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THE SCHWARZSCHILD-TYPE TWO-BODY PROBLEM: A TOPOLOGICAL VIEW

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Abstract. The Schwarzschild-type two-body problem (associated to a force function of the form $A/r + B/r^3, A, B > 0$), which models several problems of nonlinear particle dynamics, is being tackled from the standpoint of topology. The corresponding mechanical system is fully characterized, and the first integrals of energy and angular momentum are pointed out. These integrals are used to settle the invariant manifolds and the bifurcation set for the whole allowed interplay among the field parameters, the total energy level, and the angular momentum. The orbits on each manifold are interpreted in terms of physical motion. Besides recovering motions characteristic to classical models, entirely new types of motion are found.

1. Introduction

The theory of orbits in a force field characterized by a force function of the form $A/r + B/r^3$ (with r = distance of a particle to the field source; $A, B > 0$ constants) constitutes an extensively discussed subject. This theory, which models concrete problems belonging to astrophysics, stellar dynamics, celestial mechanics, astrodynamics, cosmogony, etc., was approached by various methods, both qualitative and (especially) quantitative.

Many authors studied quantitatively the motion in such a field (see, e.g., Brumberg, 1972; Chandrasekhar, 1983; Damour and Schaefer, 1986), generally in a relativistic context, showing that the analytic solution of the problem can be obtained in closed form by means of elliptic functions. But the analytic form of these solutions hides the general geometric properties of the model.

As to the rather few qualitative approaches, they dealt mainly with the regularization of motion equations (see, e.g., Saari, 1974; Belenkii, 1981; Szebehely and

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Bond, 1983; Cid *et al.*, 1983). In addition, the quoted authors used only Sundman-type transformations of time.

Stoica and Mioc, 1997 studied qualitatively the problem for any $A \neq 0$, $B \neq 0$, and provided a complete geometric and physical description of the orbits.

Following Smale's topological program (see Smale, 1970, Iacob, 1973 or Abraham and Marsden, 1981), the aim of this paper is to determine the topological type of the energy-momentum invariant manifolds, determined by the first integrals of energy and angular momentum, and the set of bifurcation points, in whose neighbourhood the topological type of the invariant manifolds is changing. Using the geometrical properties of the invariant manifolds, the types of physical motion are briefly characterized for $A, B > 0$.

2. Basic Equations

It is clear that the Schwarzschild-type two-body problem can be reduced to a central force problem (e.g. Arnold 1976). Within this framework, the motion of the particle is confined to a plane. We shall use polar coordinates (r, θ), and follow the treatment presented by Abraham and Marsden (1981, p.656).

The mechanical system with symmetry which describes the problem is (M, K, V, G), where:

$M = (0, \infty) \times S^1$ is the space of the polar coordinates (r, θ), regarded as a Riemannian manifold endowed with the metric

$$\left\langle (r_1, \theta_1, \dot{r}_1, \dot{\theta}_1), (r_2, \theta_2, \dot{r}_2, \dot{\theta}_2) \right\rangle = \dot{r}_1 \dot{r}_2 + r_1 r_2 \dot{\theta}_1 \dot{\theta}_2,$$

dots marking time-differentiation;

K is the kinetic energy of the metric above, whose expression on the cotangent bundle T^*M is

$$K(r, \theta, p_r, p_\theta) = (p_r^2 + p_\theta^2 / r^2)/2, \tag{1}$$

p_r, p_θ denoting the momenta;

V is the potential energy, given by

$$V(r, \theta) = -A/r - B/r^3; \tag{2}$$

$G = SO(2) \cong S^1$ is the Lie group that acts on M by rotations ($\cong$ denoting isomorphism). Observe that G acts by isometries and leaves V invariant (cf. Abraham and Marsden 1981).

The Hamiltonian of the system is

$$H(r, \theta, p_r, p_\theta) = (p_r^2 + p_\theta^2/r^2)/2 - A/r - B/r^3. \tag{3}$$

The momentum mapping $J : T^*M \longrightarrow \mathbf{R}$ is given by $J(r, \theta, p_r, p_\theta) = p_\theta$, and is invariant under the action of G.

Consider $\mathbf{x} = (r, \theta) \in M$ and the mapping $J_{\mathbf{x}} : T^*_{\mathbf{x}}M \longrightarrow \mathbf{R}$. The expression $J_{\mathbf{x}} : (p_r, p_\theta) \longmapsto p_\theta$ of this mapping shows that $J_{\mathbf{x}}$ is surjective for all $\mathbf{x} \in M$. In other words,

$$\Lambda = \{\mathbf{x} \in M \mid J_{\mathbf{x}} : T^*_{\mathbf{x}}M \longrightarrow \mathbf{R} \text{ is not surjective}\} = \emptyset.$$

Note that $dJ = dp_\theta$, therefore J has no critical points on T^*M.

The problem admits the first integrals of energy and angular momentum, respectively:

$$H(r, \theta, p_r, p_\theta) = K(r, \theta, p_r, p_\theta) + V(r) = h, \tag{4}$$

$$J(r, \theta, p_r, p_\theta) = p_\theta = C, \tag{5}$$

where h and C stand for the integration constants of energy and angular momentum.

