\overline{\Omega}^{\varepsilon}$, then

$$Go\sigma\in H^{1}(\Omega^{\varepsilon})^{3\times 3} \text{ and } \frac{\partial}{\partial x_k}(Go\sigma)=\left(\frac{\partial G}{\partial\tau_{ij}}o\sigma\right)\frac{\partial\sigma_{ij}}{\partial x_k}\quad\forall i,j,k\in\{1,2,3\}.$$

Proof. We have $|G(\tau)|=|\tau|^{\zeta}$ $\forall\tau\in\mathbb{S}^{\star}_{3}$. Since $|\sigma|\geq\beta$, and therefore

$$|Go\sigma|=|\sigma|^{\zeta}=|\sigma|\,|\sigma|^{\zeta-1}\leq\beta^{\zeta-1}|\sigma|,$$

as a consequence $Go\sigma\in L^{2}(\Omega^{\varepsilon})^{3\times 3}$.
Similarly,by a standard calculation of differentiation of a composition, we have

$$\begin{aligned}\left|\left(\frac{\partial G}{\partial\tau_{ij}}o\sigma\right)\frac{\partial\sigma_{ij}}{\partial x_k}\right|&=\left||\sigma|^{\zeta-1}\left((\zeta-1)\sigma^{2}_{ij}|\sigma|^{-2}+1\right)\frac{\partial\sigma_{ij}}{\partial x_k}\right|\\&\leq|\sigma|^{\zeta-1}\left|\frac{\partial\sigma_{ij}}{\partial x_k}\right|\leq\beta^{\zeta-1}\left|\frac{\partial\sigma_{ij}}{\partial x_k}\right|\end{aligned} \tag{3.9}$$

and thus $\left(\frac{\partial G}{\partial\tau_{ij}}o\sigma\right)\frac{\partial\sigma_{ij}}{\partial x_k}\in L^{2}(\Omega^{\varepsilon})^{3\times 3}$. It remains to verify that

$$\int_{\Omega^{\varepsilon}}(Go\sigma).\frac{\partial\Phi}{\partial x_k}dx=\int_{\Omega^{\varepsilon}}\left(\frac{\partial G}{\partial\tau_{ij}}o\sigma\right)\frac{\partial\sigma_{ij}}{\partial x_k}.\Phi dx\quad\forall\Phi\in\mathcal{C}^{1}_{0}(\Omega^{\varepsilon})^{3\times 3}.$$

By Friedrich Theorem (see [6, p. 265]), there exists a sequence σ_n in $\mathcal{C}^{\infty}_{0}\left(\mathbb{R}^{3}\right)^{3\times 3}$ such that $\sigma_n\to\sigma$ in $L^{2}(\Omega^{\varepsilon})^{3\times 3}$ and $\nabla\sigma_n\to\nabla\sigma$ in $L^{2}(W^{\varepsilon})^{3\times 3\times 3}$ for all open W^{ε} with $\overline{W^{\varepsilon}}\subset\Omega^{\varepsilon}$. Then, we can follow the proof with an argument similar to that used in proof of [6, Proposition 9.5]. □

Lemma 3.2. Let $\varepsilon,\zeta\in\,]0,1[$. If u^{ε}_{0} verifies the assumptions (3.3), (3.4). Then $\left(J^{\varepsilon}_{\zeta}\right)'(u^{\varepsilon}_{0})$ belong to $L^{2}(\Omega^{\varepsilon})^{3}$, moreover, there exist a constant $\gamma>0$ does not depend on Ω^{ε}, such that

$$\left\|\left(J^{\varepsilon}_{\zeta}\right)'(u^{\varepsilon}_{0})\right\|_{0,\Omega^{\varepsilon}}\leq\varepsilon^{-1}\gamma\left\|u^{\varepsilon}_{0}\right\|_{2,\Omega^{\varepsilon}}. \tag{3.11}$$

Proof. Using Green's formula in (3.6) and using the assumption (3.3), we get

$$\left\langle\left(J^{\varepsilon}_{\zeta}\right)'(u^{\varepsilon}_{0}),\phi\right\rangle=-\sqrt{2}\alpha\varepsilon^{-1}\int_{\Omega^{\varepsilon}}\left\{\operatorname{Div}\left(\Psi_{\zeta}(D(u^{\varepsilon}_{0}))\right\}\phi\,dx \tag{3.12}$$

Applying lemma 3.1 for $\sigma=D(u^{\varepsilon}_{0})$ and $\beta=\varepsilon^{-1}\eta$, clearly $\Psi_{\zeta}(D(u^{\varepsilon}_{0}))\in H^{1}(\Omega^{\varepsilon})^{3\times 3}$. By [7] we can write the Gelfand triple

$$K^{\varepsilon}_{\text{div}}\subset L^{2}(\Omega^{\varepsilon})^{3}\subset\left(K^{\varepsilon}_{\text{div}}\right)',$$

and it follows the following relation :

$$\left(\left(J_\zeta^\varepsilon\right)'(u_0^\varepsilon),\phi\right) = \left\langle \left(J_\zeta^\varepsilon\right)'(u_0^\varepsilon),\phi\right\rangle \quad \forall \phi \in \ L^2(\Omega^\varepsilon)^3.$$

