INTEGRATION OF VECTOR FUNCTIONS WITH RESPECT TO VECTOR MEASURES

ROBERT PALLU DE LA BARRIÈRE

Abstract. The algebraic theory of integration with respect to a semi-variation is outlined. It is applied to the integration of vector-valued functions with respect to a vector-valued measure. Different settings are considered (bilinear integration, Dobrakov's integral, tensor integration). Emphasis is put on convergence theorems.

The purpose of this paper is to study the integration of a vector function f with respect to a vector measure m. This problem may be settled in different settings. In his pioneering work [1], Bartle supposes f has values in a Banach space F, m has values in a Banach space E. Furthermore a bilinear map from $F \otimes E$ into a Banach space G is given and the integral of f with respect to m has values in G. A another setting is that of Dobrakov [DO] : m has values in the space $L(X,Y)$ of continuous linear operators from X into Y, f is X-valued and the integral is Y-valued. From the set-theoretical point of view, Dobrakov'setting may be considered as a particular case of Bartle's setting. Conversely it is possible to transform a problem given in the Bartle's setting into a problem in the Dobrakov's setting. But the additionnal assumptions concerning mainly the additivity of m makes the set-theoretical manipulations dangerous. For example Dobrakov supposes only the σ-additivity of $m(.)x$ for every $x \in X$ and not the σ-additivity in the strong sense.

Many other contributions appeared in the literature since the pioneering works of Bartle and Dobrakov. For example, Guessous [GU] supposes that the integral has its values in the completion of $F \otimes E$ with respect to a tensor norm (ε or π). This setting will be called the tensor integration. It is studied more recently by Jefferies and Okada [J.O] and [JE] Chapter 4.

1991 *Mathematics Subject Classification.* 46G10, 28B05.

Key words and phrases. Vector measures, semi-variation, Bochner integral.

As many proofs are similar in different settings, the authors often fail to give complete proofs, referring abruptly to previous papers. This makes the contributions less and less legible, especially if the reader wish to construct self-contained proofs. The first motivation of this paper is to avoid this discomfort.

In all the previously quoted works, a generalisation of the ordinary semi-variation of a vector measure is introduced. This semi-variation is defined as a sub-additive set-function. In [PB] under the influence of [TH] we modify the definition so as the semi-variation is defined as a semi-norm on the vector space of simple functions. Or course the restriction of our semi-variation to sets (*i.e.* characteristic functions) is the customary semi-variation. But as two different vector-semi-variations may be equal on sets, our definition is more powerfull. In particular it is possible to define the integrability of a scalar function with respect to a vector-semi-variation without taking into account the objects and setting it comes from. This is very important for the integration of vector functions with respect to vector measure because it is possible to define the integrability of a scalar function with respect to the semi-variation introduced to this special context (this semi-variation is called below "contextual semi-variation"). This enable us to define the Bochner-integrability in a convenient frame and to make easier the connection with the classical Bochner-integrability with respect to a scalar measure (for the Dobrakov integral see [DO2] and [PAN]).

Section 1 is designed to recall definitions and fundamental results. Section 2 is devoted to the algebraic theory of integrability of scalar functions with respect to a semi-variation. This theory is inspired by the work of Wilhelm ([W1,W2]) and is exposed in [PB] with more details.

Section 3 is devoted to the particular case where the semi-variation is defined on a space $(\mathcal{T})$ of $\mathcal{T}$-simple functions, $\mathcal{T}$ being a σ-algebra.

Section 4 is devoted to the integrability of vector functions.

Section 5 is devoted to the definition of the "contextual" semi-variation, that means the semi-variation adapted to the spaces E, F, G where the measure, the function to be integrated and the integral take respectively their values.

Section 6 is devoted to the customary semi-variation of a vector measure that means the semi-variation adapted to the integration of scalar functions.

Section 7 is devoted to the general setting. The bilinear form from $F \times E$ into G is denoted by $y, x \mapsto y \otimes x$ with a view of compatibility with the tensor setting. To cover simultanuously the setting of Bartle and the setting of Dobrakov, we suppose that $A \mapsto y \odot m(A)$ is a G-valued measure for every $y \in F$.

Section 8 is devoted to the case G is the completion of $F \otimes E$ with respect to the tensor norm ε or π. Some special results are derived using the Orlicz-Pettis's theorem. The covering of the classical case of the integration of vector functions with respect to a scalar measure is warranted both for Pettis and Bochner integrability.

1. Preliminaries

We first recall some definitions. A measurable space $(T, \mathcal{T})$ is a couple formed by an arbitrary set T and a σ-algebra of subsets of T. A function f on T with values in a space F is said to be simple if it is of the form $f = \sum_i \mathbf{1}_{A_i} \xi_i$ where $\{A_i\}$ is a finite $\mathcal{T}$-partition of T and $\xi_i \in F$. The vector space of F-valued simple functions will be denoted by ${}_F(\mathcal{T})$ (or by $(\mathcal{T})$ if $F = \mathbb{R}$).

We reserve the term "measure" to additive set functions which satisfie some σ-additivity property. If E is a Banach space by a E-valued measure defined on $(T, \mathcal{T})$ we mean a additive set function m on $\mathcal{T}$ such that $m(T) = \sum_n m(T_n)$ for every countable $\mathcal{T}$-partition $\{T_n\}$ of T. The convergence of the series is supposed to be valid in the norm topology. Sometimes to emphasize this property we will write "(strong) measure" instead of measure.

There are two ways to weaken the property of σ-additivity. The first one is to suppose σ-additivity for a weaker topology that the norm topology. For exemple if E is the space $L(Y, X)$ of all continuous operators from a Banach space Y into a Banach space X we may consider the strong operator topology (defined by the semi-norms $A \mapsto \|Ay\|$ for y running over Y) or the weak operator topology (defined by the semi-norms $A \mapsto |\langle x', Ay \rangle|$ for x' running over X' and y running over Y). Following the Dunford-Pettis theorem this two topologies are identical. This situation will be encountered in section 4.

An another way to weaken the property of σ-additivity is to consider two spaces E and F in duality and a E-valued set function m defined on $\mathcal{T}$ satisfying the property that $\langle m(.), y \rangle$ is a measure for every Y in F. Such a set function will be called a "weak measure" (for the duality (E, F)). We may note that this σ-additivity property may be considered as the σ-additivity for the topology $\sigma(E, F)$. In fact the two ways of weaken the property of σ-additivity differ from the expository point of view but are able to handle the same concrete situations.

An important particular case of vector measure is obtained by taking $T = \mathbb{N}$ and $\mathcal{T} = \mathcal{P}(\mathbb{N})$. With any set function m is associated the sequence $\{m(n)\}$ such that $m(n) = m(\{n\})$. The discussion of various forms of the σ-additivity property is indeed the discussion of various notions of summability of a sequence. If E is a Banach space a E-valued sequence $\{x_i\}$ is said to be summable if there exists $S \in E$ such that for every $\varepsilon > 0$ there is a finite subset of $\mathbb{N}$ such that for every finite J containing I we have $\|S - \sum_{i \in J} x_i\| \leq \varepsilon$. The sequence $\{x_i\}$ is summable iff it satisfies the so called Cauchy criterion: for every $\varepsilon > 0$ there exists a finite subset I of $\mathbb{N}$ such that $\|\sum_{i \in J} x_i\| \leq \varepsilon$ provided J is a finite subset of $\mathbb{N} \setminus I$. The Cauchy criterion can be generalized to an arbitrary locally convex vector space under the following form: for every continuous semi-norm p on E and every $\varepsilon > 0$ there exists a finite subset I of $\mathbb{N}$ such that $p\left(\sum_{i \in J} x_i\right) \leq \varepsilon$ provided J is a finite subset of $\mathbb{N} \setminus I$. This criterion may be adopted as the definition of summability of the sequence $\{x_i\}$ but doesn't imply that the series $\sum_i x_i$ converges in E but only in the completion of E (more precisely in the quasi-completion of E). For instance a sequence $\{x_i\}$ in a Banach space E is weakly summable (that means summable for the topology $\sigma(E, E')$) iff the sequence $\{\langle x_i, y\rangle\}$ is summable for every $y \in E'$. Its "sum" (generally called "weak sum") belongs to the $\sigma(E, E')$-completion of E *i.e.* E''.

To end this section let us recall that a control measure of a vector measure m is a measure μ such that $\forall A \in \mathcal{T} \quad \mu(A) = 0 \iff m(A) = 0$. Any vector measure has a control measure ([PB] theorem V.46).

2. Integrability with respect to a semi-variation

In this section we outline the algebric construction of the space of integrable functions with respect to a semi-variation as developped in [P.B.]. We refer to this treatise for detailed proofs.

Definition 2.1. Let T an arbitrary set and $\mathcal{L}$ a vector sub-lattice of $\mathbb{R}^T$. A semi-norm v on $\mathcal{L}$ is said to be a Riesz-semi-norm iff the two following conditions are satisfied:

$$0 \leq f_1 \leq f_2 \quad \Longrightarrow \quad v(f_1) \leq v(f_2)$$

$$v(f) = v(|f|)$$

A Riesz-semi-norm is said to be a semi-variation if the following property is satisfied (σ-subadditivity):

$$f, f_n \in \mathcal{L}_+ and f \leq \sum_{n=1}^{\infty} f_n \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

or equivalently:

$$f, f_n \in \mathcal{L} and |f| \leq \sum_{n=1}^{\infty} |f_n| \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

The condition $f \leq \sum_{n=1}^{\infty} f_n$ (pour $f, f_n \in \mathcal{L}_+$) means that for every $t \in T$, one have $f(t) \leq \sum_{n=1}^{\infty} f_n(t)$, this last inequality being satisfied in particular if $\sum_{n=1}^{\infty} f_n(t) = +\infty$. Our aim is to extend a semi-variation v on $\mathcal{L}$ to a space $\mathcal{L}^1(v)$ such that the latter space is complete and $\mathcal{L}$ is dense in $\mathcal{L}^1(v)$. The first step is to define the space $\mathcal{L}^*(v)$ de $\mathcal{L}$ of "not to large functions". Let us put:

$$\mathcal{L}^*(v) = \left\{ f \in I\!R^T \;\middle|\; \exists f_n \in \mathcal{L} : |f| \leq \sum_{n=1}^{\infty} |f_n| \, and \sum_{n=1}^{\infty} v(f_n) < +\infty \right\}$$

It is easy to prove that $\mathcal{L}^*(v)$ is a vector sublattice of $I\!R^T$ and contains $\mathcal{L}$. On this space, we define:

$$v^*(f) = \inf \left\{ \sum_{n=1}^{\infty} v(f_n) \;\middle|\; f_n \in \mathcal{L}, |f| \leq \sum_{n=1}^{\infty} |f_n| \right\}$$

A routine checking proves that v^* is a semi-variation on $\mathcal{L}^*(v)$ which extends v.

The second step is to define $\mathcal{L}^1(v)$ as the closure of $\mathcal{L}$ in $\mathcal{L}^*(v)$ (equipped with the semi-norm v^*). The elements of $\mathcal{L}^1(v)$ are called integrable with respect to v or v-integrable.

From the construction, it results the following theorem:

Theorem 2.2. *$\mathcal{L}^1(v)$ is a vector sub-lattice et v^* is a semi-variation on $\mathcal{L}^1(v)$.*

The completeness of $\mathcal{L}^1(v)$ is given by the following theorem whose proof is very simple.

Theorem 2.3. *Let f_n be a sequence $\in \mathcal{L}^1(v)$ such that $\sum_{n=1}^{\infty} v^*(f_n) < \infty$. Put:*

$$f(t) = \begin{cases} \sum_{n=1}^{\infty} f_n(t), \text{ if this series is absolutely convergent} \\ \text{otherwise an arbitrary value} \end{cases}$$

Then $f \in \mathcal{L}^1(v)$ and $\lim_N v^*(f - \sum_{n=1}^{N} f_n) = 0$. In other words: $\sum_{n=1}^{N} f_n$ converges to f in $\mathcal{L}^1(v)$.

Proof. We have $|f| \leq \sum_{n=1}^{\infty} |f_n|$ and therefore (by virtue of the σ-subadditivity of v^*):

$$v^*(f) \leq \sum_{n=1}^{\infty} v^*(f_n)$$

Moreover $\left|f - \sum_{n=1}^{N} f_n\right| \leq \sum_{n>N} |f_n|$, and therefore:

$$v^*(f - \sum_{n=1}^{\infty} f_n) \leq \sum_{n>N} v^*(f_n)$$

As the limit of the latter expression is 0, the proof is completed. □

Let N be the set of all t such that $\sum_{n=1}^{\infty} f_n(t)$ is not absolutely convergent. Il we modify f on N, then we get a function f' such that $v^*(f - f') = 0$. In particular $\mathbf{1}_N \in \mathcal{L}^1(v)$ et $v^*(\mathbf{1}_N) = 0$. This incite to introduce the following definitions:

Definition 2.4. Let v be a semi-variation on the vector lattice $\mathcal{L} \subset \mathbb{R}^T$. A function f is said to be v-negligible iff $v^*(f) = 0$. A set $A \subset T$ is said to be v-negligible iff $v^*(\mathbf{1}_A) = 0$. A property concerning elements of T is said to be true almost everywhere with respect to v (in short v-a.e.) if the set of $t \in T$ where it is false is v-negligible.