3. Effective Potential Energy

Eliminating p_θ between (3)+(4) and (5), one gets

$$p_r^2 = 2(h - V_C), \tag{6}$$

in which

$$V_C(r) = V(r) + C^2/(2r^2) = -A/r + C^2/(2r^2) - B/r^3 \tag{7}$$

denotes the so-called *effective potential energy.*

Settled the constant angular momentum C, one sees by (6) that the real motion is possible only in the domains $V_C(r) \leq h$, where h is a fixed total energy level.

The graph of the function $V_C = V_C(r)$ in different cases, for all values C depending on A, B is plotted in Figure 1.

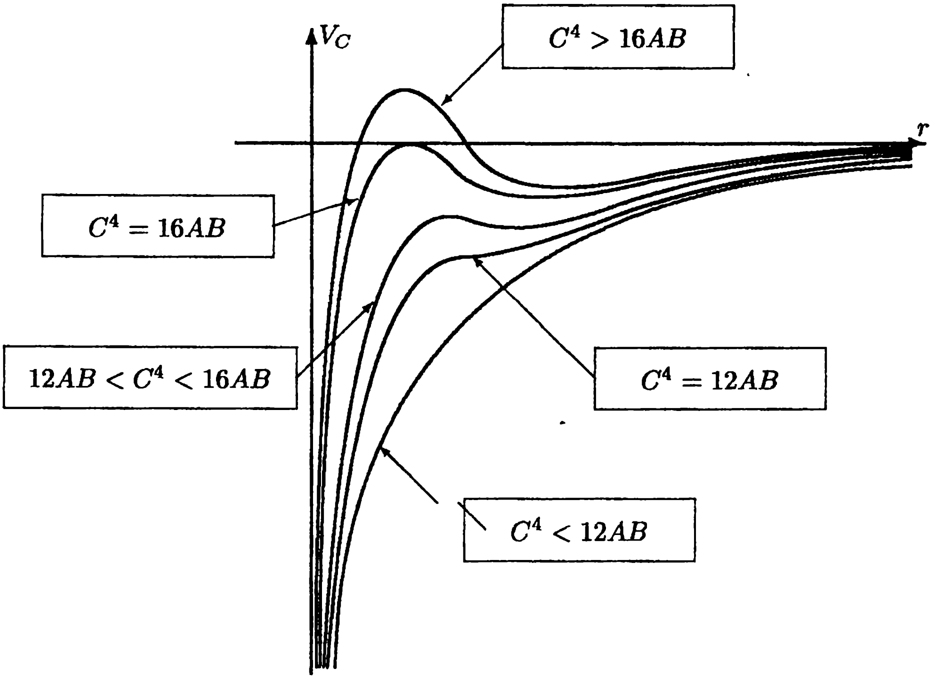

FIGURE 1. The graph of the effective potential energy:

4. Bifurcation Set and Topological Type of Invariant Manifold

To study the motion, we use the invariant manifolds $I_{h,C} = (H \times J)^{-1}(h, C)$, whose defining equations are (4) and (5). Obviously, the topological type of $I_{h,C}$ depends on the condition $V_C(r) \leq h$, and because of the rotational symmetry, each component of $I_{h,C}$ is a product, S^1 being one of the factors. Using the graphs of Figure 1, and observing their significance as regards the allowed domains for r to have real motion, we are able to identify the invariant manifolds diffeomorphic ($\approx$) with $I_{h,C}$ on which the phase curves lie.

To synthesize all possible cases, les us establish and plot the bifurcation set $H \times J$, defined as the set of couples $(h, C) \in \mathbf{R}^2$ for which the energy-momentum mapping $H \times J$ fails to be locally trivial, in other words, those points in whose neighbourhood the topological type of the invariant manifolds is changing (see e.g. Abraham and Marsden 1981).

First we determine the set of critical values $\Sigma'_{H\times J} \subseteq \Sigma_{H\times J}$, defined by the conditions

(8) $$\Sigma'_{H\times J} = \{(h,C) \in \mathbf{R}^2 \mid h = V_C(r), V'_C(r) = 0\} = \bigcup_{i=1}^{2}\{(h,C) \in \mathbf{R}^2 \mid h = (h_{cr})_i\},$$

where

(9) $$(h_{cr})_i = V_C(r_i), \qquad i = 1,2;$$

and

(10) $$r_i = \frac{C^2 + (-1)^{i+1}\sqrt{C^4 - 12AB}}{2A} \qquad i = 1,2.$$

are the critical points of the effective potential $V_C(r)$, $(V'_C(r) = 0)$.

After some computations we obtain the set of critical values:

(11) $$\Sigma'_{H\times J} = \{(h,C) \in \mathbf{R}^2 \mid 108B^2h^2 + 2C^2(18AB - C^4)h + A^2(16AB - C^4) = 0\}.$$

The graph of this curve is plotted in Figure 2 and has two components, defined for $i = 1,2$ by:

(12)
$$\{(h,C) \in \mathbf{R}^2 \mid h = (h_{cr})_i\} = \{(h,C) \in \mathbf{R}^2 \mid h = \frac{C^2(C^4 - 18AB) + (-1)^{i+1}(C^4 - 12AB)^{\frac{3}{2}}}{108B^2}\}$$

Note that the complete picture of the set of critical values is symmetric to the $C = 0$ axis, and this symmetry occurs in all the nexts.

