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# ON THE SPLINE APPROXIMATING METHODS FOR SECOND ORDER SYSTEMS OF DIFFERENTIAL EQUATIONS

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**Abstract.** One proposes an approximation method for the solution of the systems of $p$ second order differential equations by means of spline functions. There is studied the error estimation and the stability of the proposed method.

## 1. Introduction.

Consider the following system of nonlinear second order differential equations with the initial conditions:

$$\begin{cases} y_i''(x) & = & f_i(x, y_1, \dots, y_p) \\ y_i(x_0) & = & y_{i,0}, \; y_i'(x_0) = y_{i,0}', \quad i = \overline{1,p} \end{cases}$$

where $f_i \in C^r([0,1] \times \mathbb{R}^p)$, $i = \overline{1,p}$ and $r, p \in \mathbb{N}$.

The approximate solution of a system of 2 equations of second order was constructed by Th. Fawzy, Z. Ramadan and A.Ayad [1,2]. In this paper we propose a generalization of the method, for system of $p$ equations. The system (1) can be transformed in a system of $n$ equations of the first order, but the order of the method presented is $O(h^{\alpha+r+2m})$. The order of the method presented by G.Micula and Maria Micula[5], for system of $n$ equations of first order, is $O(h^{\alpha+r+m})$.

## 2. Description of the approximating method.

Let $L_i$ be the

Lipschitz constants satisfied by the functions $f_i^{(q)}$, $i = \overline{1,p}$, $q = \overline{0,r}$:

$$|f_i^{(q)}(x, y_{1,1}, \dots, y_{p,1}) - f_i^{(q)}(x, y_{1,2}, \dots, y_{p,2})| \le L_i \sum_{j=1}^{p} |y_{j,1} - y_{j,2}| \tag{1}$$

$\forall (x, y_{1,1}, \dots, y_{p,1})$, $(x, y_{1,2}, \dots, y_{p,2})$ in the domain of definition of $f_i$, $i = \overline{1,p}$.

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Let $\Delta$ be a partition of the interval $[0,1]$:

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1,\ h := x_{k+1} - x_k,\ k = \overline{0, n-1}$$

Assume that $f_i \in C^r([0,1] \times \mathbb{R}^p)$ and that the modulus of continuity of the functions $y_i^{(r+2)}$ is $\omega(y_i^{(r+2)}, h)$, and $\omega(h) = \max_{i=\overline{1,p}} \omega(y_i^{(r+2)}, h)$.

The functions $f_i^{(q-1)}$ depending of $x$, $y_1, \ldots, y_p$ are given from the following algorithm:

Set $f_i^{(0)} = f_i$ and if $f_i^{(q-1)}$ are defined, then:

$$y_i^{(q+2)}(x) = f_i^{(q)}(z) = f_{i_x}^{(q-1)}(z) + f_{i_{y_1}}^{(q-1)}(z) y_1' + \ldots + f_{i_{y_p}}^{(q-1)}(z) y_p' \tag{2}$$

where $z = (x, y_1, \ldots, y_p)$.

We define the spline functions approximating $y_i$ by $s_{i,\Delta}$, $i = \overline{1,p}$, for $x_k \leq t_m \leq t_{m-1} \leq \ldots \leq t_1 \leq t \leq x \leq x_{k+1}$, $k = \overline{0, n-1}$ a partition of the interval $[x_k, x_{k+1}]$, in the following way:

$$\begin{aligned} s_{i,\Delta}(x) &:= s_{i,k}^{[m]}(x) := s_{i,k-1}^{[m]}(x_k) + s_{i,k-1}^{'[m]}(x_k)(x - x_k) + \\ &+ \int_{x_k}^{x} \int_{x_k}^{t} f_i[t_1, s_{1,k}^{[m-1]}(t_1), \ldots, s_{p,k}^{[m-1]}(t_1)] dt_1 dt \end{aligned} \tag{3}$$

where $s_{i,-1}^{[m]}(x_0) := y_{i,0}$, $s_{i,-1}^{'[m]}(x_0) = y_{i,0}'$ and $m$ is a positiv integer number.