By comparison with (3.12), we find

$$\left(J_\zeta^\varepsilon\right)'(u_0^\varepsilon) = -\sqrt{2}\alpha\varepsilon^{-1}\mathrm{Div}\left(\Psi_\zeta(D\left(u_0^\varepsilon\right)\right).$$

But, due to fact that (3.9) we have$\|\Psi_\zeta(D\left(u_0^\varepsilon\right))\|_{1,\Omega^\varepsilon} \leq \eta^{\zeta-1}\left\|D\left(u_0^\varepsilon\right)\right\|_{1,\Omega^\varepsilon}$. Then, using Sobolev injection related to Div and D, the relation (3.11) can be easily deduced with $\gamma = \sqrt{6}\alpha\eta^{\zeta-1}$. □

3.2. Demonstration of Theorem 3.1

First, we seek to estimate the solution independently of ζ. Let $t \in [0,T]$. As

$$\left\langle \left(J_\zeta^\varepsilon\right)'\left(u_\zeta^\varepsilon\right), u_\zeta^\varepsilon\right\rangle \geq 0,$$

the equation (3.7) for $\phi = u_\zeta^\varepsilon(t)$ becomes

$$\frac{1}{2}\frac{d}{dt}\left\|u_\zeta^\varepsilon(t)\right\|_{0,\Omega^\varepsilon}^2 + a\left(u_\zeta^\varepsilon(t), u_\zeta^\varepsilon(t)\right) + l^\varepsilon\left\|u_\zeta^\varepsilon(t)\right\|_{0,\Gamma_1^\varepsilon}^2 \leq (f^\varepsilon(t), u_\zeta^\varepsilon(t)). \tag{3.13}$$

By [5] there exist a constant $C_k > 0$ such that

$$a\left(u_\zeta^\varepsilon(t), u_\zeta^\varepsilon(t)\right) + l^\varepsilon\left\|v\left(t\right)\right\|_{0,\Gamma_1^\varepsilon}^2 \geq 2\mu C_K\left\|v\left(t\right)\right\|_{1,\Omega^\varepsilon}^2 \quad \forall v\left(t\right) \in K_{\mathrm{div}}^\varepsilon.$$

Then, by the integral of (3.13) relative to t, and using a Gronwall-type argument we obtain

$$\left\|u_\zeta^\varepsilon(t)\right\|_{0,\Omega^\varepsilon}^2 + \int_0^t \left\|u_\zeta^\varepsilon(\sigma)\right\|_{1,\Omega^\varepsilon}^2 d\sigma \leq c \tag{3.14}$$

Now, we derive (3.7) in t and taking $\phi = \dfrac{\partial u_\zeta^\varepsilon}{\partial t}(t)$,

$$\begin{aligned}&\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) + a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) + l^\varepsilon\left\|\frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right\|_{0,\Gamma_1^\varepsilon}^2 \\ &+\left\langle \frac{d}{dt}\left(\ J_\zeta^\varepsilon\right)'\left(u_\zeta^\varepsilon(t)\right), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right\rangle = \left(\frac{\partial f^\varepsilon}{\partial t}(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right).\end{aligned} \tag{3.15}$$

Taking into account $K_{\mathrm{div}}^\varepsilon \subset L^2(\Omega^\varepsilon)^3 \subset \left(K_{\mathrm{div}}^\varepsilon\right)'$ and by [12], the following inequality holds: there exists a positives constants ρ and λ, such that

$$a(v,v) + \rho\left\|v\right\|_{0,\Omega^\varepsilon}^2 \geq \lambda\left\|v\right\|_{1,\Omega^\varepsilon}^2 \ \ \forall v \in K^\varepsilon.$$

We know that the operator $\left(J_\zeta^\varepsilon \right)'$ is monotonous, we have

$$
\begin{aligned}
& \left\langle \frac{d}{dt} \left(J_\zeta^\varepsilon \right)' (\phi(t)) , \phi'(t) \right\rangle \\
= \; & \int_\omega k^\varepsilon \lim_{s \to 0} \frac{\psi_\zeta(\phi_\tau(t+s)) - \psi_\zeta(\phi_\tau(t))}{s} . \frac{\phi_\tau(t+s) - \phi_\tau(t)}{s} dx' \\
& + \sqrt{2} \alpha \varepsilon^{-1} \int_{\Omega^\varepsilon} \lim_{s \to 0} \frac{\Psi_\zeta(\phi(t+s)) - \Psi_\zeta(\phi(t))}{s} . \frac{\phi(t+s) - \phi(t)}{s} dx' \\
\geq \; & 0.
\end{aligned}
$$

So, the formula (3.15) becomes

$$
\begin{aligned}
& \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(t) \right\|_{0,\Omega^\varepsilon}^2 + \lambda \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{1,\Omega^\varepsilon}^2 ds + 2 l^\varepsilon \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \\
\leq \; & \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon}^2 + (\rho + 1) \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds
\end{aligned}
\tag{3.16}
$$

But, $\dfrac{\partial u_\zeta^\varepsilon}{\partial t}(0)$ is defined by, for all $\phi \in K_{\text{div}}^\varepsilon$,

$$
\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(0), \phi \right) = (f^\varepsilon(0), \phi) - a(u_0^\varepsilon, \phi) - \left\langle \left(J_\zeta^\varepsilon \right)' (u_0^\varepsilon) , \phi \right\rangle
$$