If $\mathcal{P}$ is a property concerning functions defined on a subset of T, then a function f defined on T is said to have essentially the property $\mathcal{P}$ if there exists a negligible set A such that the restriction of f to the complement of A has the property $\mathcal{P}$.

The most usual properties of negligible functions and sets are given in the following theorem.

Theorem 2.5. *(1) If f est v-negligible and if $|g| \leq |f|$, then g is v-negligible. In particular if $A \subset T$ is v-negligible and if $B \subset A$, then B is v-negligible.*

(2) Any countable union of negligible sets is negligible.

(3) A function f is v-negligible iff $\{t \in T \mid f(t) \neq 0\}$ is v-negligible.

Theorem 2.3. appears now as a variant of the standard Lebesgue's theorem on series.

Let $\{f_n\}$ be a sequence in $\mathcal{L}^1(v)$ converging to an element f. Then there exists a subsequence $\{f_{\phi(k)}\}$ such that $\sum_k v^*(f_{\phi(k)}) < \infty$. By Theorem 2.3., $\{f_{\phi(k)}\}$ converges v-a.e. to f. Hence every converging sequence in $\mathcal{L}^1(v)$ has a subsequence converging v-a.e.. On the other hand remark that if $\{f_n\}$ is a convergent sequence in $\mathcal{L}^1(v)$ which converges pointwise to a function f, then $\{f_n\}$ converges to f in $\mathcal{L}^1(v)$.

Notation. The quotient of $\mathcal{L}^1(v)$ with respect to v^* is denoted by $L^1(v)$. It is a Banach space.

We shall now define an important class of semi-variations.

Definition 2.6. A semi-variation v on a lattice $\mathcal{L}$ is said to be exhaustive [1] iff it satisfies one of the equivalent following properties:

(1) every increasing and majorized sequence $\{f_n\}$ is a Cauchy sequence with respect to v.

(2)

$$\left[f_n \in \mathcal{L}_+ \ , \ f \in \mathcal{L}_+ \ , \ \sum_{n=1}^{\infty} f_n \le f \right] \Longrightarrow \lim_n v(f_n) = 0$$

The equivalence between the two properties is easy to prove.

The natural norm on $\mathbf{l}^1$ or $\mathbf{c}_0$ is exhaustive. On the other hand the natural norm on $\mathbf{l}^\infty$ is not exhaustive.

A somewhat technical proof leads to the following result:

Theorem 2.7. *If v is an exhaustive semi-variation, then v^* is exhaustive on $\mathcal{L}^1(v)$.*

The main result concerning exhaustive semi-variations is the theorem of dominated convergence. First remark that if v is a exhaustive semi-variation on $\mathcal{L}$ and $\{f_n\}$ is a positive increasing majorized sequence in $\mathcal{L}^1(v)$, then putting $f(t) = \lim_n f_n(t)$, we have $f \in \mathcal{L}^1(v)$ and f is the limit of f_n in $\mathcal{L}^1(v)$. In fact $\{f_n\}$ is a Cauchy sequence by virtue of the exhaustivity of v . As an easy consequence, if $\{f_n\}$ is a decreasing sequence in $\mathcal{L}^1(v)$ converging pointwise to 0, then this sequence converges to 0 in $\mathcal{L}^1(v)$.

Theorem 2.8. *[Theorem of dominated convergence] Let v be an exhaustive semi-variation on $\mathcal{L}$. Let $f_n, h \in \mathcal{L}^1(v)$ such that $|f_n| \le h$. If f_n converges pointwise to a function f, then f_n converges to f in $\mathcal{L}^1(v)$.*

Proof. Put $\quad g_n = \sup\{|f_p - f_q| \mid p, q \ge n\}$.

We have: $\quad g_n = \lim_m g_{n,m}$ with: $\qquad g_{n,m} = \sup\{|f_p - f_q| \mid n \le p, q \le m\}$

[1] The term "exhaustive" has been suggested to me by [KA]. I am far from affirming it is the best one. Wilhelm uses the term "saturable". Swartz uses the term "strongly bounded". Bartle ([?]) speeks of the *-property. Lewis [LE] uses a similar concept under the name "variationnel semiregularity". A Banach lattice is said to be "order bounded" if condition (2) is satisfied. Of course all this terms refers to Sdifferent settings. One can only say that the are used to define similar properties.

On the other hand $0 \leq g_{m,n} \leq 2h$, $g_{n,m} \in \mathcal{L}^1(v)$ and the sequence $m \to g_{n,m}$ is increasing. Therefore (see the preliminary remarks), $g_n \in \mathcal{L}^1(v)$ for all n. Furthermore the sequence $\{g_n\}$ is decreasing and converge pointwise to 0. Now $|f_n - f_k| \leq g_n$ for all $k \geq n$ and therefore $|f_n - f| \leq g_n$. Hence $f_n - f \in \mathcal{L}^*(v)$, $f \in \mathcal{L}^*(v)$ and $v^*(f_n - f))$ converges to 0. □

It is common in Convex Analysis to extend convex functions defined on a convex subset into a function defined on the whole space and taking the value $+\infty$ outside the domain of definition of the original function. Adopting the notation used in Convex Analysis, for every $]-\infty, +\infty]$-valued convex function f, we shall denote by dom(f) the "effective domain" of f, i.e. $\{t \mid f(t) \neq +\infty\}$.

Definition 2.9. Let T be an arbitrary set and $\mathcal{L}$ a vector sub-lattice of $\mathbb{R}^T$. One call extended semi-variation on $\mathcal{L}$ any sub-linear symmetric function v defined on $\mathcal{L}$ with values in $[0, +\infty]$ such that:

$$0 \leq f_1 \leq f_2 \quad \Longrightarrow \quad v(f_1) \leq v(f_2)$$

$$v(f) = v(|f|)$$

$$f, f_n \in \mathcal{L}_+ and f \leq \sum_{n=1}^{\infty} f_n \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

or equivalently

$$f, f_n \in \mathcal{L} and |(|f) \leq \sum_{n=1}^{\infty} |f_n| \quad \Longrightarrow \quad v(f) \leq \sum_{n=1}^{\infty} v(f_n)$$

It is easy to verify that dom(v) is a vector sub-lattice of $\mathcal{L}$ and that the restriction of v to dom(v) is a semi-variation as defined in Definitions 2.1. An extended semi-variation is said to be exhaustive if its restriction to dom(f) is exhaustive.

Theorem 2.10. *Any pointwise supremum v of a family $\{v_i\}$ of extended semi-variations is an extended semi-variation.*

Proof. The two first properties in Definition 2.9 are plain. Now suppose:

$$f, f_n \in \mathcal{L}_+ and f \leq \sum_{n=1}^{\infty} f_n$$

For every i, we have $v_i(f) \leq \sum_{n=1}^{\infty} v_i(f_n)$. But for every f, we have $v(f) = \sup v_i(f)$. Therefore $v_i(f) \leq \sum_n v(f_n)$. Taking the pointwise supremum in the left hand member gives: $v(f) \leq \sum_n v(f_n)$. □

It's worthwhile to note that the pointwise supremum of a family of exhaustive extended semi-variations may fail to be exhaustive. For instance let T be a compact set. For every $t \in T$ the function $f \mapsto |f(t)|$ is an exhaustive semi-variation on the space $C(T)$, but the semi-variation $f \mapsto \sup_{t \in T} |f(t)|$ is not exhaustive unless T is finite.

3. Semi-variations on measurable spaces

Let $(T, \mathcal{T})$ be a measurable space. $\mathcal{E}(\mathcal{T})$ or $\mathcal{E}$ denotes the space of $\mathcal{T}$-simple functions. We shall apply the results of Section 2 to the case $\mathcal{L} = \mathcal{E}$.

Theorem 3.1. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}$. For every function $f \in \mathcal{L}^1(v)$, there exists a function $f' \in \mathcal{L}^1(v)$, $\mathcal{T}$-measurable such that $v^*(f - f') = 0$*

Proof. Let us témporarily denote by $\mathcal{L}^{*\mathcal{T}}$ and $\mathcal{L}^{1,\mathcal{T}}$ respectively the subspaces of $\mathcal{T}$-measurable elements of $\mathcal{L}^*(v)$ and $\mathcal{L}^1(v)$. Let us look at Theoreme 2.3. Suppose $f_n \in \mathcal{L}^{1,\mathcal{T}}$. The set N where the series $\sum_{n=0}^{\infty} f_n(t)$ doesn't converge absolutely belongs to $\mathcal{T}$. If we put $f(t) = 0$ for $t \in N$, then $f \in \mathcal{L}^{1,\mathcal{T}}$. Consequently $\mathcal{L}^{1,\mathcal{T}}$ is complete. Now let $f \in \mathcal{L}^1(v)$. There exists a sequence $f_n \in \mathcal{E}$ such that $\lim_n v^*(f - f_n) = 0$. As $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^{1,\mathcal{T}}$, it converges to a element $f' \in \mathcal{L}^{1,\mathcal{T}}$ and we have $v^*(f - f') = 0$. □

Note that the quotient of $\mathcal{L}^{1,\mathcal{T}}$ by v^* is $L^1(v)$.

Definition 3.2. A measurable space $(T, \mathcal{T})$ is said to be complete with respect to a semi-variation v on $\mathcal{E}(\mathcal{T})$ if

$$A, B \in \mathcal{T} \ , \ B \subset A \ , \ v(\mathbf{1}_A) = 0 \quad \Longrightarrow \quad B \in \mathcal{T}$$

Lemma 3.3. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}$. Let A be v-negligible. Then there exists $B \in \mathcal{T}$ such that $A \subset B$ and $v(\mathbf{1}_B) = 0$*

Proof. Suppose $v^*(\mathbf{1}_A) = 0$. Let $\varepsilon > 0$. According to the definition of v^*, there exists a sequence $\{f_n\}$ of elements of $\mathcal{E}_+(\mathcal{T})$ such that $\mathbf{1}_A \leq \sum_{n \in \mathbb{N}} f_n$ and $\sum_{n \in \mathbb{N}} v(f_n) \leq \varepsilon$. Put

$B = \{t \in T \mid \sum_{n \in \mathbb{N}} f_n(t) \geq 1\}$. Then $B \in \mathcal{T}$ and $\mathbf{1}_B \leq \sum_{n \in \mathbb{N}} f_n$. Therefore according to the σ-subadditivity of v: $v(\mathbf{1}_B) \leq \sum_{n \in \mathbb{N}} v(f_n)$, and $v(\mathbf{1}_B) \leq \varepsilon$.

Taking $\varepsilon = 1/n$ ($n \in \mathbb{N}$), we get a sequence of elements $B_n \in \mathcal{T}$ such that $A \subset B_n$ and $v(\mathbf{1}_B) \leq 1/n$. Taking $B = \bigcap_{n \in \mathbb{N}} B_n$, we get $A \subset B$ and $v(\mathbf{1}_B) = 0$. □

Theorem 3.4. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}$. Define $\mathcal{T}'$ as follows:*

$$A \in \mathcal{T}' \iff \exists B \in \mathcal{T} \text{ such that } v^*(A \triangledown B) = 0$$

Then $\mathcal{T}'$ is a σ-algebra and $(T, \mathcal{T}')$ is complete with respect to v.

Proof. Let $A \in \mathcal{T}'$ and $B \in \mathcal{T}$ such that $v^*(A \triangledown B) = 0$. We have $A \triangledown B = A^c \triangledown B^c$ and therefore $A^c \in \mathcal{T}'$.

Now let $\{A_n\}$ be a sequence of elements of $\mathcal{T}'$ and, for every n, $B_n \in \mathcal{T}$ such that $v^*(A_n \triangledown B_n) = 0$. We have $\bigcup_n A_n \triangledown \bigcup_n B_n \subset \bigcup_n A_n \triangledown B_n$. That implies $\bigcup_n A_n \in \mathcal{T}'$. Hence $\mathcal{T}'$ is a σ-algebra. Plainly this σ-algebra is complete with respect to v. □

Proposition 3.5. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}(\mathcal{T})$. Then we have $\mathcal{T}' = \{A \subset T \mid \mathbf{1}_A \in \mathcal{L}^1(v)\}$*

Proof. If $A \in \mathcal{T}'$ there exists $B \in \mathcal{T}$ such that $v^*(\mathbf{1}_A - \mathbf{1}_B) = 0$. Therefore $\mathbf{1}_A \in \mathcal{L}^1(v)$.