In (4) we use the following $m$ iterations:

$$s_{i,k}^{[0]}(x) := s_{i,k-1}^{[m]}(x_k) + s_{i,k-1}^{'[m]}(x_k)(x - x_k) + \sum_{j=0}^{r} \frac{(x - x_k)^{j+2}}{(j+2)!} M_{i,k}^{(j)} \tag{4}$$

$$\begin{aligned} s_{i,k}^{[j]}(x) &:= s_{i,k-1}^{[m]}(x_k) + s_{i,k-1}^{'[m]}(x_k)(x - x_k) + \\ &+ \int_{x_k}^{x} \int_{x_k}^{t} f_i(t_{m-j+1}, s_{1,k}^{[j-1]}(t_{m-j+1}), \ldots, s_{p,k}^{[j-1]}(t_{m-j+1})) dt_{m-j+1} dt \\ M_{i,k}^{(j)} &:= f_i^{(j)}(x_k, s_{1,k-1}^{[m]}(x_k), \ldots, s_{p,k-1}^{[m]}(x_k)),\ j = \overline{1,m} \end{aligned} \tag{5}$$

It is clear by construction that $s_{i,\Delta} \in C^1[0,1]$, $i = \overline{1,p}$.

## 3. Error estimation and convergence.

The following notations will be used $y_{i,k}^{(j)} := y_i^{(j)}(x_k)$ for $i = \overline{1,p}$, $j = \overline{0, r+1}$ and $k = \overline{1, n-1}$. The exact solution $y_i := y_i^{[m]}$, $i = \overline{1,p}$ of (1) can be written in the following form:

- By Taylor's expansion, for $y_i^{[0]}$, holds:

$$y_i^{[0]}(x) = \sum_{j=0}^{r+1} \frac{y_{i,k}^{(j)}}{j!}(x-x_k)^j + \frac{y_i^{(r+2)}(\xi_{i,k})}{(r+2)!}(x-x_k)^{r+2} \tag{6}$$

where $\xi_{i,k} \in ]x_k, x_{k+1}[$, $i = \overline{1,p}$.

- For $1 \le j \le m$ the exact solution $y_i^{[j]}$ is given by:

$$\begin{aligned} y_i^{[j]}(x) \;&:=\; y_{i,k} + y'_{i,k}(x-x_k) + \\ &+ \int_{x_k}^{x}\int_{x_k}^{t} f_i(t_{m-j+1}, y_1^{[j-1]}(t_{m-j+1}), \ldots, y_p^{[j-1]}(t_{m-j+1}))dt_{m-j+1}dt \end{aligned} \tag{7}$$

where $i = \overline{1,p}$, , $j = \overline{1,m}$, $k = \overline{0,n-1}$

The error is defined by the usual way, for $i = \overline{1,p}$, $k = \overline{0,n-1}$:

$$\begin{aligned} e_i(x) := |y_i(x) - s_{i,\Delta}(x)| \quad &, \quad e'_i(x) := |y'_i(x) - s'_{i,\Delta}(x)| \\ e_{i,k} := |y_{i,k} - s_{i,\Delta}(x_k)| \quad &, \quad e'_{i,k} := |y'_{i,k} - s'_{i,\Delta}(x_k)| \end{aligned} \tag{8}$$

**Lemma 3.1.** *[1] Let $\alpha$ and $\beta$ be nonnegative real numbers, $\beta \neq 1$ and $\{A_i\}_{i=0}^{k}$ be a sequence satisfying the conditions:*

$$A_0 \ge 0,\ A_{i+1} \le \alpha + \beta A_i,\ i = 0, 1, \ldots, k$$

*then the following inequality holds:*

$$A_{k+1} \le \beta^{k+1}A_0 + \alpha\frac{\beta^{k+1}-1}{\beta-1}$$

**Lemma 3.2.** *[1] Let $\alpha$ and $\beta$ be positive real numbers , and $\{A_i\}_{i=1}^{m}$ be a sequence satisfying :*

$$A_1 \ge 0,\ A_i \le \alpha + \beta A_{i+1},\ i = 1, \ldots, m-1.$$

*then*

$$A_1 \le \beta^{m-1}A_m + \alpha\sum_{i=0}^{m-2}\beta^i$$

**Definition 3.1.** For any $u \in [x_k, x_{k+1}]$, $k = \overline{0,n-1}$, $j = \overline{1,m}$ we define the operator $T_{kj}$ by:

$$T_{kj}(u) := \sum_{i=1}^{p} |y_i^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)|,$$

whose norm is defined by:

$$||T_{kj}|| := \max_{u\in[x_k,x_{k+1}]}\{T_{kj}(u)\}$$

**Lemma 3.3.** *For any $u \in [x_k, x_{k+1}]$, $k = \overline{0, n-1}$, $j = \overline{1, m}$, the following estimations:*

$$||T_{km}|| \leq \{1 + pL\sum_{j=0}^{r} \frac{1}{(j+2)!}\} \sum_{i=1}^{p} e_{i,k} + \sum_{i=1}^{p} e'_{i,k} + p\,\omega(h)\frac{h^{r+2}}{(r+2)!} \tag{9}$$