Consequently, we deduce that

$$
\frac{\partial u_\zeta^\varepsilon}{\partial t}(0) = f^\varepsilon(0) - A(u_0^\varepsilon) - \left(J_\zeta^\varepsilon \right)' (u_0^\varepsilon) \text{ in } L^2 \left(\Omega^\varepsilon \right)^3 \tag{3.17}
$$

where $A(u_0^\varepsilon) \in \mathcal{L} \left(K_{\text{div}}^\varepsilon ; K_{\text{div}}^{\varepsilon'} \right)$ is given by Riesz's representation theorem,

$$
\langle A(u_0^\varepsilon), \phi \rangle = a(u_0^\varepsilon, \phi).
$$

According to lemma 3.2 and the assumptions (3.1) , we have

$$
\left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon} \leq \text{cte (independent of } \zeta).
$$

This, joined to (3.16) and using a Gronwall lemma, shows that

$$
\left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(t) \right\|_{0,\Omega^\varepsilon}^2 + \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{1,\Omega^\varepsilon}^2 ds + l^\varepsilon \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \leq c. \tag{3.18}
$$

By (3.14) and (3.18), we can extract from u_ζ^ε a sequence denoted u_δ^ε such that the following convergences in $L^\infty \left(0, T; L^2 \left(\Omega^\varepsilon \right)^3 \right) \cap L^2 \left(0, T; H^1 \left(\Omega^\varepsilon \right)^3 \right)$:

$$
u_\delta^\varepsilon \longrightarrow u^\varepsilon, \;\; \frac{\partial u_\delta^\varepsilon}{\partial t} \longrightarrow \frac{\partial u^\varepsilon}{\partial t}.
$$

We deduce from equation (3.7) that

$$\left(\frac{\partial u_\delta^\varepsilon}{\partial t}, \phi - u_\delta^\varepsilon\right) + a\left(u_\delta^\varepsilon, \phi - u_\delta^\varepsilon\right) + l^\varepsilon \int_{\Gamma_1^\varepsilon} u_\delta^\varepsilon \left(\phi - u_\delta^\varepsilon\right) d\tau + J_\delta^\varepsilon\left(\phi\right)$$
$$+J_\delta^\varepsilon\left(u_\delta^\varepsilon\right) - \left(f^\varepsilon, \phi - u_\delta^\varepsilon\right) = J_\delta^\varepsilon\left(\phi\right) - J_\delta^\varepsilon\left(u_\delta^\varepsilon\right) - \left\langle \left(J_\delta^\varepsilon\right)'\left(u_\delta^\varepsilon\right), \phi - u_\delta^\varepsilon \right\rangle \geq 0$$

Finally, passing to the limit in δ as in [12], and using the semi-continuous inferior of the function $u \to \int_0^T \breve{a}(u,u)dt$ and $v \to \int_0^T J^\varepsilon\left(v\right)dt$ for $L^2\left(0, T; K_{\mathrm{div}}^\varepsilon\right)$ with the weak topology, to obtain (2.9)-(2.10).
The proof of uniqueness is analogous to [8], and this concludes the proof of theorem 3.1. □

4. Some estimates and convergence

4.1. The rescaled problem

To estimate the solutions $\{u^\varepsilon, p^\varepsilon\}$ we use the scaling $z = x_3/\varepsilon$ and the following fixed domains

$$\begin{aligned} \Omega &= \left\{(x', z) \in \mathbb{R}^3 : (x', 0) \in \omega, \quad 0 < z < h(x')\right\}, \\ Q &= \Omega \times]0, T[\,. \end{aligned}$$

We denote by Γ_1 is the upper boundary of the equation $z = h\left(x\right)$ and Γ_L is the lateral boundary. This rescaling maps the spaces K^ε, $K_{\mathrm{div}}^\varepsilon$ and $L_0^2(\Omega^\varepsilon)$ onto the spaces K, K_{div} and $L_0^2(\Omega)$ respectively, are defined by:

$$\begin{aligned} K &= \left\{\phi \in H^1(\Omega)^3 : \phi = 0 \text{ on } \Gamma_L,\ \phi.\nu = 0 \text{ on } \omega \cup\ \Gamma_1\right\}, \\ K_{\mathrm{div}} &= \left\{\phi \in K : \mathrm{div}(\phi) = 0 \text{ in } \Omega\ \right\}, \\ L_0^2(\Omega) &= \left\{q \in L^2(\Omega) : \int_\Omega q\ dx = 0\right\}. \end{aligned}$$

We denote by $\widehat{u}^\varepsilon = \left(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon, \widehat{u}_3^\varepsilon\right)$ and $\widehat{p}^\varepsilon$ the rescaling of the solution by $\{u^\varepsilon, p^\varepsilon\}$ of problem (2.9)-(2.10). For any $(x', z, t) \in Q$, we set

$$\begin{aligned} \widehat{u}_i^\varepsilon\left(x', z, t\right) &= u_i^\varepsilon\left(x', x_3, t\right)\ i = 1, 2,\ \widehat{u}_3^\varepsilon\left(x', z, t\right) = \varepsilon^{-1} u_3^\varepsilon\left(x', x_3, t\right), \\ \left(\widehat{u}_0^\varepsilon\right)_i\left(x', z\right) &= \left(u_0^\varepsilon\right)_i\left(x', x_3\right)\ i = 1, 2,\ \left(\widehat{u}_0^\varepsilon\right)_3\left(x', z\right) = \varepsilon^{-1}\left(u_0^\varepsilon\right)_3\left(x', x_3\right), \\ \widehat{p}^\varepsilon\left(x', z, t\right) &= \varepsilon^2 p^\varepsilon\left(x', x_3, t\right), \end{aligned}$$

and defining the rescaled force by

$$f^\varepsilon\left(x', x_3, t\right) = \varepsilon^{-2} \widehat{f}\left(x', z, t\right).$$