Conversely suppose $\mathbf{1}_A \in \mathcal{L}^1(v)$. Then there exist $N \in \mathcal{T}$ and a $\mathcal{T}$-measurable function g such that $\mathbf{1}_A = g$ on $T \setminus N$. Supposing $g = 0$ on N, we have $g = \mathbf{1}_B$ and $v^*(A \triangledown B) = 0$ *i.e.* $A \in \mathcal{T}'$. □

Definition 3.6. Let $(T, \mathcal{T})$ be a measurable space and v be a semi-variation on (T). A real function f is said to be v-measurable if it is the limit v-a.e. of a sequence of $\mathcal{T}$-simple functions.

Theorem 3.7. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}(\mathcal{T})$. Then f is v-measurable iff f is measurable with respect to the σ-algebras $\mathcal{T}'$ (theorem 3.4.) and* $\mathbf{Bor}(\mathbb{R})$ *(in short $\mathcal{T}'$-measurable).*

Proof. Plainly every $\mathcal{T}'$-simple function coincides v-a.e. with a $\mathcal{T}$-simple function. Suppose f is $\mathcal{T}'$-measurable and therefore the limit v-a.e. of $\mathcal{T}'$-simple functions. Then f is v-a.e. the limit of $\mathcal{T}$-simple functions.

Conversely suppose f is v-a.e. the limit of $\mathcal{T}$-simple functions f_n. Let $N \in \mathcal{T}$ such that $v^*(\mathbf{1}_N) = 0$ and $\mathbf{1}_{T\setminus N} f_n$ converges to $\mathbf{1}_{T\setminus N} f$ on T. As $\mathbf{1}_{T\setminus N} f_n$ is $\mathcal{T}$-measurable, $\mathbf{1}_{T\setminus N} f$ is $\mathcal{T}$-measurable. For every $A \in \mathbf{Bor}(\mathbb{N})$ we have

$$f^{-1}(A) \bigtriangledown (\mathbf{1}_{T\setminus N} f)^{-1}(A) \subset N$$

Therefore $f^{-1}(A) \in \mathcal{T}'$. Hence f is $\mathcal{T}'$-measurable. □

Consequently any pointwise limit of v-measurable functions is v-measurable.

Proposition 3.8. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E})(\mathcal{T})$. For every v-integrable function f and every bounded v-measurable function g the function fg is v-integrable.*

Proof. Let g be v-measurable and bounded. Define g_n as follows:

$$g_n(t) = h2^{-n} \text{ if } f(t) \in \{h2^{-n}, (h+1)2^{-n}\}$$

Then $g_n \in (\mathcal{T}')$ and $|g_n - g_m| \leq 2^{-n}$ if $m > n$. On the other hand let $\{f_n\}$ be a sequence of elements of $\mathcal{E}(\mathcal{T}')$ such that f_n converges v-a.e. and $\lim_n v(f - f_n) = 0$. Pick $k, K \in \mathbb{R}$ such that $\forall n \in \mathbb{N} \quad v(f_n) \leq K$ and $|g_n| \leq k$. Then we have:

$$f_n g_n - f_m g_m = (f_n - f_m) g_n + f_m (g_n - g_m)$$

and therefore:

$$|f_n g_n - f_m g_m| = k\,|f_n - f_m| + |f_m|\,2^{-n}$$

and

$$v(f_n g_n - f_m g_m) = kv(f_n - f_m) + K2^{-n}$$

Hence $\lim_n v(f_n f_n - f_m g_m) = 0$. The sequence $\{f_n g_n\}$ is a Cauchy sequence in $\mathcal{L}^1(v)$ and converges v-p.p. to fg. We conclude that $fg \in \mathcal{L}^1(v)$. □

Theorem 3.9. *Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $\mathcal{E}(\mathcal{T})$. Suppose $f \in \mathcal{L}^1(v)$ and g v-measurable. If $|g| \leq |f|$ then $g \in \mathcal{L}^1(v)$.*

Proof. We have $g = f(g/f)$ (with $g(t)/f(t) = 1$ if $f(t) = 0$). By the preceding proposition g is v-integrable. □

Definition 3.10. Let $(T, \mathcal{T})$ be a measurable space and v be an exhaustive semi-variation on $(\mathcal{E}, \mathcal{T})$. A sequence $\{f_n\}$ of v-measurable functions is said to converge v-almost uniformly to a function f iff for every $\varepsilon > 0$, there exists a v-measurable set A_ε such that $v^*(T \setminus A_\varepsilon) \leq \varepsilon$ and f_n converges uniformly to f on A_ε.

Theorem 3.11. *[Egorov] Let $(T, \mathcal{T})$ be a measurable space and v be an exhaustive semi-variation on $(\mathcal{E}, \mathcal{T})$. A sequence $\{f_n\}$ of v-measurable functions converges v-a.e. to a function f iff $\{f_n\}$ converges to f v-almost uniformly.*

PB. Thorme II.31. □

4. Integrability of vector functions

Definition 4.1. Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $(\mathcal{T})$. Let E be a Banach space. An E-valued function f is said to be v-measurable if f equals v-a.e. the limit of a sequence of $\mathcal{T}$-simple functions.

If f is E-valued v-measurable function then $\|f(.)\|_E$ is v-measurable.

Definition 4.2. Let $(T, \mathcal{T})$ be a measurable space and v a semi-variation on $(\mathcal{T})$. Let E be a Banach space. Then $\mathcal{L}^1_E(v)$ denotes the space of all v-measurable functions such that $\|f(.)\|_E \in \mathcal{L}^1(v)$.

Theorem 4.3. *$\mathcal{L}^1_E(v)$ is a vector space.*

Proof. Let us prove

$$f, g \in \mathcal{L}^1_E(v) \quad \Longrightarrow \quad f + g \in \mathcal{L}^1_E(v)$$

We have $\|(f+g)(.)\| \leq \|f(.)\| + \|g(.)\|$. Hence $\|(f+g)(.)\|$ is v-measurable and majorized by an integrable function and therefore is integrable. □

The space $\mathcal{L}^1_E(v)$ is endowed with the semi-norm $f \mapsto \| \, \|f(.)\|_E \, \|_{\mathcal{L}^1(v)}$.

The quotient of $\mathcal{L}^1_E(v)$ by the v-a.e. equality will be denoted by $L^1_E(v)$.

Theorem 4.4. *Let v be a semi-variation on $(\mathcal{T})$. Then the space $\mathcal{L}^1_F(v)$ is complete.*

Proof. Let us prove that every sequence $\{f_n\}$ of members of $\mathcal{L}^1_F(v)$ with:

$$\sum_{n \in \mathbb{N}} \|f_n\|_{\mathcal{L}^1_F(v)} < \infty$$

has a sum in $\mathcal{L}^1_F(v)$. Put $g_n = \|f_n(.)\|$. By definition $\|f_n\|_{\mathcal{L}^1_F(v)} = \|g_n\|_{\mathcal{L}^1(v)}$ and therefore:

$$\sum_{n\in\mathbb{N}} \|g_n\|_{\mathcal{L}^1(v)} < \infty$$

Lebesgue's theorem on series yields:

$$\sum_{n\in\mathbb{N}} |g_n(t)| < \infty \qquad v-a.e.$$

Hence there exists a v-negligible set N such that for every $t \notin N$, we have:

$$\sum_{n\in\mathbb{N}} \|f_n(t)\|_F < \infty$$

The space F being complete, for every $t \notin N$ there exists $f(t) \in F$ with

$$f(t) = \sum_{n\in\mathbb{N}} f_n(t)$$

For $t \notin N$, put $k_n(t) = \left\|\sum_{p\leq n} f_p(t)\right\|_F$ and $k(t) = \|f(t)\|_F$. If a $\mathcal{T}$-measurable positive function is majorized by an element of $\mathcal{L}^1(v)$ then this function belongs to $\mathcal{L}^1(v)$. Hence we have $k_n \in \mathcal{L}_1(v)$ because $k_n(t) \leq \sum_{p\leq n} \|f_p(t)\|$. On the other hand $k(t) = \lim_n k_n(t)$ for all $t \notin N$.

For $m \leq n$, we have:

$$\begin{aligned}
|k_n(t) - k_m(t)| &= \left| \left\|\sum_{p\leq n} f_p(t)\right\|_F - \left\|\sum_{p\leq m} f_p(t)\right\|_F \right| \\
&\leq \left\|\sum_{p\leq n} f_p(t) - \sum_{p\leq m} f_p(t)\right\|_F \\
&= \left\|\sum_{p=m+1}^{n} f_p(t)\right\|_F
\end{aligned}$$

Hence:

$$\|k_n - k_m\|_{\mathcal{L}_1(v)} \leq \left\| \left\|\sum_{p=m+1}^{n} f_p(.)\right\|_F \right\|_{\mathcal{L}_1(v)} = \left\|\sum_{p=m+1}^{n} f_p\right\|_{\mathcal{L}^1_F(v)}$$

That implies the sequence $\{k_n \mid n \in \mathbb{N}\}$ is a Cauchy sequence in $\mathcal{L}^1(v)$. As $k_n(t)$ converges to $k(t)$ for all $t \notin N$, we have $k \in \mathcal{L}^1(v)$, which proves $f \in \mathcal{L}^1_F(v)$). It remains to prove that the sequence $\{f_n\}$ converges to f in $\mathcal{L}^1_F(v)$. We have:

$$\left\|f - \sum_{n=1}^{N} f_n\right\|_{\mathcal{L}^1_F(v)} = \left\|\sum_{n>N} f_n\right\|_{\mathcal{L}^1_F(v)} = \left\|\left\|\sum_{n>N} f_n(.)\right\|\right\|_{\mathcal{L}^1(v)} \leq \left\|\sum_{n>N} g_n\right\|_{\mathcal{L}^1(v)}$$

As the limit of the last expression is null, the theorem is proven. □

Theorem 4.5. *[Theorem of dominated convergence] Let v be an exhaustive semi-variation. Suppose $\{f_n\}$ is a sequence in $\mathcal{L}^1_E(v)$ and $h \in \mathcal{L}^1_+(v)$ such that $\|f_n(.)\| \leq h$ v-a.e.. If $f_n(t)$ converges v-a.e. to a function f, then $f \in \mathcal{L}^1_E(v)$ and f_n converges to f in $\mathcal{L}^1_E(v)$.*

Proof. Applying Theorem 2.8 to the function $\|f_n(.)\|$ proves $\|f(.)\|$ is integrable so $f \in \mathcal{L}^1_E(v)$. Applying this theorem to the function $\|f_n(.) - f(.)\|$ proves that f_n converges to f in $\mathcal{L}^1_E(v)$. □

5. The contextual semi-variation

Let $(T, \mathcal{T})$ be a measurable space. We are given three Banach spaces E, F and G and a bilinear continuous application $y, x \mapsto y \odot x$ of $F \times E$ into G. The 4-uple $(E, F, G, \odot)$ will be called a bilinear context. Let us consider a mapping m of $\mathcal{T}$ into E such that for every $y \in F$, $A \mapsto y \odot m(A)$ is a (strong) measure on $\mathcal{T}$ with values in G. A particular case of such a mapping is an E-valued (strong) measure.

If f is a $\mathcal{T}$-simple functions with values in F, then its integral with respect to m is defined by

$$\int f \odot m = \sum_i y_i \odot m(A_i)$$

if $\{A_i\}$ is a finite partition of T and $f = \sum_i \mathbf{1}_{A_i} y_i$. Define $\int_A f \odot m = \int (\mathbf{1}_A f) \odot m$. Then the application $A \mapsto \int_A f \odot m$ is a (strong) measure with values in G which is denoted by $f \odot m$.

Let us first define a semi-variation of m which we shall call "contextual" semi-variation because it depends on the context $(E, F, G, \odot)$. For every $h \in$ put:

$$\underline{w}(h) = \sup \left\{ \left\| \int f \odot m \right\|_G \mid f \in_F, \|f(.)\| \leq |h| \right\}$$

and $\underline{w}(A) = \underline{w}(\mathbf{1}_A)$. Then

$$\underline{w}(A) = \sup \left\{ \left\| \sum_j y_j \odot m(B_j) \right\| \mid \{B_j\} \text{ finite partition of } A, y_j \in F, \|y_j\| \leq 1 \right\}$$

For every partition $\{A_i \mid i \in \mathbb{N}\}$ of A, one have:

$$\underline{w}(A) \leq \sum_i \underline{w}(A_i)$$

Indeed let $\{B_j\}$ be a finite partition of A and $y_j \in \mathfrak{B}(F)$. We have:

$$\begin{aligned}
\sum_j \|y_j \odot m(B_j)\| &= \left\|\sum_j \sum_i y_j \odot m(B_j \cap A_i)\right\| \\
&= \left\|\sum_i \sum_j y_j \odot m(B_j \cap A_i)\right\| \\
&\leq \sum_i \left\|\sum_j y_j \odot m(B_j \cap A_i)\right\| \\
&\leq \sum_i \underline{w}(A_i)
\end{aligned}$$

This gives the announced inequality. More precisely the following result holds:

Theorem 5.1. *$\underline{w}$ is an extended semi-variation.*

Proof. (0) An easy checking gives:

$$0 \leq f_1 \leq f_2 \quad \Longrightarrow \quad v(f_1) \leq v(f_2)$$

(1) Let us first prove that $\{A_n\}$ being an increasing sequence in $\mathcal{T}$ whose union is T, then $\underline{w}(h) = \lim_n \underline{w}(\mathbf{1}_{A_n} h)$. We may suppose $h \geq 0$.