*with $L = \max\{L_1, L_2, \dots, L_p\}$, and*

$$||T_{k1}|| \leq a\sum_{i=1}^{p} e_{i,k} + b\sum_{i=1}^{p} e'_{i,k} + c\,h^{2m+r}\omega(h) \tag{10}$$

*hold, where a, b, c are constants independent of h:*

$$\begin{aligned} a &= \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^{j} + 2\left(\frac{pL}{2}\right)^{m}\sum_{j=0}^{r}\frac{1}{(j+2)!} \\ b &= \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^{j} \\ c &= \frac{p^m}{(r+2)!}\left(\frac{L}{2}\right)^{m-1} \end{aligned}$$

*Proof.* Using (5) and (7) we get:

$$\begin{aligned} |y_i^{[0]}(u) - s_{i,k}^{[0]}(u)| &\leq |y_{i,k} - s_{i,k-1}^{[m]}(x_k)| + |y'_{i,k} - s'^{[m]}_{i,k-1}(x_k)||x - x_k| + \\ &+ \sum_{j=0}^{r-1}\frac{|x-x_k|^{j+2}}{(j+2)!}|y_{i,k}^{(j+2)} - M_{i,k}^{(j)}| + \frac{|x-x_k|^{r+2}}{(r+2)!}|y_i^{(r+2)}(\xi_{i,k}) - M_{i,k}^{(r)}| \end{aligned} \tag{11}$$

From (9) and (2), we can see that:

$$|y_{i,k}^{(j+2)} - M_{i,k}^{(j)}| \leq L_i\{\sum_{j=1}^{p}|y_{j,k} - s_{j,k-1}^{[m]}(x_k)|\} \leq L_i\sum_{j=1}^{p} e_{j,k} \tag{12}$$

$$|y_i^{(r+2)}(\xi_{i,k}) - M_{i,k}^{(r)}| \leq \omega(y_i^{(r+2)}, h) + L_i\sum_{j=1}^{p} e_{j,k} \tag{13}$$

where $\omega(y_i^{(r+2)}, h)$ is the modulus of continuity of function $y^{(r+2)}$. Using (13) and (14) in (12), we obtain:

$$\begin{aligned} \max_{u\in[x_k,x_{k+1}]}|y_i^{[0]}(u) - s_{i,k}^{[0]}(u)| &\leq e_{i,k} + he'_{i,k} + L_i\sum_{j=1}^{p} e_{j,k}\sum_{l=0}^{r}\frac{h^{l+2}}{(l+2)!} + \\ + \frac{h^{r+2}}{(r+2)!}\omega(h) &\leq e_{i,k} + e'_{i,k} + L_i\sum_{j=1}^{p} e_{j,k}\sum_{l=0}^{r}\frac{1}{(l+2)!} + \frac{\omega(h)h^{r+2}}{(r+2)!} \end{aligned} \tag{14}$$

Adding in (15) for $i = \overline{1, p}$, we get:

$$||T_{km}|| \leq \{1 + pL\sum_{j=0}^{r}\frac{1}{(j+2)!}\}\sum_{i=1}^{p} e_{i,k} + \sum_{i=1}^{p} e'_{i,k} + p\frac{h^{r+2}}{(r+2)!}\omega(h) \tag{15}$$