To meet our needs in paragraph 4.2, according to [5] we must assume

$$\begin{cases} \mu C(\Gamma_1^\varepsilon) \leq l^\varepsilon \\ \text{where } C(\Gamma_1^\varepsilon) = 2\|\frac{\partial}{\partial x_2} h^\varepsilon\|_{\mathcal{C}(\bar{\omega})}(1 + \|\frac{\partial}{\partial x_1} h^\varepsilon\|^2_{\mathcal{C}(\bar{\omega})} \\ l^\varepsilon = \varepsilon^{-1}\ l \ \text{ and } l \text{ be not dependent on } \varepsilon \end{cases} \tag{4.1}$$

One can check that $\{\widehat{u}^\varepsilon, \widehat{p}^\varepsilon\}$ solves the rescaled problem

$$\left.\begin{array}{c}\displaystyle\sum_{i=1,2}\varepsilon^2\left(\frac{\partial\widehat{u}_i^\varepsilon}{\partial t},\widehat{\phi}_i-\widehat{u}_i^\varepsilon\right)+\varepsilon^4\left(\frac{\partial\widehat{u}_3^\varepsilon}{\partial t},\widehat{\phi}_3-\widehat{u}_3^\varepsilon\right)+\widehat{a}\left(\widehat{u}^\varepsilon,\widehat{\phi}-\widehat{u}^\varepsilon\right)\\ \displaystyle-\sum_{i=1,2}\int_\Omega\widehat{p}^\varepsilon\frac{\partial(\widehat{\phi}_i-\widehat{u}_i^\varepsilon)}{\partial x_i}dx'dz-\int_\Omega\frac{1}{\varepsilon}\widehat{p}^\varepsilon\frac{\partial(\widehat{\phi}_3-\widehat{u}_3^\varepsilon)}{\partial z}dx'dz\\ \displaystyle+\sum_{i=1,2}l\int_{\Gamma_1}\widehat{u}_i^\varepsilon\left(\widehat{\phi}_i-\widehat{u}_i^\varepsilon\right)d\tau+l\int_{\Gamma_1}\varepsilon^2\widehat{u}_3^\varepsilon\left(\widehat{\phi}_3-\widehat{u}_3^\varepsilon\right)d\tau\\ \displaystyle+\sqrt{2}\alpha\varepsilon^{-1}\int_\Omega\left(\left|\widetilde{D}\left(\widehat{\phi}\right)\right|-\left|\widetilde{D}\left(\widehat{u}_\tau^\varepsilon\right)\right|\right)dx+\int_\omega k\left(\left|\widehat{\phi}_\tau\right|-\left|(\widehat{u}^\varepsilon)_\tau\right|\right)dx'\\ \displaystyle\geq\sum_{i=1,2}\int_\Omega\widehat{f}_i\left(\widehat{\phi}_i-\widehat{u}_i^\varepsilon\right)dx'dz+\varepsilon\int_\Omega\widehat{f}_3\left(\widehat{\phi}_3-\widehat{u}_3^\varepsilon\right)dx'dz\\ \forall\widehat{\phi}\in K,\ \forall t\in\,]0,T[\,,\\ \\ \widehat{u}^\varepsilon(0)=\widehat{u}_0^\varepsilon,\end{array}\right\}\tag{4.2}$$

where

$$\begin{aligned}\widehat{a}\left(\widehat{u}^\varepsilon(t),\widehat{\phi}-\widehat{u}^\varepsilon(t)\right)=&\sum_{i,j=1,2}\int_\Omega\varepsilon^2\mu\left(\frac{\partial\widehat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\widehat{u}_j^\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial x_j}(\widehat{\phi}_i-\widehat{u}_i^\varepsilon)dx'dz\\&+\sum_{i=1,2}\int_\Omega\mu\left(\frac{\partial\widehat{u}_i^\varepsilon}{\partial z}+\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial z}(\widehat{\phi}_i-\widehat{u}_i^\varepsilon)dx'dz\\&+\int_\Omega 2\mu\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial z}\frac{\partial(\widehat{\phi}_3-\widehat{u}_3^\varepsilon)}{\partial z}dx'dz\\&+\sum_{j=1,2}\int_\Omega\mu\varepsilon^2\left(\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_j}+\frac{\partial\widehat{u}_j^\varepsilon}{\partial z}\right)\frac{\partial}{\partial x_j}(\widehat{\phi}_3-\widehat{u}_3^\varepsilon)dx'dz,\end{aligned}$$

and

$$\left|\widetilde{D}(v)\right|=\left[\frac{1}{4}\sum_{i,j=1}^{2}\varepsilon^2\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)^2+\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial v_i}{\partial z}+\varepsilon^2\frac{\partial v_3}{\partial x_i}\right)^2+\varepsilon^2\left(\frac{\partial v_3}{\partial z}\right)^2\right]^{\frac{1}{2}}.$$