The complete bifurcation set is:

(13) $$\Sigma_{H\times J} = \Sigma'_{H\times J} \cup \{(h,C) \in \mathbf{R}^2 \mid h = 0\}.$$

For different values of the energy and angular momentum constants we found seven cases. The corresponding sets in the (h,C) plane are noted in Figure 2 with (a), (b), ..., (g). In different cases the topological type of the invariant manifolds and the type of orbits in the configuration space is:

(a) If the energy and angular momentum constants are in the domain $\{(h,C) \in \mathbf{R}^2 \mid h \geq 0, h > (h_{cr})_1(C)\}$, then the invariant manifold is diffeomorphic with the reunion of two disjont cylinders $(I_{h,\mu} \approx S^0 \times S^1 \times \mathbf{R})$, and the corresponding orbits in configuration

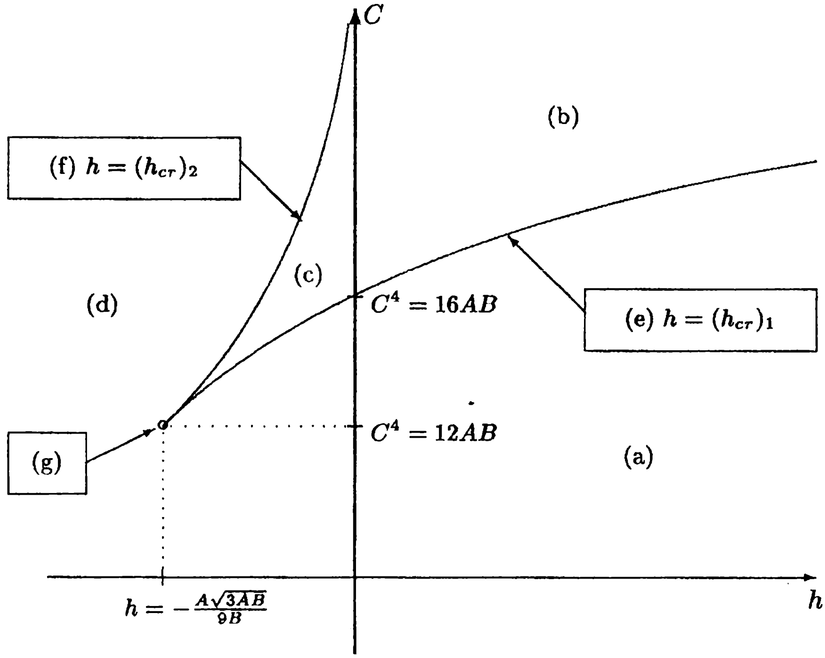

FIGURE 2. The bifurcation set.

space are ejecting from collision and tending to infinity ($0 \to \infty$), or coming from infinity and tending to collisions ($\infty \to 0$).

(b) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid h \geq 0, h < (h_{cr})_1(C)\}$, then $I_{h,C} \approx S^0 \times S^1 \times \mathbf{R}$, but in this case the orbits are comming from infinity and then tending back to infinity ($\infty \to \infty$), or are ejecting from collision and tending back to collision ($0 \to 0$).

(c) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid h < 0, (h_{cr})_2 < h < (h_{cr})_1\}$, then $I_{h,C} \approx (S^1 \times \mathbf{R}) \cup (S^1 \times S^1)$, is the disjont reunion of a cylinder and a torus. The orbits type is ($0 \to 0$), or there are periodic or quasiperiodic orbits (P/QP).

(d) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid (h < 0, h < (h_{cr})_2)$ or $(h < 0, h > (h_{cr})_1)\}$, then $I_{h,C} \approx (S^1 \times \mathbf{R})$ (one cylinder), and the orbits are of the ($0 \to 0$) type.

(e) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid (h = (h_{cr})_1), C^4 > 12AB\}$, then $I_{h,C}$ is diffeomorphic with the reunion of two cylinders which are intersecting in a circle. In this case unstable equilibrium orbits (UE) may exist, or the orbits are of the type ($0 \to$ UE), (UE $\to 0$), ($\infty \to$ UE), (UE $\to \infty$).

(f) If $(h, C) \in \{(h, C) \in \mathbf{R}^2 \mid (h = (h_{cr})_2), C^4 > 12AB\}$ then $I_{h,C} \approx (S^1 \times \mathbf{R}) \cup S^1$ (disjont reunion of a cylinder and a circle), and the orbits are of type $(0 \to 0)$ or (SE), stable equilibrium.

(g) If $C^4 = 12AB, h = (h_{cr})_1 = (h_{cr})_2 = -\frac{A\sqrt{3AB}}{9B}$, then $I_{h,C}$ is homeomorphic (and not diffeomorphic in this case) with $S^1 \times \mathbf{R}$, and the orbits are of type (UE), $(0 \to \text{UE})$ or $(\text{UE} \to 0)$.

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