For computing $||T_{kj}||$, we use (6), (8) and (2):

$$|y_i^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)| \leq |y_{i,k} - s_{i,k-1}^{[m]}(x_k)| + |y'_{i,k} - s'^{[m]}_{i,k-1}(x_k)||x - x_k| +$$

$$+L_i \int_{x_k}^{x} \int_{x_k}^{t} \{\sum_{i=1}^{p} |y_i^{[m-j-1]}(t_{j+1}) - s_{i,k}^{[m-j-1]}(t_{j+1})|\} dt_{j+1} dt$$

$$\max_{u \in [x_k, x_{k+1}]} |y_i^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)| \leq e_{i,k} + he'_{i,k} + L_i ||T_{k(j+1)}|| \int_{x_k}^{x} \int_{x_k}^{t} dt_{j+1} dt$$

and the result is:

$$||T_{kj}|| \leq \sum_{i=1}^{p} e_{i,k} + \sum_{i=1}^{p} e'_{i,k} + pL\frac{h^2}{2}||T_{k(j+1)}|| \tag{16}$$

Applying Lemma 3.2 we get from (17):

$$||T_{k1}|| \leq \left(\frac{pL}{2}\right)^{m-1} h^{2m-2} ||T_{km}|| + \sum_{i=1}^{p} (e_{i,k} + e'_{i,k}) \sum_{j=0}^{m-2} \left(\frac{pL}{2}\right)^j \tag{17}$$

and, using (16), it can be shown that:

$$\begin{aligned} ||T_{k1}|| \quad &\leq \quad \{\sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^j + 2\left(\frac{pL}{2}\right)^m \sum_{j=0}^{r} \frac{1}{(j+2)!}\} \sum_{i=1}^{p} e_{i,k} + \\ &+ \quad \sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^j \sum_{i=1}^{p} e'_{i,k} + p^m \left(\frac{L}{2}\right)^{m-1} \frac{1}{(r+2)!} \omega(h) h^{2m+r} \\ &\leq a \quad \sum_{i=1}^{p} e_{i,k} + b \sum_{i=1}^{p} e'_{i,k} + ch^{2m+r} \omega(h) \end{aligned}$$

for

$$a = \sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^j + 2\left(\frac{pL}{2}\right)^m \sum_{j=0}^{r} \frac{1}{(i+2)!}$$

$$b = \sum_{j=0}^{m-1} \left(\frac{pL}{2}\right)^j, \; c = \frac{p^m}{(r+2)!} \left(\frac{L}{2}\right)^{m-1}$$

□

**Lemma 3.4.** *For $e_i$, $e'_i$ defined in* (9), *there exist constants* $\{d_{i1}\}$, $\{d_{i2}\}$, $\{d_{i3}\}$, $\{d_{i4}\}$, $\{d_{i5}\}$, $\{c_{i1}\}$, $\{c_{i2}\}$, $i = \overline{1,p}$ *independent of h such that the following inequalities hold:*

$$\begin{aligned} e_i(x) &\leq (1 + d_{i1}h)e_{i,k} + hd_{i1}\sum_{\substack{j=1\\ j\neq i}}^{p} e_{j,k} + hd_{i2}e'_{i,k} + \\ &+ hd_{i3}\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} + c_{i1}h^{2m+r+2}\omega(h) \end{aligned} \tag{18}$$

$$\begin{aligned} e'_i(x) &\leq hd_{i4}e_{i,k} + hd_{i4}\sum_{\substack{j=1\\ j\neq i}}^{p} e_{j,k} + (1 + hd_{i5})e'_{i,k} + \\ &+ hd_{i5}\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} + c_{i2}h^{2m+r+1}\omega(h) \end{aligned} \tag{19}$$

*Proof.* Using (8), (6) and (11) we estimate:

$$\begin{aligned} e_i(x) &= \leq |y_{i,k} - s^{[m]}_{i,k-1}(x_k)| + |y'_{i,k} - s'^{[m]}_{i,k-1}(x_k)|(x - x_k) + \\ &+ L_i\int_{x_k}^{x}\int_{x_k}^{t}\{\sum_{j=1}^{p}|y_j^{[m-1]}(t_1) - s_{j,k}^{[m-1]}(t_1)|\}dt_1 dt \leq \\ &\leq e_{i,k} + he'_{i,k} + L_i||T_{k1}||\int_{x_k}^{x}\int_{x_k}^{t} dt_1 dt \leq \\ &\leq (1 + \frac{L_i a}{2})he_{i,k} + \frac{L_i a}{2}h\sum_{\substack{j=1\\ j\neq i}}^{p} e_{j,k} + (1 + \frac{L_i b}{2})he'_{i,k} + \\ &+ \frac{L_i b}{2}h\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} + \frac{L_i c}{2}h^{2m+r+2}\omega(h) \end{aligned}$$