4.2. Estimates of solutions

We have the following estimate theorem

Theorem 4.1. *Assume that* (4.1) *hold, and let* $\{u^\varepsilon, p^\varepsilon\}$ *be a solution of problem (2.9)-(2.10). Then, there exist three constants* C, $\tilde{C}$ and $\tilde{C}'$ *independents of* ε *such that*

$$\begin{aligned}&\sum_{i=1}^{2}\left(\|\varepsilon\widehat{u}_i^\varepsilon(t)\|_{0,\Omega}^2+\int_0^t\left\|\frac{\partial\widehat{u}_i^\varepsilon}{\partial z}(s)\right\|_{0,\Omega}^2ds+\int_0^t\left\|\varepsilon^2\frac{\partial\widehat{u}_3^\varepsilon}{\partial x_i}(s)\right\|_{0,\Omega}^2ds\right)+\\&\left\|\varepsilon^2\widehat{u}_3^\varepsilon(t)\right\|_{0,\Omega}^2+\int_0^t\left\|\varepsilon\frac{\partial\widehat{u}_3^\varepsilon}{\partial z}(s)\right\|_{0,\Omega}^2ds+\sum_{i,j=1}^{2}\int_0^t\left\|\varepsilon\frac{\partial\widehat{u}_i^\varepsilon}{\partial x_j}(s)\right\|_{0,\Omega}^2ds\leq C,\end{aligned}\tag{4.3}$$

$$\sum_{i=1,2}\|\widehat{u}_i^\varepsilon\|_{L^2(Q)}^2+\|\varepsilon\widehat{u}_3^\varepsilon\|_{L^2(Q)}^2\leq\tilde{C},\tag{4.4}$$

$$\sum_{i=1,2} \|\varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial t}\|^2_{L^2(Q)} + \|\varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial t}\|^2_{L^2(Q)} \leq \tilde{C}'. \tag{4.5}$$

Proof. From [8], we recall the following inequalities (Poincaré, Korn and Young respectively)

$$\|u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} \leq 2\overline{h}^2 \varepsilon^2 \|\nabla u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} + 2\overline{h}\varepsilon \int_{\Gamma_1^\varepsilon} \|u^\varepsilon(t)\|^2_{0,\Gamma_1^\varepsilon} d\tau, \tag{4.6}$$

$$\mu \|\nabla u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} \leq a(u^\varepsilon(t), u^\varepsilon(t)) + \mu C(\Gamma_1^\varepsilon) \int_{\Gamma_1^\varepsilon} \|u^\varepsilon(t)\|^2_{0,\Gamma_1^\varepsilon} d\tau, \tag{4.7}$$

$$ab \leq \theta^2 \frac{a^2}{2} + \theta^{-2} \frac{b^2}{2}, \ \forall (a,b) \in \mathbb{R}^2, \ \forall \theta \in \mathbb{R}^*.$$

Integrating (2.9) over $[0,t]$ and choosing $\phi = 0$, we have

$$\begin{aligned} \frac{1}{2} \|u^\varepsilon(t)\|^2_{0,\Omega^\varepsilon} + \int_0^t a(u^\varepsilon(s), u^\varepsilon(s)) ds + l^\varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds \\ \leq \frac{1}{2} \|u_0^\varepsilon\|^2_{0,\Omega} + \int_0^t (f^\varepsilon(s), u^\varepsilon(s)) \, ds. \end{aligned} \tag{4.8}$$

Hence, by using Hölder, Poincaré and Young inequalities for $\theta = \sqrt{\mu/2}$,

$$a = \|\nabla u^\varepsilon(s)\|_{0,\Omega^\varepsilon} \ \text{ and } b = \varepsilon \overline{h} \|f^\varepsilon(s)\|_{0,\Omega^\varepsilon},$$

then $\theta = \sqrt{l^\varepsilon/2}$, $a = \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon}$ and $b = \sqrt{\overline{h}\varepsilon} \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon}$, respectively, we get

$$\begin{aligned} \left| \int_0^t (f^\varepsilon(s), u^\varepsilon(s)) ds \right| \leq \frac{\mu}{4} \int_0^t \|\nabla u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds + \frac{2\varepsilon^2 \overline{h}^2}{\mu} \int_0^t \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds \\ + \frac{l^\varepsilon}{4} \int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds + \frac{2\overline{h}\varepsilon}{l^\varepsilon} \int_0^t \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds. \end{aligned} \tag{4.9}$$

Ignoring the first term of (4.8) and combining (4.1), (4.7) and (4.9) we infer

$$\begin{aligned} \frac{\mu}{4} \int_0^t \|\nabla u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds + \frac{l^\varepsilon}{4} \int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds \\ \leq \frac{1}{2} \|u_0^\varepsilon\|^2_{0,\Omega} + \left(\frac{2\varepsilon^2 \overline{h}^2}{\mu} + \frac{2\overline{h}\varepsilon}{l^\varepsilon} \right) \int_0^t \|f^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds, \end{aligned}$$

multiplying the last inequality by $4\varepsilon^2$ and passing to the fixed domain in the right hand, we get

$$\varepsilon^2 \int_0^t \|\nabla u^\varepsilon(s)\|^2_{0,\Omega^\varepsilon} ds + \varepsilon \int_0^t \|u^\varepsilon(s)\|^2_{0,\Gamma_1^\varepsilon} ds \leq C \tag{4.10}$$

where C does not depend on ε.