(1a) Suppose $\underline{w}(h) < \infty$. Let $\varepsilon > 0$. There exists a function $f = \sum_{i\in I} \mathbf{1}_{B_i} y_i$, where $\{B_i\}$ is a finite partition of T, $\{y_i\}$ a family of members of F, such that $\|\mathbf{1}_{A_i}(.)y_i\| \leq h$ et $\underline{w}(h) - \varepsilon \leq \|\sum_i y_i \odot m(B_i)\|$. For every $n \in \mathbb{N}$, put $f_n = \mathbf{1}_{A_n} f$, *i.e.* : $f_n = \sum_i \mathbf{1}_{B_i \cap A_n} y_i$. We have $\int f_n \odot m = \sum_i y_i \odot m(B_i \cap A_n)$. For every $i \in I$, we have $\lim_n y_i \odot m(B_i \cap A_n) = y_i \odot m(B_i)$, because $y_i \odot m(.)$ is a strong measure. Therefore there exists N such that

$$n \geq N \quad \Longrightarrow \quad \left\|\int f \odot m - \int f_n \odot m\right\| \leq \varepsilon$$

Hence $n \geq N \quad \Longrightarrow \quad \underline{w}(h) - 2\varepsilon \leq \int f_n \odot m$ with $\|f_n(.)\| \leq h$. The assertion is proved.

(1b) Suppose $\underline{w}(h) = \infty$. Let $M > 0$. There exists a function $f = \sum_{i\in I} \mathbf{1}_{B_i} y_i$, where $\{B_i\}$ is a finite partition of T, $\{y_i\}$ a family of members of F, such that $\|\mathbf{1}_{A_i}(.)y_i\| \leq h$ et $\left\|\int f \odot m\right\| \geq M$. Defining $f_n = f\mathbf{1}_{A_n}$, one sees that there exists N such that

$$n \geq N \quad \Longrightarrow \quad \left\|\int f \odot m - \int f_n \odot m\right\| \leq \varepsilon$$

Therefore $n \geq N \quad \Longrightarrow \quad \int f_n \odot m \geq M - \varepsilon$. The assertion is proved.

(2) Let us prove that

$$h, h_n \in_+ \ , \ \sum_n h_n \geq h \quad \Longrightarrow \quad \sum_n \underline{w}(h_n) \geq \underline{w}(h)$$

Let $g_n = \sum_{p=0}^n h_p$ et $A_n = \{t \in T \mid g_n(t) \geq (1-\varepsilon)h(t)\}$. The sequence $\{A_n\}$ is increasing and its union is T. We have $g_n \geq (1-\varepsilon)\mathbf{1}_{A_n} h$, and therefore $\underline{w}(g_n) \geq (1-\varepsilon)\underline{w}(\mathbf{1}_{A_n} h)$.

Furthermore $\underline{w}(g_n) \leq \sum_{p=0}^n \underline{w}(h_p)$. Hence

$$\sum_{p=0}^n \underline{w}(h_p) \geq (1-\varepsilon)\underline{w}(\mathbf{1}_{A_n} h)$$

Making n converge to $+\infty$, we get:

$$\sum_p \underline{w}(h_p) \geq (1-\varepsilon)\underline{w}(h)$$

As ε is arbitrary:

$$\sum_p \underline{w}(h_p) \geq \underline{w}(h)$$

□

Proposition 5.2. *Suppose m is a (strong) E-valued measure. If $f \in_F$, then*

$$\left\| \int f \odot m \right\| \leq \int \|f(.)\| \operatorname{var}(m)$$

Proof. Let $f = \sum_i \xi_i \mathbf{1}_{A_i}$ with $\xi_i \in F$ and $\{A_i\}$ a finite $\mathcal{T}$-partition of T. We have $\int f \otimes m = \sum_i \xi_i \odot m(A_i)$, and therefore

$$\begin{aligned} \left\| \int f \odot m \right\| &\leq \sum_i \|\xi_i \odot m(A_i)\| \\ &= \sum_i \|\xi_i\| \, \|m(A_i\| \\ &\leq \sum_i \|\xi_i\| \operatorname{var}(m)(A_i) \\ &= \int \|f(.)\| \ \operatorname{var}(m) \end{aligned}$$

□

Corollary 5.3. *Suppose m is a (strong) E-valued measure. The contextual semi-variation $\underline{w}$ is majorized by* $\operatorname{var}(m)$. *In particular if* $\operatorname{var}(m)$ *is σ-finite, then the same is true for $\underline{w}$.*

In the case $\underline{w}$ is exhaustive, we may introduce an important class of integrable functions.

Definition 5.4. Let $\underline{w}$ be exhaustive. A function f from T into F is said to be Bochner-integrable iff $f \in \mathcal{L}^1_F(\underline{w})$.

Note that we have restricted the notion of Bochner-integrability to the case $\underline{w}$ is exhaustive. Without this restriction the definition of Bochner-integrability would lead to pathological facts as we shall see further.

As we have

$$\forall A \in \mathcal{T}, f \in_F \qquad (f \odot m)A \leq \underline{w}(f(.))$$

and as $_F$ is dense in $\mathcal{L}^1_F(\underline{w})$, the map $A \mapsto (f \odot m)(A)$ may be extended by continuity to $\mathcal{L}^1_F(\underline{w})$ and the map $A \mapsto (f \odot m)(A)$ is a vector measure. We may put:

$$\text{(BOCHNER)} \int_A f \odot m = (f \odot m)(A)$$

Theorem 4.5. may be translated into the following theorem.

Theorem 5.5. *[Theorem of dominated convergence] Let $\underline{w}$ be exhaustive. Suppose $\{f_n\}$ is a sequence of Bochner-integrable functions and $h \in \mathcal{L}^1_+(\underline{w})$ such that $\|f_n(.)\| \leq h$ $\underline{w}$-a.e.. If $f_n(t)$ converges $\underline{w}$-a.e.to a function f, then f is Bochner-integrable and f_n converges to f in $\mathcal{L}^1_E(\underline{w})$.*

6. The intrinsic semi-variation of a vector measure

Suppose m is an E-valued measure. Taking $F = \mathbb{R}$ and $G = E$, the contextual semi-variation $\underline{w}$ is called the intrinsic (or scalar) semi-variation of m and is denoted by $m^\bullet$. In other words we put:

$$\forall h \in \qquad m^\bullet(h) = \sup\left\{ \left\| \int f\ m \right\| \ \middle|\ f \in_E, |f| \leq |h| \right\}$$

and $m^\bullet(A) = m^\bullet(\mathbf{1}_A)$.

When no context is present, the name "semi-variation" will refer to the intrinsic semi-variation.

The following theorem is fundamental in the theory of vector measures.

Theorem 6.1. *The semi-variation $m^\bullet$ of a measure m is exhaustive.*

Proof. [PB] Theorem VI.10. □

Let v be a semi-variation on $(T, \mathcal{T})$ and j be the canonical mapping of $(T, \mathcal{T})$ into $L^1(v)$. It is easy to prove that if j is a measure then its semi-variation is v and that if v is exhaustive then j is a measure. As a consequence we have the next Proposition. Let us first give a notation. If v be a semi-variation on $(T, \mathcal{T})$, ∇v denotes the set of all real measures (considered as linear forms on $(T, \mathcal{T})$) majorized by v. By the Hahn-Banach theorem the following formula holds:

$$\forall f \in (T, \mathcal{T}) \qquad v(f) = \sup \left\{ \int f\mu \mid \mu \in \nabla v \right\}$$

Proposition 6.2. *Let v be a semi-variation on $(T, \mathcal{T})$. The following properties are equivalent:*

(1) v is exhaustive

(2) For every decreasing sequence $\{A_n\}$ of elements of $\mathcal{T}$ with empty intersection, one has $\lim_n v(A_n) = 0$

(3) For every countable $\mathcal{T}$-partition $\{T_n\}$ of T, one has $\lim_n v(T_n) = 0$.

(4) ∇v is relatively weakly compact. [2]

Proof. (1) implies (2) by the theorem of dominated convergence.

Suppose (3) fails. Then we can find $\delta > 0$ and a countable family $\{B_n\}$ of disjoint elements of $\mathcal{T}$ such that $v(B_n) \geq \delta$. By replacing B_0 by $B_0 \cap (T \setminus \cup_n B_n)$ we obtain a partition $\{B_n\}$ such that $v(B_n) \geq \delta$. Putting $A_n = \cup_{p \geq n} B_n$ we contradict (2). Hence (2) implies (3).

Suppose (3) holds. Let $\{A_n\}$ be a sequence of elements of $\mathcal{T}$ with empty intersection. Putting $T_n = A_n \setminus A_{n+1}$, we obtain a partition $\{T_n\}$. Hence $\lim_n v(T_n) = 0$ and a fortiori $\lim_n v(A_n) = 0$. Hence (3) implies (2).

Suppose (2) holds. For every countable $\mathcal{T}$-partition $\{T_n\}$ of T, one has

$$\lim_n v(T \setminus \bigcup_{n \leq N} T_n) = 0$$

That means the canonical mapping j of $(T, \mathcal{T})$ into $L^1(v)$ is a measure and v is its semi-variation. Therefore v is exhaustive. Hence (2) implies (1). The equivalence between (3)

[2] People working with set-defined semi-variations would be interested by the set ∂v of all scalar measures μ such that $\mu(A) \leq v(A)$ for all $A \in \mathcal{T}$. As $\nabla v \subset \partial v$ if ∂v is relatively weakly compact then the same is true for ∇v. Conversely if ∇v is relatively weakly compact then (2) shows that for every decreasing sequence $\{A_n\}$ of elements of $\mathcal{T}$ with empty intersection, one has $\lim_n \mu(A_n) = 0$ uniformly with respect to μ if μ runs over ∂v and that means ∂v is relatively weakly compact.

and (4) is a direct consequence of the Hahn-Banach theorem:

$$v(T \setminus \bigcup_{n \le N} T_n) = \sup \left\{ \mu(T \setminus \bigcup_{n \le N} T_n) \;\middle|\; \mu \in \nabla v \right\}$$

and the equivalence between uniform σ-additivity and relative weak compactness in the space of real measures. □

As a byproduct of the proof we have the following result:

Lemma 6.3. *Let v be a semi-variation on $(T, \mathcal{T})$. If v is not exhaustive, then there exists $\delta > 0$ and a countable partition $\{T_n\}$ of T such that $v(T_n) \ge \delta$*

Let us remark that if $\{m_n\}$ is a sequence of vector measures defined on $(T, \mathcal{T})$ then $m_n(A)$ converges to 0 uniformly with respect to A iff $m_n^\bullet(T)$ converges to 0. The following result plays an essential role in the next sections.

Theorem 6.4. *[Vitali-Hahn-Saks] Let $(T, \mathcal{T}, \mu)$ be a measure space, E a Banach space and $\{m_n\}$ a sequence of E-valued measures. It is assumed*

(1) all the m_n are absolutely continuous with respect to μ.

(2) for every $A \in \mathcal{T}$, the sequence $\{m_n(A)\}$ converges.

Then:

$$\forall \varepsilon > 0 \; \exists \eta \text{ such that } \quad \mu(A) \le \eta \quad \Longrightarrow \quad \|m_n(A)\| \le \varepsilon \text{ for all } n \in \mathbb{N}$$

Equivalently:

$$\forall \varepsilon > 0 \; \exists \eta \text{ such that } \quad \mu(A) \le \eta \quad \Longrightarrow \quad m_n^\bullet(A) \le \varepsilon \text{ for all } n \in \mathbb{N}$$

Proof. Cf [DS] III.7.2. □

Corollary 6.5. *With the hypothesis of Vitali-Hahn-Saks theorem, the sequence $\{m_n\}$ is uniformly σ-additive.*

Corollary 6.6. *With the hypothesis of Vitali-Hahn-Saks theorem, if one puts*

$$\forall A \in \mathcal{T} \qquad m(A) = \lim_n m_n(A)$$

then m is an E-valued measure.