So, for $d_{i1} = \frac{L_i a}{2}$, , $d_{i2} = 1 + d_{i3}$, $d_{i3} = \frac{L_i b}{2}$, $c_{i1} = \frac{L_i c}{2}$ we obtain (19). Similarly we prove for $e'_i$. □

Using the matrix notation:

$$\begin{aligned} E(x) &:= (e_1(x), \dots, e_p(x), e'_1(x), \dots, e'_p(x))^T \\ E_k &:= (e_{1,k}, \dots, e_{p,k}, e'_{1,k}, \dots, e'_{p,k})^T \\ C &:= (c_{11}, c_{21}, \dots, c_{p1}, c_{12}, c_{22}, \dots, c_{p2})^T \end{aligned}$$

from the Lemma 3.4 we can write:

$$E(x) \leq (I + hA)E_k + Ch^{2m+r+1}\omega(h) \tag{20}$$

where I is the unit matrix of order $2p \times 2p$ and

$$A = \begin{pmatrix} d_{11} & \dots & d_{11} & d_{12} & d_{13} & \dots \\ d_{21} & \dots & d_{21} & d_{22} & d_{23} & \dots \\ & & & & \dots & \\ d_{14} & \dots & d_{14} & d_{15} & d_{15} & \dots \\ d_{24} & \dots & d_{24} & d_{25} & d_{25} & \dots \\ & & & & & \end{pmatrix}$$

If for the matrix $M = (m_{ij})$ we defined the norma by:

$$||M|| := \max_i \sum_j |m_{ij}|$$

then on the basis of (21) we can write:

$$||E(x)|| \leq (1 + h||A||)||E_k|| + ||C||h^{2m+r+1}\omega(h)$$

The inequality holds for any $x \in [0, 1]$. Setting $x = x_{k+1}$ it follows:

$$||E_{(k+1)}|| \leq (1 + h||A||)||E_k|| + ||C||h^{2m+r+1}\omega(h).$$

Using Lemma 3.1 and noting that $||E_0|| = 0$, we get:

$$\begin{aligned} ||E(x)|| &\leq (1 + h||A||)^{k+1}||E_0|| + ||C||h^{2m+r+1}\omega(h)\frac{(1 + h||A||)^{k-1} - 1}{1 + h||A|| - 1} \leq \\ &\leq \frac{||C||}{||A||}(e^{||A||} - 1)h^{2m+r}\omega(h) \end{aligned}$$

Now it follows strighforward:

$$e_i^{(j)}(x) \leq B_0\omega(h)h^{2m+r}, \text{for } i = \overline{0,p},\ j = 0,1 \tag{21}$$

where $B_0 := \dfrac{||C||}{||A||}(e^{||A||} - 1)$ is a constant independent of $h$.

We estimate the difference $|y_i^{(q+2)}(x) - s_i^{(q+2)}(x)|$, $q = \overline{0,r}$, $i = \overline{1,p}$

$$\begin{aligned} &|y_i^{(q+2)}(x) - s_i^{(q+2)}(x)| := |\frac{d^{q+2}}{dx^{q+2}}y_i^{[m]}(x) - \frac{d^{q+2}}{dx^{q+2}}s_{i,k}^{[m]}(x)| = \\ = \ &|f_i^{(q)}(t_1, y_1^{[m-1]}(t_1), ..., y_p^{[m-1]}(t_1)) - -f_i^{(q)}(t_1, s_{1,k}^{[m-1]}(t_1), ..., s_{p,k}^{[m-1]}(t_1))| \leq \\ \leq \ &L_i||T_{k1}|| \leq L_i\{a\sum_{j=1}^{p} e_{j,k} + b\sum_{j=1}^{p} e'_{j,k} + ch^{2m+r}\omega(h)\} \leq B_{i1}h^{2m+r}\omega(h) \end{aligned} \tag{22}$$

for $B_{i1} = L_i[p(a+b)B_0 + c]$ a constant independent of h.