We change again to the fixed domain in the first term of inequality (4.10), we find (4.3). From (4.6) and (4.10), it is easy to obtain a constant $\tilde{C} = \max(2\overline{h}^2, 2\overline{h})C$, such that

$$\varepsilon^{-1} \int_0^t \|u^\varepsilon(s)\|_{0,\Omega^\varepsilon}^2 \, ds \leq \tilde{C}$$

In fact, the last estimate is equivalent to (4.4).
Now, from (3.15) and as

$$\left\langle \left(J_\zeta^\varepsilon\right)'' \left(u_\zeta^\varepsilon\right), \frac{\partial}{\partial t} u_\zeta^\varepsilon \right\rangle \geq 0,$$

we have

$$\begin{aligned} &\frac{1}{2} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(t) \right\|_{0,\Omega^\varepsilon}^2 + \int_0^t a\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right) ds + l^\varepsilon \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \\ &\quad \leq \tfrac{1}{2} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon}^2 + \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon} ds \end{aligned} \tag{4.11}$$

By applying the inequality (4.6) for $\dfrac{\partial}{\partial t} u_\zeta^\varepsilon$ and the Young successively, we get

$$\begin{aligned} \left| \int_0^t \left(\frac{\partial f^\varepsilon}{\partial t}(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right) ds \right| \quad &\leq \frac{\mu}{8} \int_0^t \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \frac{4\varepsilon^2 \overline{h}^2}{\mu} \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds \\ &+ \frac{3l^\varepsilon}{4} \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds + \frac{2\overline{h}\varepsilon}{3l^\varepsilon} \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds. \end{aligned} \tag{4.12}$$

From (4.11), (4.12) and using (4.7) we obtain

$$\begin{aligned} &\frac{1}{2} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(t) \right\|_{0,\Omega^\varepsilon}^2 + \frac{\mu}{16} \int_0^t \left\| \nabla \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \frac{l^\varepsilon}{16} \int_0^t \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \leq \\ &\frac{1}{2} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon}^2 + \frac{4\varepsilon^2 \overline{h}^2}{\mu} \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \frac{2\overline{h}\varepsilon}{3l^\varepsilon} \int_0^t \left\| \frac{\partial f^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds \end{aligned} \tag{4.13}$$

We must estimate $\dfrac{\partial u_\zeta^\varepsilon}{\partial t}(0)$. Starting from the equation (3.17) and taking into account the assumptions (3.1), (3.3), then applying lemma 3.2, we conclude

$$\begin{aligned} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon} \quad &\leq \quad \|f^\varepsilon(0)\|_{0,\Omega^\varepsilon} + \|A(u_0^\varepsilon)\|_{0,\Omega^\varepsilon} + \left\| \left(J_\zeta^\varepsilon\right)' (u_0^\varepsilon) \right\|_{0,\Omega^\varepsilon} \\ &\leq \quad \|f^\varepsilon(0)\|_{0,\Omega^\varepsilon} + 2\sqrt{3}\mu \left\| u_0^\varepsilon \right\|_{2,\Omega^\varepsilon} + \varepsilon^{-1}\gamma \left\| u_0^\varepsilon \right\|_{2,\Omega^\varepsilon}. \end{aligned}$$

We recall that $\varepsilon \in]0,1[$, by multiplying the last inequality by $\varepsilon^{\frac{5}{2}}$ and the fact that $\varepsilon^3 \|u_0^\varepsilon\|_{2,\Omega^\varepsilon}^2 \leq \|\widehat{u}_0\|_{2,\Omega}^2$, we deduce

$$\varepsilon^{\frac{5}{2}} \left\| \frac{\partial u_\zeta^\varepsilon}{\partial t}(0) \right\|_{0,\Omega^\varepsilon} \leq c_0 \tag{4.14}$$

with

$$c_0 = \left\| \widehat{f}(0) \right\|_{0,\Omega} + 2\sqrt{3}\mu \|\widehat{u}_0\|_{2,\Omega} + \gamma \|\widehat{u}_0\|_{2,\Omega}.$$

Consequently, it follows from (4.13)-(4.14) and passing to the limit when $\zeta \to 0$, we find (after multiplying by $2\varepsilon^5$)

$$\frac{\mu}{8}\varepsilon^5 \int_0^t \left\| \nabla \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds + \frac{l}{8}\varepsilon^4 \int_0^t \left\| \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_{0,\Gamma_1^\varepsilon}^2 ds \leq C' \tag{4.15}$$

with

$$C' = (c_0)^2 + \frac{8\overline{h}^2}{\mu} \left\| \frac{\partial}{\partial t} \widehat{f} \right\|_{L^2(Q)}^2 + \frac{4\overline{h}}{3l} \left\| \frac{\partial}{\partial t} \widehat{f} \right\|_{L^2(Q)}^2$$

is a constant independent of ε.
We apply the inequality (4.6) for $\dfrac{\partial u^\varepsilon}{\partial t}$ in the estimate (4.15), that implies that there exists a constant $\tilde{C}'$ independent of ε such that

$$\varepsilon \int_0^t \left\| \frac{\partial u^\varepsilon}{\partial t}(s) \right\|_{0,\Omega^\varepsilon}^2 ds \leq \tilde{C}'.$$