Proof. Cf [DS] IV.10.6. □

Remaark 6.7. The theorem of Vitali-Hahn-Saks and its corollaries may be easily deduced from the case of scalar measures (e.g. [PB] Theorem XV.9). Suppose first E is separable. Then there exists a sequence $\{\xi_h\}$ in $\mathfrak{B}_{E'}$ such that $\|x\| = \sup_h \langle x, \xi_h\rangle$ for every $x \in E$. The measures $\langle m_n(.), \xi_h\rangle$ satisfie the hypothesis of the Vitali-Hahn-Saks theorem for scalar measures. Therefore

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad |\langle m_n(A), \xi_h\rangle| \leq \varepsilon$$

and consequently

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad \|m_n(A)\| \leq \varepsilon$$

Consequently (E being supposed to be separable) the vector measures m_n are uniformly σ-additive.

In the case E is not supposed to be separable let $\{A_k\}$ be a countable partition of T and $\mathcal{T}'$ the σ-algebra generated by this partition. The restrictions of the measures $\{m_n\}$ to $\mathcal{T}'$ have their values in a separable space and therefore are uniformly σ-additive. But this implies the $\{m_n\}$ are also uniformly σ-additive as well as the $\{\langle m_n(.), y\rangle \mid n \in \mathbb{N}, \|y\| \leq 1\}$. Hence this family of measures is relatively weakly compact. By the Dunford-Pettis theorem ([PB] theorem VII.18) we have

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad \forall n \in \mathbb{N}\ \|y\| \leq 1 \quad |\langle m_n(A), y\rangle| \leq \varepsilon$$

and therefore

$$\forall \varepsilon \quad \exists \eta \qquad \mu(A) \leq \eta \quad \Longrightarrow \quad \forall n \in \mathbb{N} \quad \|m_n(A\| \leq \varepsilon$$

Example 6.8. Let F and E two Banach spaces and suppose $F \subset E'$ and that F is a norming subspace for E. For $y \in F$, $x \in E$ put $y \odot x = \langle y, x\rangle$. Suppose m is an E-valued measure. Let us compute the contextual semi-variation $\underline{w}$.

By Corollory 5.3., we know that $\underline{w} \leq \mathrm{var}(m)$. Let us prove that the equality holds.

Let $h \in_+$ with $h = \sum_j \mathbf{1}_{B_j} z_j$ where $\{B_j\}$ is a $\mathcal{T}$ finite partition of T and $z_j \in \mathbb{R}_+$. We have

$$\underline{w}(h) = \sup \left\{ \sum_{ij} \langle \xi_{ij}, m(A_{ij})\rangle \right\}$$

where the sup is taken over all finite partitions $\{A_{ij}\}$ of B_j and $\|\xi_{ij}\| \leq z_j$. For every i, j we have

$$\sup\left\{\langle \xi_{ij}, m(A_{ij})\rangle \;\Big|\; \|\xi_{ij}\| \leq z_j\right\} = z_j \, \|m(A_{ij})\|$$

and therefore

$$\begin{aligned}\underline{w}(h) &= \sup\left\{\sum_{ij} z_j \, \|m(A_{i,j})\| \;\Big|\; \{A_{ij}\} \text{ finite partition of } B_j\right\}\\ &= \sup\left\{\sum_j z_j \sum_i \|m(A_{ij})\| \;\Big|\; \{A_{ij}\} \text{ finite partition of } B_j\right\}\\ &= \sum_j z_j \mathrm{var}(m)(B_j) = \int h \ \mathrm{var}(m)\end{aligned}$$

Finally we obtain the announced equality: $\underline{w} = \mathrm{var}(m)$. The integral of a function f with respect to m may be denoted by $\int \langle f \ , \ m\rangle$.

Example 6.9. Dinculeanu considers in his treatise [DIN] the following situation. Two Banach spaces are given as well as a $L(Y, X)$-valued additive set function m. It is supposed that for every $y \in Y$, $m(.)y$ is σ-additive, *i.e.* is a X-valued measure. This property is nothing but the σ-additivity of m for the strong operator topology on $L(Y, X)$ (see section 1).

Given $f \in_F$ its integral with respect to m is given by

$$\int f \odot m = \sum_i m(A_i)\xi_i \quad \text{if} \quad f = \sum \xi_i \mathbf{1}_{A_i}$$

with $\{A_i\}$ being a finite $\mathcal{T}$-partition of T and $\xi_i \in F$. This setting has been intensively studied by Dobrakov([DO1] and [DO2]) so we will refer to it as the Dobrakov setting. We suppose that the mapping $m(.)y$ is σ-additive for every $y \in Y$ so the Dobrakov setting is a particular case of the bilinear one with $E = L(Y, X), F = Y, G = X, y \odot u = uy$.

Following the Orlicz-Pettis theorem the assumption on m is equivalent to the following: for every $y \in Y$ and every $x' \in X'$ the mapping $\langle m(.)y, x'\rangle$ is a measure. That means m is a weak measure for the duality $(L(Y, X), Y \otimes X')$.

Let Z be a subspace of X' norming for X. For every $z \in Z$ the mapping $y, u \mapsto \langle z, uy\rangle$ is a bilinear form. We may consider the measure m_z such that $m_z(A) = zm(A)$ for every $A \in \mathcal{T}$. For every simple function f we have:

$$\int f \ m_z = \left\langle z, \int f \odot m\right\rangle$$

Let us compute the contextual semi-variation $\underline{w}$ of m. We have for $h \in_+$:

$$\begin{aligned}
\underline{w}(h) &= \sup\left\{\left\|\int f \odot m\right\| \;\Big|\; \|f(.)\| \le h\right\} \\
&= \sup\left\{\left\langle z, \int f \odot m\right\rangle \;\Big|\; \|f(.)\| \le h, \|z\| \le 1\right\} \\
&= \sup\left\{\left\|\int \langle f, m_z\rangle\right\| \;\Big|\; \|f(.)\| \le h, \|z\| \le 1\right\} \\
&= \sup\left\{\int h \operatorname{var}(m_z) \;\Big|\; \|z\| \le 1\right\}
\end{aligned}$$

We will now show how a problem of integration in the bilinear setting may be transformed into an equivalent one in the Dobrakov setting. Let $(E, F, G, \odot)$ a bilinear context and m a E-valued set function such that $y \odot m(.)$ is a G-valued measure for every $y \in F$. For every $A \in \mathcal{T}$ let $p(A) \in L(F, G)$ defined by $p(A)y = y \odot m(A)$ for every $y \in F$. For $u \in L(F, G)$ and $y \in f$ put $y \underline{\odot} u = \hat{u}y$ Suppose $f \in_F$ with $f = \sum_i \xi_i \mathbf{1}_{A_i}$ where $\{A_i\}$ is a finite $\mathcal{T}$-partition of T and $\xi_i \in F$. We have:

$$\int f \underline{\odot} p = \sum_i \xi_i \underline{\odot} p(A_i) = \sum_i p(A_i)\xi_i = \sum_i \xi_i \odot m(A_i) = \int f \odot m$$

Let us compare the contextual semi-variations $\underline{w}$ and $\overline{w}$ of m and p. We have for every $h \in_+$:

$$\begin{aligned}
\overline{w}(h) &= \sup\left\{\left\|\sum_i p(A_i)\xi_i\right\| \;\Big|\; \sum_i \|\xi_i\| \mathbf{1}_{A_i} \le h\right\} \\
&= \sup\left\{\|\xi_i \odot m(A_i)\| \;\Big|\; \sum_i \|\xi_i\| \mathbf{1}_{A_i} \le h\right\} \\
&= \underline{w}(h)
\end{aligned}$$

Hence $\overline{w} = \underline{w}$.

If m is an E-valued (strong) measure, then p is a $L(F, G)$-valued (strong) measure. Indeed we have

$$\|p(A)\| = \sup\{p(A)y \mid y \in \mathfrak{B}_F\} = \sup\{y \odot m(A) \mid y \in \mathfrak{B}_F\} \le \|\odot\| \, \|m(A)\|$$

But we can only prove the inequality $p^\bullet \le \|\odot\| \, m^\bullet$.

Example 6.10. The following example stems from [DO1] (example 7). Put $T = \mathbb{N}$ and $\mathcal{T} = \mathcal{P}(\mathbb{N})$. Use the Dobrakov setting with $X = \mathbf{l}^1$, $Y = \mathbf{c}_0$. Suppose we have a bounded sequence $\{\xi_n\}$ in $\mathbf{c_0}$. For every $A \subset I\!N$ define $m(A) \in L(\mathbf{l^1}, \mathbf{c_0})$ by $m(A)x = \sum_{t\in A} x_t\xi_t$. For every $x \in \mathbf{l^1}$, $m(.)x$ is a $\mathbf{c_0}$-valued measure. Furthermore we have

$$\|m(A)\| = \sup\left\{\left\|\sum_{t\in A} x_t\xi_t\right\| \mid x \in \mathbf{l^1}, \|x\| \leq 1\right\} = \sup\{\|\xi_t\| \mid t \in A\}$$

Hence if $\lim_n \|\xi_n\| = 0$ then m is a $L(\mathbf{l^1}, \mathbf{c_0})$-valued measure.

By choosing appropriately the sequence $\{\xi_i\}$ we can manage to get the contextual semi-variation $\underline{w}$ being finite but not exhaustive. For example choosing

$$\begin{aligned} \xi_1 &= [1,0,0,\ldots] \\ \xi_2 = \xi_3 &= [0,1/2,0,0,\ldots] \\ \xi_4 = \xi_5 = \xi_6 &= [0,0,1/3,0,0,\ldots] \\ \ldots &= \ldots \end{aligned}$$

Then one finds that for any $h \in_+$ one has:

$$\underline{w}(h) = \sup\left\{\left\|\sum_{t\in I\!N}(f(t))_t\xi_t\right\| \mid f \in_{l^1}\ ,\ \ \|f(.)\| \leq h\right\} = \left\|\sum_{t\in I\!N} h(t)\xi_t\right\|$$

In particular:

$$\underline{w}(A) = \left\|\sum_{t\in A}\xi_t\right\|$$

We get $\underline{w}([k, k+1, k+2, \ldots] = 1$ for every $k \in I\!N$ and therefore the following condition fails: $\lim \underline{w}(A_k) = 0$ whenever $\{A_k\}$ is decreasing and has empty intersection.

7. **Bilinear Integration**

In this section we consider a bilinear context $(E, F, G, \odot)$. Given an E-valued set function such that $y \odot m(.)$ is a G-valued measure for all $y \in F$ we define the integrability and the integral of a F-valued function with respect to m. The contextual semi-variation is denoted by $\underline{w}$.

Lemma 7.1. *Let $\{\nu_n\}$ be a sequence of (strong) vector measure on a measurable space $(T, \mathcal{T})$. There exists a positive measure μ on $(T, \mathcal{T})$ such that $\mu(A) = 0 \Longleftrightarrow \nu_n^\bullet(A) = 0$ for every n.*

Proof. Let μ_n be a control measure of ν_n. It suffices to take $\mu = \sum_n 2^{-n}\mu_n / \|\mu_n\|$. □

Lemma 7.2. *Suppose $\underline{w}$ σ-finite. Let $\{f_n\}$ be a sequence of simple functions converging $\underline{w}$-a.e. to 0. Suppose for every $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges. Then it converges to 0, uniformly with respect to A.*

Proof. Let $N \in \mathcal{T}$ such that $\underline{w}(N) = 0$ and f_n converges on $T \setminus N$ to f. Replacing f_n by $\mathbf{1}_{T\setminus N} f_n$, doesn't modify $f_n \odot m$ and brings us to the case f_n converges everywhere to 0. Let μ be the measure associated with the sequence $\{f_n \odot m\}$ by the preceding lemma. As the measures $f_n \odot m$ are absolutely convergent with respect to μ, following the Vitali-Hahn-Saks theorem, there exists η such that

$$\mu(A) \leq \eta \quad \Longrightarrow \quad \|(f_n \odot m)(A)\| \leq \varepsilon/2$$

Now by the Egorov's theorem, as f_n converges to 0 $\underline{w}$-a.e. (and therefore μ-a.e.), there exists C_η such that $\mu(T \setminus C_\eta) \leq \eta$ and f_n converges uniformly to 0 on C_η.

As $\underline{w}$ is σ-finite we may suppose $\underline{w}(C_\eta) < \infty$. Indeed let us consider a countable partition $\{T_k\}$ of T such that $\underline{w}(T_k) < \infty$. Put $S_n = \bigcup_{h \leq k} T_h$ and $R_k = \bigcup_{h>n} T_h$. We first take C'_η such that $\mu(T \setminus C'_\eta) \leq \eta/2$ and $\{f_n\}$ converges uniformly to C'_η. For k large enough, one have $\mu(C'_\eta \cap R_k) \leq \varepsilon/2$. Taking $C_\eta = C'_\eta \cap S_k$, we obtain $\mu(T \setminus C_\eta) \leq \eta$ and $\{f_n\}$ converges uniformly to C_η.