Thus, we proved the following result:

**Theorem 3.1.** *Let $(y_1, \ldots, y_p)$ be the exact solution of the problem* (1) *and $(s_{1,\Delta} \ldots, s_{p,\Delta})$ be the approximate solution for the problem* (1). *If $f \in C^r([0,1] \times \mathbb{R}^p)$, then the following estimations hold for $x \in [0,1]$:*

$$|y_i^{(q)}(x) - s_{i,\Delta}^{(q)}(x)| \leq B_{i2}\omega(h)h^{2m+r} = O(h^{2m+r+\alpha})$$

*where $q = \overline{0, r+2}$ and $B_{i2}$, $i = \overline{1,p}$ are constants independent of h.*

## 4. Stability of the method

For new initial conditions,
$y_i(x_0) = y_{i,0}^*$, $y_0'(x_0) = y_{i,0}'^*$ , we defined the approximate solution by:

$$\begin{aligned} w_{i,\Delta}(x) &:= w_{i,k-1}^{[m]}(x_k) + w_{i,k-1}'^{[m]}(x_k)(x - x_k) + \\ &+ \int_{x_k}^{x}\int_{x_k}^{t} f_i[t_1, w_{1,k}^{[m-1]}(t_1), \ldots, w_{p,k}^{[m-1]}(t_1)]dt_1 dt \end{aligned} \tag{23}$$

where $w_{i,-1}^{[m]}(x_0) := y_{i,0}^*$ and $w_{i,-1}'^{[m]}(x_0) = y_{i,0}'^*$, $i = \overline{1,p}$, $k = \overline{0, n-1}$.

In (24) we use $m$ iterations, for $x_k \leq t_m \leq \ldots \leq t_1 \leq t \leq x \leq x_{k+1}$, like in (5) and (6).

We use the following notations, for $i = \overline{1,p}$, $k = \overline{0,n-1}$:

$$\begin{aligned} e_i^*(x) &:= |w_i(x) - s_i(x)|,\ e_i'^*(x) := |w_i'(x) - s_i'(x)| \\ e_{i,k}^* &:= |w_i(x_k) - s_i(x_k)|,\ e_{i,k}'^* := |w_i'(x_k) - s_i'(x_k)| \\ M_{i,k}^{*(j)} &:= f_i^{(j)}(x_k, w_{1,k-1}^{[m]}, \ldots, w_{p,k-1}^{[m]}(x_k)),\ j = \overline{1,m} \end{aligned}$$

and we define the operator:

$$T_{kj}^*(u) := \sum_{i=1}^{p} |w_{i,k}^{[m-j]}(u) - s_{i,k}^{[m-j]}(u)|, u \in [x_k, x_{k+1}],\ k = \overline{0, n-1},\ j = \overline{1,m}$$

with the norm:

$$||T_{kj}^*|| = \max_{u \in [x_k, x_{k+1}]} \{T_{kj}^*(u)\}$$

**Lemma 4.1.** *For any $u \in [x_k, x_{k+1}]$, $k = \overline{0, n-1}$, the estimations*

$$||T^*_{km}|| \leq \{1 + pL\sum_{j=0}^{r} \frac{1}{(j+2)!}\}\sum_{i=1}^{p} e^*_{i,k} + \sum_{i=1}^{p} e^{*'}_{i,k} \tag{24}$$

$$||T^*_{k1}|| \leq a\sum_{i=1}^{p} e^*_{i,k} + b\sum_{i=1}^{p} e^{*'}_{i,k} \tag{25}$$

*hold, where*

$$a = \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^j + 2\left(\frac{pL}{2}\right)^m \sum_{j=0}^{r} \frac{1}{(i+2)!}$$

$$b = \sum_{j=0}^{m-1}\left(\frac{pL}{2}\right)^j$$

The proof is similarly with the proof of Lemma 3.3.

**Lemma 4.2.** *For $e^*_i$, $e^{*'}_i$, $i = \overline{1,p}$ above defined, the following inequalities hold:*

$$e^*_i(x) \leq (1 + d_{i1}h)e^*_{i,k} + hd_{i1}\sum_{\substack{j=1\\ j\neq i}}^{p} e^*_{j,k} + hd_{i2}e^{*'}_{i,k} + hd_{i3}\sum_{\substack{j=1\\ j\neq i}}^{p} e'_{j,k} \tag{26}$$

$$e^{*'}_i(x) \leq hd_{i4}e^*_{i,k} + hd_{i4}\sum_{\substack{j=1\\ j\neq i}}^{p} e^*_{j,k} + (1 + hd_{i5})e^{*'}_{i,k} + hd_{i5}\sum_{\substack{j=1\\ j\neq i}}^{p} e^{*'}_{j,k} \tag{27}$$

*where the constants are defined in Lemma* 3.4.