Finally, passing this estimate to the fixed domain Ω to get (4.5). □

Theorem 4.2. *Under the hypotheses of theorem 4.1 there exists a constant C independent of ε such that*

$$\left\| \frac{\partial \hat{p}^\varepsilon}{\partial x_i} \right\|_{L^2(0,T,H^{-1}(\Omega))} \leq C, \ i = 1,2 \text{ and } \left\| \frac{\partial \hat{p}^\varepsilon}{\partial z} \right\|_{L^2(0,T,H^{-1}(\Omega))} \leq C\varepsilon. \tag{4.16}$$

Proof. Let ξ in $L^2(0,T,H_0^1(\Omega))$, putting in (4.2) $\phi = \widehat{u}^\varepsilon + \tilde{\xi}$, where $\tilde{\xi} = (\xi,0,0)$ or $\tilde{\xi} = (0,\xi,0)$ and integrating over $[0,t]$ we find for $i = 1,2$,

$$\int_0^t \left(\frac{\partial \widehat{p}^\varepsilon}{\partial x_i}(s), \xi(s) \right) ds \leq \int_0^t \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial t}(s), \xi(s) \right) ds$$

$$+\mu \sum_{i,j=1,2} \int_0^t \int_\Omega \varepsilon^2 \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \widehat{u}_j^\varepsilon}{\partial x_i} \right)(s) \frac{\partial \xi}{\partial x_j}(s)\, dx' dz ds$$

$$+\sqrt{2}\alpha \int_0^t \int_\Omega \left(\left| \widetilde{D}\left(\widehat{u}^\varepsilon + \tilde{\xi} \right) \right| - \left| \widetilde{D}\left(\widehat{u}^\varepsilon \right) \right| \right) dx' dz ds$$

$$+\mu \sum_{i=1,2} \int_0^t \int_\Omega \left(\frac{\partial \widehat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_i} \right)(s) \frac{\partial \xi}{\partial z}(s)\, dx' dz ds - \int_0^t \left(\widehat{f}_i(s), \xi(s) \right) ds.$$

The Hölder inequality and estimates (4.3)-(4.5) show the continuity of the linear functional

$$\xi \to \int_0^t \left(\frac{\partial \widehat{p}^\varepsilon}{\partial x_i}(s), \xi(s) \right) ds,$$

which proves (4.16) for $i = 1, 2$. In addition, case $i = 3$ follows from the choice $\phi = \widehat{u}^\varepsilon(t) \pm \xi$ with $\xi \equiv (0, 0, \xi)$. □

4.3. Convergence u^ε and p^ε

To establish a limit solution of the problem, we introduce the following space,

$$V_z = \left\{ v = (v_1, v_2) \in L^2(\Omega)^2 : \frac{\partial v}{\partial z} \in L^2(\Omega)^2;\ v = 0 \text{ on } \Gamma_L \right\}.$$

From [6], $L^2(0, T, V_z)$ is a Banach space. We show the following result:

Theorem 4.3. *Under the hypotheses of theorem 4.1, for any solution $\{u^\varepsilon, p^\varepsilon\}$, there exist $u^\star = (u_1^\star, u_2^\star) \in L^2(0, T, V_z)$ and $p^\star \in L^2(0, T, L_0^2(\Omega))$ such that when ε tends to 0 we have the following convergences in $L^2(0, T, V_z)$:*

$$(\widehat{u}_1^\varepsilon, \widehat{u}_2^\varepsilon) \rightharpoonup (u_1^\star, u_2^\star), \quad \varepsilon^2 \left(\frac{\partial}{\partial t} \widehat{u}_1^\varepsilon, \frac{\partial}{\partial t} \widehat{u}_2^\varepsilon \right) \rightharpoonup 0 \tag{4.17}$$

the following convergences in $L^2(Q)$:

$$\varepsilon \widehat{u}_3^\varepsilon \rightharpoonup 0, \quad \varepsilon^3 \frac{\partial \widehat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0, \quad \varepsilon \frac{\partial \widehat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad \varepsilon^2 \frac{\partial \widehat{u}_3^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad \varepsilon \frac{\partial \widehat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \tag{4.18}$$

$(1 \le i, j \le 2)$, *and the convergence $\widehat{p}^\varepsilon \rightharpoonup p^\star$ in $L^2(0, T, L_0^2(\Omega))$.*
Moreover, $p^\star$ depends only on x'.

Proof. In particular (4.3), (4.4) we have

$$\left\| \frac{\partial \widehat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(Q)}^2 \le C \text{ and } \|\widehat{u}_i^\varepsilon\|_{L^2(Q)}^2 \le \tilde{C},$$

for $i = 1, 2$, we deduce the first convergence of (4.17). Similarly, from (4.15) and (4.5) we find the second. For the rest of the proof, we use the same steps in the stationary case as in [1, 5]. □