Now have:

$$(f_n \odot m)^\bullet(T) \leq (f_n \odot m)^\bullet(C_\eta) + (f_n \odot m)^\bullet(T \setminus C_\eta)$$

But

$$(f_n \odot m)^\bullet(C_\eta) \leq \underline{w}(\mathbf{1}_{C_\eta} \|f_n(.)\|)$$

There exists N such that

$$n \geq N \quad \Longrightarrow \quad \|f_n(t)\| \leq \varepsilon/2\underline{w}(C_\eta) \quad \text{for all} \quad t \in C_\eta$$

That gives

$$n \geq N \quad \Longrightarrow \quad (f_n \odot m)^\bullet(T) \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

□

Definition 7.3. Suppose $\underline{w}$ is σ-finite. A function f from T into F is said to be integrable iff there exists a sequence $\{f_n\}$ of simple functions such that

(1) f_n converges $\underline{w}$-a.e. to f

(2) For every $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges.

The preceding lemma ensures that if two sequences $\{f_n\}$ and $\{f'_n\}$ fulfill the conditions of the preceding definition, then for every $A \in \mathcal{T}$, the two sequences $\{(f_n \odot m)(A)\}$ and $\{(f'_n \odot m)(A)\}$ have the same limit. If f is an integrable function, one may put:

$$\int_A f \odot m = \lim_n (f_n \odot m)(A) \qquad \int f \odot m = \lim_n (f_n \odot m)(T)$$

A glance at Corollary 6.6. shows that the mapping $A \mapsto \int_A f \odot m$ is a G-valued measure. This measure will be denoted by $f \odot m$.

Any integrable function f is $\underline{w}$-measurable (*i.e.* is $\underline{w}$-a.e. a limit of simple functions) as well as the function $t \mapsto \|f(t)\|$.

Lemma 7.4. *The conditions of the preceding definition are supposed to be satisfied. Then $(f_n \odot m)(A)$ converges uniformly to $(f \odot m)(A)$ for $A \in \mathcal{T}$.*

Proof. Let μ the measure associated with the sequence $\{f_n \odot m\}$ by lemma 7.1.. Let $\varepsilon > 0$. There is η such that

$$\mu(A) \leq \eta \quad \Longrightarrow \quad (f_n \odot m)^\bullet(A) \leq \varepsilon \text{ and } (f \odot m)^\bullet)(A) \leq \varepsilon$$

There exists C_η such that $\mu(T \setminus C_\eta) \leq \eta$ and f_n converges uniformly to f on C_η. As $\underline{w}$ is σ-finite, we may suppose $\underline{w}(C_\eta) < \infty$. Then we have

$$((f_n - f) \odot m)^\bullet(T) \leq ((f_n - f) \odot m)^\bullet(C_\eta) + (f_n \odot m)^\bullet(T \setminus C_\eta) + (f \odot m)^\bullet(T \setminus C_\eta)$$

There exists N such that

$$n > N, t \in C_\eta \quad \Longrightarrow \quad \|f(t) - f_n(t)\| \leq \varepsilon$$

Then we have:

$$((f_n - f) \odot m)^\bullet(T) \leq \varepsilon \underline{w}(C_\eta) + 2\varepsilon$$

and therefore $\lim_n((f_n - f) \odot m)^\bullet(T) = 0$. □ □

Lemma 7.5. *Let f be integrable and $C \in \mathcal{T}$. Then $\mathbf{1}_C f$ is integrable and*

$$\forall A \in \mathcal{T} \qquad (\mathbf{1}_C f \odot m)(A) = (f \odot m)(A \cap C)$$

Proof. Let $\{f_n\}$ be a sequence as considered in Definition 7.3.. One easily checks that $(\mathbf{1}_C f_n \odot m)(A) = (f_n \odot m)(A \cap C)$. Therefore $\lim_n(\mathbf{1}_C f_n \odot m)(A)$ exists for all $A \in \mathcal{T}$. Therefore $\mathbf{1}_C f$ is integrable and passing to the limit gives the announced equality. □ □

Theorem 7.6. *Let $\underline{w}$ be σ-finite and exhaustive. Then every Bochner-integrable function is integrable.*

Proof. (1) Suppose first $\underline{w}$ is finite. Then by [PB],Théorème VIII.13., ${}_F$ is dense in $\mathcal{L}^1_F(\underline{w})$. Let $f \in \mathcal{L}^1_F(\underline{w})$. There exists $f_n \in_F$ which converges to f in $\mathcal{L}^1_F(\underline{w})$. By extracting a sub-sequence, we may suppose that f_n converges to f $\underline{w}$-p.p.. But for all $A \in \mathcal{T}$, we have:

$$\|(f_n \odot m)(A) - (f_p \odot m)(A)\| \leq \underline{w}(\|f_n(.) - f_p(.)\|) = \|f_n - f_p\|_{\mathcal{L}^1_F(\underline{w})}$$

Hence the sequence $\{(f_n \odot m)(A)\}$ is a Cauchy sequence in F. That proves f is integrable.

(2) Suppose now $\underline{w}$ is σ-finite. Let $\{S_n\}$ a increasing sequence of members of $\mathcal{T}$ such that for every n, $\underline{w}(S_n) < \infty$ et $\cup_n S_n = T$. Let $f \in \mathcal{L}^1_F(\underline{w})$. For every n, there is $f_n \in_F$, null outside S_n such that $\|f - f_n\|_{\mathcal{L}^1_F(\underline{w})} \leq 1/n$. The proof is achieved as in (1) by using the sequence $\{f_n\}$. □

Theorem 5.5. takes the following form:

Theorem 7.7. *[Theorem of dominated convergence] Let $\underline{w}$ be σ-finite and exhaustive. Suppose $\{f_n\}$ is a sequence of Bochner-integrable F-valued functions and $h \in \mathcal{L}^1_+(\underline{w})$ such that $\|f_n(.)\| \leq h$ $\underline{w}$-a.e.. If $f_n(t)$ converges $\underline{w}$-a.e. to a function f, then f is Bochner-integrable and f_n converges to f in $\mathcal{L}^1_E(\underline{w})$. Moreover $\int f_n \odot m$ converges to $\int f \odot m$ and $((f_n - f) \odot m)^\bullet(T)$ converges to 0.*

Example 7.8. Take $T = \mathbb{N}, \mathcal{T} = \mathcal{P}(\mathbb{N})$. Then f is integrable iff the sequence $\{f(k) \odot m(k)\}$ is summable.

Proof. (1) Suppose $\{f(k) \odot m(k)\}$ is summable. For $n \in \mathbb{N}$, define $f_n(k) = f(k)$ if $t \leq n$, $= 0$ otherwise. Then $f_n \in_F$ and $f_n(k)$ converges to $f(k)$ for all k. As $f_n = \mathbf{1}_{[0,n]}f$, we have by lemma 7.5.:

$$(f_n \odot m)(A) = \sum_{k \leq n, k \in A} f(k) \odot m(k)$$

Hence f is integrable and

$$(f \odot m)(A) = \lim_n (f_n \odot m)(A)$$

In particular

$$(f \odot m)(k) = \lim_n (f_n \odot m)(k) = f(k) \odot m(k)$$

(2) Suppose f integrable. Then $f \odot m$ is a summable sequence. By lemma 7.5. we have $(f \odot m)(k) = ((\mathbf{1}_{\{k\}} f) \odot m)(\mathbb{N})$, *i.e.* $(f \odot m)(k) = f(k) \odot m(k)$. Hence the sequence $\{f(k) \odot m(k)\}$ is summable. □

Lemma 7.9. *$\underline{w}$ is supposed to be σ-finite. Let $\{f_n\}$ be a sequence of integrable functions converging $\underline{w}$-a.e. to 0. Suppose the sequence $\{(f_n \odot m)(A)\}$ converges for every $A \in \mathcal{T}$. Then this sequence converges to 0, uniformly with respect to A.*

Proof. As for lemma 7.2.. □

Theorem 7.10. *[First theorem of convergence] The semi-variation $\underline{w}$ is supposed to be σ-finite. Let $\{f_n\}$ be a sequence of integrable functions such that:*

(1) f_n converges $\underline{w}$-a.e. to a function f

(2) For every $A \in \mathcal{T}$, $(f_n \odot m)(A)$ converges in F

Then f is integrable and $(f_n \odot m)(A)$ converges to $(f \odot m)(A)$ uniformly with respect to A. In other words $\lim_n((f_n - f) \odot m)^\bullet(T) = 0$.

Proof. Let $\{h_n\}$ a sequence of simple functions converging $\underline{w}$-a.e. to f. Let μ as defined by lemma 7.1. using the measures $\{f_n \odot m \mid n \in \mathbb{N}\} \bigcup \{h_n \odot m \mid n \in \mathbb{N}\}$. Then $f_n - h_n$ converges to 0 $\underline{w}$-a.e. and therefore μ-a.e..Let $\{D_k\}$ be an increasing sequence of members of $\mathcal{T}$ such that $f_n - h_n$ converges uniformly to 0 on D_k and $\lim_k \mu(T \setminus D_k) = 0$. As $\underline{w}$ is σ-finite, we may suppose $\underline{w}(D_k) < \infty$. Put $D = \bigcup_k D_k$ and $N = T \setminus D$. For every k, choose n_k such that $\sup\{\|f_{n_k}(t) - h_{n_k}(t)\| \mid t \in D_k\} \leq 1/k\underline{w}(D_k)$. Put $g_k = \mathbf{1}_{D_k \cup N} h_{n_k}$. Then $\{g_k\}$ is a sequence of simple functions which converges $\underline{w}$-a.e. to f .

For $A \subset N$ and every $n \in \mathbb{N}$ we have $(f_n \odot m)(A) = 0$ and $(h_n \odot m)(A) = 0$ and therefore:

$$\begin{aligned}
((g_k - f_{n_k}) \odot m)^\bullet(T) &= ((g_k - f_{n_k}) \odot m)^\bullet(D) \\
&\leq ((g_k - f_{n_k}) \odot m)^\bullet(D_k) + ((g_k - f_{n_k}) \odot m)^\bullet(D \setminus D_k) \\
&= ((g_k - f_{n_k}) \odot m)^\bullet(D_k) + (f_{n_k} \odot m)^\bullet(D \setminus D_k) \\
&\leq 1/k + (f_{n_k} \odot m)^\bullet(D \setminus D_k)
\end{aligned}$$

Let us apply the Vitali-Hahn-Saks theorem (Theorem 6.4.) to the sequence of measures $\{f_{n_k} \odot m \mid k \in \mathbb{N}\}$. For every $\varepsilon > 0$, there is η such that $\mu(C) \leq \eta \Longrightarrow (f_{n_k} \odot m)^\bullet(C) \leq \varepsilon$. But there exists K such that $k \geq K \Longrightarrow \mu(D \setminus D_k) \leq \eta$. Then we have $k \geq K \Longrightarrow (f_{n_k} \odot m)^\bullet(D \setminus D_k) \leq \varepsilon$. That proves $\lim_k (f_{n_k} \odot m)^\bullet(D \setminus D_k) = 0$.

It follows that $\lim_k((g_k - f_{n_k}) \odot m)^\bullet(T) = 0$. Then for every $A \in \mathcal{T}$, we have

$$\lim_k((g_k \odot m)(A) - (f_{n_k} \odot m)(A)) = 0$$

The sequence $\{(g_k \odot m)(A)\}$ converges and

$$\lim_k(g_k \odot m)(A) = \lim_k(f_{n_k} \odot m)(A) = \lim_n(f_n \odot m)(A)$$

This proves that f is integrable.

For every integrable function f, put $p(f) = (f \odot m)^\bullet(T)$. One has $p(f - f_{n_k}) \leq p(f - g_k) + p(g_k - f_{n_k})$ and therefore $\lim_k p(f - f_{n_k}) = 0$. But applying lemma 7.9. to the double sequence $\{f_n - f_{n'}\}$, one gets $\lim_{n,n'} p(f_n - f_{n'}) = 0$. Therefore $\lim_n p(f - f_n) = 0$. □ □

Lemma 7.11. *$\underline{w}$ is supposed to be σ-finite. Let f be a measurable function with values in F and $\{f_n\}$ be a sequence of simple functions such that:*

(1) f_n converges $\underline{w}$-a.e. to f

(2) for every $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges.