The proof is similarly with the proof of Lemma 3.4.

Using the matrix notation

$$E^*(x) := (e^*_1(x), \ldots, e^*_p(x), e^{*'}_1(x), \ldots, e^{*'}_p(x))^T$$

$$E^*_k := (e^*_{1,k}, \ldots, e^*_{p,k}, e^{*'}_{1,k}, \ldots, e^{*'}_{p,k})^T$$

then, the estimations $(27-28)$ can be write in the following form:

$$E^*(x) \leq (I + hA)E^*_k$$

where I and A are defined matrix. Applying Lemma 3.1, we get:

$$||E^*(x)|| \leq (1 + h||A||)||E^*_k|| \leq \left(1 + \frac{||A||}{n}\right)^n ||E^*_0|| \leq$$

$$\leq e^{||A||}||E^*_0|| \leq B^*||E^*_0||$$

where $B^* = e^{||A||}$ is a constant independent of h.

Hence:

$$e_i^*(x) \leq B^*\|E_0^*\| \tag{28}$$
$$e_i'^*(x) \leq B^*\|E_0^*\|$$

for $i = \overline{1,p}$.

For $|w_i^{(q+2)}(x) - s_i^{(q+2)}(x)|$, $q = \overline{0,r}$ we obtain, like in (23):

$$|w_i^{(q+2)}(x) - s_i^{(q+2)}(x)| \leq B_{i1}^*\|E_0^*\|$$

where $B_{i1}^* = pL_i(a+b)B^*$ is a constant independent of $h$.

Thus we proved the following result:

**Theorem 4.1.** *Let* $(s_1,\dots,s_p)$ *be the approximate solution of the problem* (1) *with the initial conditions* $y_i(x_0) = y_{i,0}$, $y_i'(x_0) = y_{i,0}'$, $i = \overline{1,p}$ *and let* $(w_1,\dots,w_p)$ *be the approximate spline solution for the same system, but with the initial conditions:* $y_i(x_0) = y_{i,0}^*$, $y_i'(x_0) = y_{i,0}^{*'}$, $i = \overline{1,p}$. *Then the inequalities:*

$$|w_i^{(q)}(x) - s_i^{(q)}(x)| \leq B_{i2}\|E_0^*\|$$

*hold for all* $x \in [0,1]$, $q = \overline{0,r+2}$, *where* $B_{i2}$, $i = \overline{1,p}$, *are constants independent of* $h$ *and* $\|E_0^*\| = \max_i\{|y_{i,0} - y_{i,0}^*|, |y_{i,0}' - y_{i,0}'^*|\}$.

## 5. Numerical example

. Consider the following system of differential equations, for $p = 2$.

$$\begin{cases} y'' = y + z - e^{-x}, & y(0) = 1, \quad y'(0) = 0 \\ z'' = y + z - e^{x}, & z(0) = 1, \quad z'(0) = 0 \end{cases}$$

The method is tested using this example in the interval $[0,1]$ with step 0.1 , where $r = 0$, $m = 1$. The result are tabulated at $x = 1$.

The analytical solution is:

$$y = e^x - x$$
$$z = e^{-x} + x$$

To test the stability of the method, we solve the above example with the new initial conditions:

$$y(0) = 1.000001, \quad y'(0) = 0.000001$$
$$z(0) = 1.000001, \quad z'(0) = 0.000001$$

The results are:

| | The convergence | | The stability |
|---|---|---|---|
| $e_1$ | $1.48222E-05$ | $e_1^*$ | $3.28131E-06$ |
| $e_1'$ | $1.15977E-04$ | $e_1^{*\prime}$ | $4.48373E-06$ |
| $e_1''$ | $2.41293E-03$ | $e_1^{*\prime\prime}$ | $6.53131E-06$ |
| $e_2$ | $1.48222E-05$ | $e_2^*$ | $3.28133E-06$ |
| $e_2'$ | $1.15977E-04$ | $e_2^{*\prime}$ | $4.48370E-06$ |
| $e_2''$ | $2.41293E-03$ | $e_2^{*\prime\prime}$ | $6.53132E-06$ |

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