5. On the limit model

By a classical semi continuity argument and using the convergence results of the theorem 4.3, we deduce that (4.2) leads to the system

$$\left.\begin{array}{c}
\sum_{i=1}^{2}\mu\int_{\Omega}\frac{\partial u_i^\star}{\partial z}(t)\,\frac{\partial}{\partial z}(\widehat{\phi}_i-u_i^\star(t))dx'dz \\
-\int_{\Omega}p^\star(x',t)\left(\frac{\partial\widehat{\phi}_1}{\partial x_1}+\frac{\partial\widehat{\phi}_2}{\partial x_2}\right)dx'dz \\
-\int_{\omega}p^\star(x',t)\left(\widehat{\phi}_1(x',h(x'))\frac{\partial h}{\partial x_1}+\widehat{\phi}_2(x',h(x'))\frac{\partial h}{\partial x_2}\right)dx' \\
+\sum_{i=1}^{2}l\int_{\Gamma_1}u_i^\star(t)\left(\widehat{\phi}_i-u_i^\star(t)\right)d\tau \\
+\alpha\int_{\Omega}\left(\left|\frac{\partial\widehat{\phi}}{\partial z}\right|-\left|\frac{\partial u^\star}{\partial z}(t)\right|\right)dx'dz+\int_{\omega}k(|\widehat{\phi}|-|u^\star(t)|)dx' \\
\geq\sum_{i=1}^{2}(\widehat{f}_i(t),\widehat{\phi}_i-u_i^\star(t))\quad\forall\widehat{\phi}\in\Pi(K),\forall t\in\,]0,T[, \\
\\
u_i^\star(x',z,0)=\widehat{u}_{0,i},\quad i=1,2
\end{array}\right\}\tag{5.1}$$

where

$$\Pi(K)=\left\{\left(\widehat{\phi}_1,\widehat{\phi}_2\right)\in H^1(\Omega)^2:\widehat{\phi}=\left(\widehat{\phi}_1,\widehat{\phi}_2,\widehat{\phi}_3\right)\in K\right\}.$$

Theorem 5.1. *Under the assumptions of theorem 4.1, the limit solution $\{u^\star,\ p^\star\}$ satisfies:*

$$-\frac{\partial}{\partial z}\,\sigma_i^\star(t)=\widehat{f}_i(t)-\frac{\partial}{\partial x_i}p^\star(t),\ i=1,2,\ \textit{in}\ L^2(\Omega),\tag{5.2}$$

$$u_i^\star(0)=\widehat{u}_{0,i},\ i=1,2\tag{5.3}$$

for a.e. $t\in\,]0,T[$, where $\sigma^\star=(\sigma_i^\star)_{i=1,2}$ checks the constitutive law of Bingham fluid, as follows

$$\left\{\begin{array}{l}
\sigma^\star=\mu\frac{\partial u^\star}{\partial z}+\alpha\frac{\partial u^\star/\partial z}{|\partial u^\star/\partial z|},\ \text{ if }\left|\frac{\partial u^\star}{\partial z}\right|\neq 0, \\
\qquad|\sigma^\star|\leq\alpha,\qquad\text{if }\left|\frac{\partial u^\star}{\partial z}\right|=0
\end{array}\right.\quad\text{in } Q\tag{5.4}$$

Proof. Let $\psi=(\psi_1,\psi_2)\in H_0^1(\Omega)^2$, putting in (5.1) $\widehat{\phi}=u^\star(t)\pm\lambda\psi$ ($\lambda>0$) and dividing the inequality obtained by λ, as λ tends to zero, for any t it follows that

$$\sum_{i=1}^{2}\mu\int_{\Omega}\frac{\partial u_i^\star}{\partial z}(t)\,\frac{\partial}{\partial z}\psi dx'dz-\int_{\Omega}p^\star(x',t)\left(\frac{\partial\psi_1}{\partial x_1}+\frac{\partial\psi_2}{\partial x_2}\right)dx'dz$$
$$+\sum_{i=1}^{2}\alpha\int_{\Omega}\left\{\left|\frac{\partial u^\star}{\partial z}(t)\right|^{-1}\frac{\partial u_i^\star}{\partial z}(t)\right\}\frac{\partial}{\partial z}\psi_i dx'dz=\sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i(t)\,\psi_i dx'dz$$

when

$$\left|\frac{\partial u^{\star}}{\partial z}(t)\right| \neq 0.$$

By Green's formula, we obtain

$$-\sum_{i=1}^{2}\int_{\Omega}\mu\frac{\partial^2 u_i^{\star}}{\partial z^2}(t)\,\psi_i dx' + \sum_{i=1}^{2}\int_{\Omega}\frac{\partial p^{\star}}{\partial x_i}(x',t)\psi_i dx'dz$$
$$-\sum_{i=1}^{2}\alpha\int_{\Omega}\frac{\partial}{\partial z}\left\{\left|\frac{\partial u^{\star}}{\partial z}(t)\right|^{-1}\frac{\partial u_i^{\star}}{\partial z}(t)\right\}\psi_i dx'dz = \sum_{i=1}^{2}\int_{\Omega}\widehat{f}_i(t)\,\psi_i dx'dz.$$

Therefore, from this equality and fact that $\widehat{f} \in L^2(Q)$ we get (5.2). Similarly, the second case of (5.4) can be recovered by [7]. The condition (5.3) is a consequence directly of (4.17), (4.18) and the condition $\widehat{u}^{\varepsilon}(0) = \widehat{u}_0^{\varepsilon}$. □

Now we are in a position to deduce the equations corresponding for problem (5.1)-(5.4).

Remark 5.1. Note that the term related to inertia effects does not exist in the limit equation in (5.2), means that the limit problem (5.2) - (5.4) is in equilibrium at each time instant. Therefore, the Reynolds equation is obtained in a manner similar to the stationary case as in [1], and from [2] the Tresca boundary condition can be recovered. Indeed, the case $\alpha = 0$ corresponds to the Stokes flow, and has been studied in [8].

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