Let given a sequence of simple real functions $\{\lambda_n\}$ such that $\lambda_n(t) \leq 1$ and $\lim_n \lambda_n(t) = 1$ for every $t \in T$. Then for every $A \in \mathcal{T}$, one has $\lim_n(f_n \odot m)(A) = \lim_n((\lambda_n f_n) \odot m)(A)$

Proof. Let μ as defined by lemma 7.1. using the measures $\{f_n \odot m\}$ Let $\varepsilon > 0$. There is η such that $\mu(A) \leq \eta \Longrightarrow \|f_n \odot m)(A)\| \leq \varepsilon$. Then there exists C_η such that $\mu(C_\eta) \leq \eta$ and on C_η, $\lambda_n(t)$ converges uniformly to 1 and $f_n(t)$ converges uniformly to $f(t)$. Furthermore we may suppose $\underline{w}(C_\eta) < \infty$. Then we have

$$(f_n \odot m)(A) - ((\lambda_n f_n) \odot m)(A) = ((1 - \lambda_n f_n) \odot m)(A)$$

$$= ((1 - \lambda_n) f_n \odot m)(A \cap C_\eta) - (\lambda_n f_n \odot m)(A \cap (T \setminus C_\eta)) + (f_n \odot m)(A \cap (T \setminus C_\eta))$$

The following inequalities hold:

$$\|((1 - \lambda_n) f_n \odot m)(A \cap C_\eta)\| \leq \underline{w}(C_\eta) \sup_{t \in T}(\lambda_n(t) - 1)$$

$$\|(\lambda_n f_n \odot m)(A \cap (T \setminus C_\eta))\| \leq (f_n \odot m)^\bullet(T \setminus C_\eta) \leq \varepsilon$$

$$\|(f_n \odot m)(A \cap (T \setminus C_\eta))\| \leq (f_n \odot m)^\bullet(T \setminus C_\eta) \leq \varepsilon$$

There exists N such that $n \geq N \implies \sup_{t\in T}(\lambda_n(t) - 1) \leq \varepsilon/\underline{w}(C_\eta)$. For $n \geq N$, we get:

$$\|(f_n \odot m)(A) - ((\lambda_n f_n) \odot m)(A)\| \leq 3\varepsilon$$

This allows to conclude. □ □

Lemma 7.12. *$\underline{w}$ is supposed to be σ-finite. For every integrable function f, there is a sequence $\{f_n\}$ of simple functions such that*

(1) f_n converges $\underline{w}$-a.e. to f

(2) for all $A \in \mathcal{T}$, the sequence $\{(f_n \odot m)(A)\}$ converges.

(3) for all n, $\|f_n(.)\| \leq \|f(.)\|$.

Proof. Let $\{f_n(t)\}$ a sequence which fulfills the 2 first conditions. Let us consider a sequence $\{k_n\}$ of simple functions such that $0 \leq k_n(t) \leq \|f_n(t)\|$ et $\lim_n k_n(t) = \|f(t)\|$ m-a.e. and define $\lambda_n(t) = \inf\{1, k_n(t)/\|f_n(t)\|\}$. By replacing $f_n(t)$ with $\lambda_n(t)f_n(t)$, we get a sequence which by the preceding lemma fulfills the 3 conditions. □

Proposition 7.13. *$\underline{w}$ is supposed to be σ-finite. Let f be an integrable function with values in F and $h \in$ such that $\|f(.)\| \leq h$. Then*

$$\left\|\int f \odot m\right\| \leq \underline{w}(h)$$

Proof. Let $\{f_n\}$ a sequence as defined by lemma 7.12.. One has: $\|f_n(.)\| \leq h$, and therefore $\left\|\int f_n \odot m\right\| \leq \underline{w}(h)$ The announced inequality is obtained by passing to the limit. □

Corollary 7.14. *For every $h \in_+$, we have :*

$$\underline{w}(h) = \sup\left\{\left\|\int f \odot m\right\| \;\middle|\; f \text{ integrable}, \|f(.)\| \leq h\right\}$$

Recall that in a Banach space E, a sequence $\{x_n\}$ is said to be weakly summable iff the sequence $\{\langle x_n, y\rangle\}$ is summable for every $y \in E'$. This definition is equivalent to the following property: the linear mapping $t \mapsto \sum_n t_n x_n$ from $\mathbf{c_{00}}$ (the space of eventually null sequences) into E is continuous. Consequently there is a bijection from the space of weakly summable sequence onto the space $L(\mathbf{c_{00}}, E)$. A sequence $\{x_n\}$ and a linear mapping ψ corresponds to each other by this isomorphism iff $\psi(t) = \sum_n t_n x_n$ for every $t \in \mathbf{c_0}$.

Proposition 7.15. *Suppose $\underline{w}$ is finite. Then for every countable $\mathcal{T}$-partition $\{T_n\}$ of T and every sequence $\{y_n\}$ in $\mathfrak{B}_F$, the sequence $\{y_n \odot m(T_n)\}$ is weakly summable.*

Proof. For every finite subset A of $\mathbb{N}$ and every $t \in \mathbf{c_{00}}$ vanishing outside A, we have:

$$\sum_{n\in A} t_n(y_n \odot m(T_n)) = \sum_{n\in A} (t_n y_n) \odot m(T_n) \leq \underline{w}(A) \leq \underline{w}(T) < \infty$$

□

Lemma 7.16. *Suppose $\underline{w}$ is finite but not exhaustive. Then there exists a $\mathcal{T}$-partition $\{T_n\}$ of T and a sequence $\{y_n\}$ in $\mathfrak{B}_F$ such that $\{y_n \odot m(T_n)\}$ is weakly summable but not summable.*

Proof. Following Lemma 6.3. there exists $\delta > 0$ and a $\mathcal{T}$-partition of T such that $\underline{w}(T_n) > \delta$. Following the definition of $\underline{w}$ there exists for every $n \in \mathbb{N}$ a finite family $\{y_{n,k} \mid k \in K_n\}$ of elements of $\mathfrak{B}_F$ and a finite $\mathcal{T}$-partition $\{T_{n,k} \mid k \in K_n\}$ of T_n such that

$$\left\| \sum_{k\in K_n} y_{n,k} \odot m(T_{n,k}) \right\| > \delta$$

The sequence $\{y_{n,k} \odot m(T_{n,k})\}$ is weakly summable but not summable.. □

Recall that if $\underline{w}$ is finite and exhaustive then every bounded $\underline{w}$-measurable function is Bochner integrable hence integrable. The following corollary proves that this property characterizes exhaustivity.

Corollary 7.17. *Suppose $\underline{w}$ is finite but not exhaustive. Then there exists a bounded measurable non integrable function.*

Proof. Put $f(t) = y_n$ for $t \in T_n$. Suppose f is integrable. Then we have $y_n \odot m(T_n) = (f \odot m)(T_n)$ and the sequence $\{y_n \odot m(T_n)\}$ is summable. We get a contradiction. □

Recall that a Banach space is said to have the Bessaga-Pelszyńsky property iff every weakly summable sequence is summable. The classical theorem of Bessaga-Pelszyńsky states that a Banach space has the Bessaga-Pelszyńsky property if and only if it contains no copy of $\mathbf{c_0}$.

Corollary 7.18. *If G has the Bessaga-Pelszyńsky property and if $\underline{w}$ is finite, then $\underline{w}$ is exhaustive.*

We give more characterisations of the Bessaga-Pelszyńsky property in the following theorem.

Theorem 7.19. *For a Banach space G the following properties are equivalent:*

(0) G has the Bessaga-Pelszyńsky property

(1) G contains no copy of $\mathbf{c_0}$

(2) For any linear context $(E, F, G, \odot)$ *such that the contextual semi-variation* $\underline{w}$ *is finite,* $\underline{w}$ *is exhaustive*

(3) For any linear context $(E, F, G, \odot)$ *such that the contextual semi-variation* $\underline{w}$ *is finite, every* $\underline{w}$*-measurable bounded function is integrable.*

Proof. (0) $\Longleftrightarrow$ (1) is the Bessaga-Pelszyńsky theorem.

(1) $\Longrightarrow$ (2) : if (2) fails the (3) fails by virtue of corollary 7.17.

(2) $\Longrightarrow$ (3) : indeed every bounded $\underline{w}$-measurable function is Bochner integrable

(3) $\Longrightarrow$ (2) by corollary 7.17.

(2) $\Longrightarrow$ (1) : Example 6.10. shows that if $G = \mathbf{c_0}$, then there exists a bilinear context for which $\underline{w}$ is not exhaustive. This remains true if G contains a copy of $\mathbf{c_0}$. Hence if (1) fails then (2) fails too. □

Swartz ([SW2] Theorem 1) proved the following result: Let X be a arbitrary infinite dimentional space. The there exists a sequence $\{m_n\}$ in $L(X, \mathbf{c_0})$ such that $\{m_n x\}$ is a summable sequence in $\mathbf{c_0}$ for every $x \in X$ and a bounded sequence $\{\xi_n\}$ in X such that the sequence $\{m_n \xi_n\}$ is not summable. This result may be translated easily into a sharpening of the part (2) $\Longrightarrow$ (1) in the preceding theorem.

We now go on to a convergence theorem of Vitali type. The following theorem ([DU] proposition I.1.17 and corollary I.5.4) is usefull.

Theorem 7.20. *Let* $\{\nu_i\}$ *be an arbitrary family of G-valued measures. Suppose that for all* i, ν_i *is absolutely continuous with respect to a positive measure* μ. *Then the following properties are equivalent:*

(1) $\{\nu_i\}$ *is uniformly* σ*-additive*

(2) For every decreasing sequence $\{A_k\}$ *in* $\mathcal{T}$ *whose intersection is empty, the sequence* $\{\nu_i(A_k)\}$ *converges to 0 uniformly with respect to* i

(3) For every $\varepsilon > 0$ *there exists* η *such that*

$$\mu(A) \leq \eta \Longrightarrow \nu_i^\bullet(A) \leq \varepsilon$$

Proposition 7.21. *Suppose $\underline{w}$ is σ-finite. Let $\{f_n\}$ be a sequence of F-valued integrable functions such that f_n converges $\underline{w}$-a.e. to a function f. If the sequence $\{f_n \odot m\}$ is uniformly σ-additive, then the sequence $\{(f_n \odot m)(A)\}$ converges for every $A \in \mathcal{T}$.*

Proof. Let μ the measure associated with the sequence $\{f_n \odot m\}$ by lemma 7.1. Let $\varepsilon > 0$. There exists η such that $\mu(A) \leq \eta \Longrightarrow \|(f_n \odot m)(A)\| \leq \varepsilon$. Then there exists C_η such that $\mu(T \setminus C_\eta) \leq \eta$ and that f_n converges uniformly to f on C_η. As $\underline{w}$ is σ-additive, it may be assumed that $\underline{w}(C_\eta) < \infty$. We have:

$$(f_n \odot m)(A) - (f_p \odot m)(A) = ((f_n - f_p) \odot m)(A \cap C_\eta)$$
$$+(f_n \odot m)(A \cap (T \setminus C_\eta)) + (f_p \odot m)(A \cap (T \setminus (C_\eta))$$

and the following majorations:

$$\|((f_n - f_p) \odot m)(A \cap C_\eta)\| \leq \underline{w}(C_\eta) \sup_{t \in A \cap C_\eta} \|f_n(t) - f_p(t)\|$$
$$\|(f_n \odot m)(A \cap (T \setminus C_\eta)\| \leq (f_n \odot m)^\bullet(T \setminus C_\eta)$$
$$\|(f_p \odot m)(A \cap (T \setminus C_\eta)\| \leq (f_p \odot m)^\bullet(T \setminus C_\eta)$$

There exists N such that $n \geq N \Longrightarrow \sup_{t \in A \cap C_\eta} \|f_n(t) - f_p(t)\| \leq \varepsilon$. Then

$$n \geq N \quad \Longrightarrow \quad \|(f_n \odot m)(A) - (f_p \odot m)(A)\| \leq \underline{w}(C_\eta)\varepsilon + 2\varepsilon$$

Hence the sequence $\{(f_n \odot m)(A)\}$ is a Cauchy sequence for every $A \in \mathcal{T}$. □

Corollary 7.22. *Suppose $\underline{w}$ is σ-finite. Then a F-valued function f is integrable iff there exists a sequence $\{f_n\}$ of simple functions such that:*

(1) f_n converges $\underline{w}$-a.e. to f

(2) the measures $f_n \odot m$ are uniformly σ-additive.

Proof. If f is integrable, then there exists a sequence $\{f_n\}$ of simple functions such that $(f_n \odot m)(A)$ converges for every $A \in \mathcal{T}$. The Vitali-Hahn-Saks theorem asserts that (2) holds.

Conversely if (1) and (2) hold then proposition 7.17. asserts that $(f_n \odot m)(A)$ converges for every $A \in \mathcal{T}$. Hence f is integrable. □

Theorem 7.23. *[Convergence theorem of Vitali type] Suppose $\underline{w}$ is σ-finite. Le $\{f_n\}$ be a sequence of integrable functions such that:*

(1) f_n converges $\underline{w}$.a.e. to a function f

(2) the measures $f_n \odot m$ are uniformly additive

Then f is integrable and $(f_n \odot m)(A)$ converges uniformly to $(f \odot m)(A)$. In other words $\lim_n((f_n - f) \odot m)^\bullet(T) = 0$.

Proof. Proposition 7.21. asserts that the hypothesis of theorem 7.10. are satisfied. □

To end this section we state a result that permits integration by pieces.

Proposition 7.24. *Let $\{T_k\}$ be a countable $\mathcal{T}$-partition of T. Suppose a function f is such that $1_{T_k} f$ is integrable for every $k \in \mathbb{N}$. Then f is integrable iff the sequence $\{\int 1_{A_k} f \odot m\}$ is summable whenever $A_k \in \mathcal{T}$ and $A_k \subset T_k$.*

Proof. It is easy to prove that the condition is necessary. Suppose it is satisfied. Put $f_{(n)} = \sum_{k \leq n} 1_{T_k} f$. Then $f_{(n)}$ converges everywhere to f. For $A \in \mathcal{T}$ we have:

$$(f_{(n)} \odot m)(A) = \sum_{k \leq n} \int (1_{A \cap T_k} f) \odot m$$

Hypothetically the right hand member converges for $n \to \infty$. Following Theorem 7.10. f is integrable. □

8. Tensor integration

In this section we shall first be concerned with the special case where m is a vector measure and G is the vector space $F \widehat{\otimes}_\varepsilon E$ (the completed space of $F \otimes E$ with respect of the norm ε). The symbol $\odot$ has to be replaced by $\otimes$. After which we will replace the ε-norm with the π-norm. We put $\mathcal{L}^1_F(m) = \mathcal{L}^1_F(m^\bullet)$.

A essential tool will be the following notion that enables us to state generalizations of the Orlicz-Pettis's theorem. Let E be a Banach space and H be a subset of E'. Then H is said to have the Orlicz-Pettis's property iff every E-valued sequence x such that the sequence $\{\langle x_n, y\rangle\}$ is summable for every $y \in H$ is summable in E. The classical Orlicz-Pettis's theorem asserts that E' has the Orlicz-Pettis's property. This notion is discussed in [TH], Appendice II. The following proposition is an easy consequence of the definitions.

Proposition 8.1. *Let $(T, \mathcal{T})$ be a measurable space, E a Banach space and H a subset of E'. Let m be a map from $\mathcal{T}$ into E such that the map $A \mapsto \langle m(A), y\rangle$ is a measure for* *all $y \in H$. If H has the Orlicz-Pettis's property with respect to E, then m is a "strong measure", i.e. for all countable $\mathcal{T}$-partition $\{T_n\}$ of T, one has*

$$m(T) = \sum_n m(T_n)$$

for the (norm) convergence in E.

Let $f \in_F$. If $f = \sum \xi_i \mathbf{1}_{A_i}$, ($\{A_i\}$ being a finite partition of T and $\xi_i \in F$), then:

$$\int f \otimes m = \sum \xi_i \otimes m(A_i)$$

Lemma 8.2. *Let $f \in_F$. Then:*

$$\left\| \int f \otimes m \right\|_\varepsilon \leq \|f\|_{\mathcal{L}^1_F(m)}$$

Proof. One has:

$$\begin{aligned}
\|f \otimes m\|_\varepsilon &= \sup \left\{ \sum_i \langle \xi_i, y' \rangle \langle m(A_i), x' \rangle \mid x' \in \mathfrak{B}(E'), y' \in \mathfrak{B}(F') \right\} \\
&= \sup \left\{ \int \langle f(.), y' \rangle \, m_{x'} \mid x' \in \mathfrak{B}(E'), y' \in \mathfrak{B}(F') \right\} \\
&\leq \sup \left\{ \int |\langle f(.), y' \rangle| \, |m_{x'}| \mid x' \in \mathfrak{B}(E'), y' \in \mathfrak{B}(F') \right\} \\
&\leq \sup \left\{ \int \|f(.)\| \, |m_{x'}| \mid x' \in \mathfrak{B}(E') \right\} \\
&= m^\bullet(\|f(.)\|)
\end{aligned}$$

□

Proposition 8.3. *For $F \otimes E$ endowed with the ε-norm, the contextual semi-norm $\underline{w}$ is $m^\bullet$.*

Proof. By lemma 8.2., for every $f \in_F$, we have $\left\| \int f \otimes m \right\|_\varepsilon \leq m^\bullet(\|f(.)\|)$ and therefore $\underline{w}(h) \leq m^\bullet(h)$ for every $h \in_{\mathbf{R}}$.

For the converse inequality, let us pick $y \in F$ such that $\|y\| = 1$ and consider the measure $(F \otimes E)$-valued measure $y \otimes m$ such that $(y \otimes m)(A) = y \otimes m(A)$ for all $A \in \mathcal{T}$.

A easy checking gives: $(y \otimes m)^\bullet = m^\bullet$. On the other hand:

$$\begin{aligned}
(y \otimes m)^\bullet(h) &= \sup\left\{\left\|\int f\; y \otimes m\right\| \;\Big|\; |f| \leq h\right\} \\
&= \sup\left\{\sum_i z_i (y \otimes m)(A_i) \;\Big|\; \left|\sum_i z_i \mathbf{1}_{A_i}\right| \leq h\right\} \\
&= \sup\left\{\left\|\sum_i (z_i y) \otimes m(A_i)\right\| \;\Big|\; \left|\sum_i z_i \mathbf{1}_{A_i}\right| \leq h\right\} \\
&= \sup\left\{\left\|\int (\sum_i z_i \mathbf{1}_{A_i} y) \otimes m\right\| \;\Big|\; \left\|\sum_i z_i \mathbf{1}_{A_i}(.) y\right\| \leq h\right\} \\
&\leq \underline{w}(h)
\end{aligned}$$

This proves $m^\bullet \leq \underline{w}$. □

Consequently the following definition agrees with Definition 5.4..

Definition 8.4. A F-valued function f is said to be Bochner-integrable with respect to the E-valued measure m iff $f \in \mathcal{L}^1_F(m)$.

Definition 8.5. A F-valued function is said to be scalarly integrable with respect to the E-valued measure m iff $\langle f(.), y' \rangle \in \mathcal{L}^1(m_{x'})$ for all $x' \in E'$ and all $y' \in F'$.

In this case for all $x' \in E'$, all $y' \in F'$ and all $A \in \mathcal{T}$, ${}_{\mathrm{w}}\!\int_A f \otimes m$ denotes the element of $(E' \otimes F')^*$ such that

$$\left\langle {}_{\mathrm{w}}\!\int_A f \otimes m, x' \otimes y' \right\rangle = \int_A \langle f(.), y' \rangle\, m_{x'}$$

A F-valued m-measurable function f is said to be ε-Pettis-intégrable iff:

(1) f is m-scalarly integrable

(2) ${}_{\mathrm{w}}\!\int_A f \otimes m \in F \,\widehat{\otimes}_\varepsilon\, E$ for all $A \in \mathcal{T}$.

Example 8.6. Every Bochner-integrable function is ε-Pettis-integrable.

Plainly if f is ε-Pettis-integrable, then the map $A \mapsto {}_{\mathrm{w}}\!\int_A f \otimes m$ is a weak measure for the duality $(F \,\widehat{\otimes}_\varepsilon\, E, F' \otimes E')$. But we have more:

Theorem 8.7. *If f is ε-Pettis-integrable, then $A \mapsto {}_{\mathrm{w}}\!\int_A f \otimes m$ is a strong measure with values in $F \,\widehat{\otimes}_\varepsilon\, E$. This measure will be denoted by $(f \otimes m)_\varepsilon$.*

Proof. Indeed following [TH] (Appendice II, Corollaire II.7), $F' \otimes E'$ has the Orlicz-Pettis-property for $F \,\widehat{\otimes}_\varepsilon\, E$. □

Theorem 8.8. *Any F-valued ε-Pettis-integrable function is integrable in the sense of Definition 7.3. and conversely.*

Proof. (1) Suppose f is ε-Pettis-integrable. If is Bochner-integrable, consider a sequence $\{g_n\}$ in ${}_F$ such that $f = \lim_n g_n$ in $\mathcal{L}^1_F(m)$. By extracting a subsequence we may suppose g_n converges to f m-a.e.. Then f is integrable in the sense of definition 7.3.. Consider now the general case. For every ε-Pettis-integrable function f let us put $p(f) = (f \otimes m)^\bullet_\varepsilon(T)$ Let us take a countable $\mathcal{T}$-partition $\{T_k\}$ such that que $1_{T_k} f \in \mathcal{L}^1_F(m)$ for all k. For every k, there exists a sequence $\{g_{n,k} \mid n \in \mathbb{N}\}$ in ${}_F(m)$ such that $g_{n,k}$ vanishes outside T_k, $g_{n,k}$ converges m-a.e. to $f 1_{T_k}$ and $p(f 1_{T_k} - g_{n,k}) \leq \frac{1}{n} 2^{-k}$. Let us put $S_k = \cup_{h \leq k} T_h$ and $R_k = T \setminus S_k$.

For all n, let $K(n)$ be such that $p(f 1_{R_{K(n)}}) \leq \frac{1}{n}$. Put $g_n = \sum_{k \leq K(n)} g_{n,k}$. Then g_n converges m-a.e. to f and we have:

$$\begin{aligned} p(f - g_n) &\leq p(f 1_{R_{K(n)}} - g_n) + p(f 1_{R_{K(n)}}) \\ &\leq \sum_{k \leq K(n)} p(f 1_{T_k} - g_{n,k}) + p(f 1_{R_{K(n)}}) \\ &\leq 2/n \end{aligned}$$

As the sequence $\{g_n\}$ converges m-a.e. and $\lim_n p(f - g_n) = 0$, f is integrable in the sense of Definition 7.3..

(2) Suppose now f is integrable in the sense of Definition 7.3.. Pick $x' \in E'$ and $y' \in F'$. One have $\lim_n \langle f_n(t), y' \rangle = \langle f(t), y' \rangle$ m-p.p.. On the other hand for every $A \in \mathcal{T}$, the sequence $n \mapsto \int_A \langle f_n(.), y' \rangle \, m_{x'}$ converges. Hence by [PB] (théorème 15.10), $\langle f_n(.), y' \rangle \in \mathcal{L}^1(m_{x'})$ and

$$\lim_n \int_A \langle f_n(.), y' \rangle \, m_{x'} = \int_A \langle f(.), y' \rangle \, m_{x'}$$

So f is scalarly integrable.

For $A \in \mathcal{T}$, put $P(A) = \lim_n \int_A f_n \otimes m$. One have $P(A) \in F \widehat{\otimes}_\varepsilon E$. By the Vitali-Hahn-Saks theorem, P is a strong measure. Moreover for all $x' \in E'$ and all $y' \in F'$,

$$\langle P(A), x' \otimes y' \rangle = \lim_n \int_A \langle f_n(.), y' \rangle \, m_{x'} = \int_A \langle f(.), y' \rangle \, m_{x'}$$

Therefore f est ε-Pettis-integrable. □

Example 8.9. If $E = \mathbb{R}$, *i.e.* if m is a scalar measure, then Bochner integrability of a F-valued function in the sense of definition 8.4. coincides with Bochner integrability of a F-valued function in the usual sense for m-measurable functions. The ε-Pettis-integrability is nothing but the usual Pettis-integrability. A glance at theorem 8.8 shows that a F-valued function f is Pettis integrable if there exists a sequence $\{f_n\}$ of simple functions converging m-a.e. to f such that $\int_A f_n m$ converges for every $A \in \mathcal{T}$.

Example 8.10. Taking $F = \mathbb{R}$ we obtain the following definition. If m is a E-valued measure a scalar m-measurable function f is Pettis-integrable iff

(1) $f \in \mathcal{L}^1(m_{x'})$ for every $x' \in E'$

(2) defining ${}_{\mathrm{w}}\!\int_A fm \in E'^*$ by

$$\forall x' \in E', A \in \mathcal{T} \qquad \left\langle {}_{\mathrm{w}}\!\int_A fm, x' \right\rangle = \int_A fm_{x'}$$

we have ${}_{\mathrm{w}}\!\int_A fm \in E$ for all $A \in \mathcal{T}$.

By theorem 8.8. this definition is equivalent to the following: there exists a sequence $\{f_n\}$ of $\mathcal{T}$-simple functions such that f_n converges m-a.e. to f and $\int_A f_n m$ converges for every $A \in \mathcal{T}$. The standard theory of integrability of scalar functions with respect to a vector measure shows the above definition is in fact equivalent to the Bochner-integrability *i.e.* to the property $f \in \mathcal{L}^1(m)$.

We go on to the case $F \otimes E$ is equipped with the π-norm. We suppose that F has the the approximation property, so as $F \widehat{\otimes}_\pi E$ is a subspace of $F \widehat{\otimes}_\varepsilon E$.

Definition 8.11. A F-valued m-measurable function will be said to be π-Pettis-integrable if it is ε-Pettis-integrable and if furthermore $f \otimes m$ is a $(F \widehat{\otimes}_\pi E)$-valued (strong) measure.

For f to be ε-Pettis-intégrable, it is not sufficient $f \otimes m$ to have its values in $F \widehat{\otimes}_\pi E$. It is necessary $f \otimes m$ to be strongly σ addditive. The following theorem gives a very general condition for this condition to be fulfilled.

Theorem 8.12. *If $F' \otimes E'$ has the Orlicz-Pettis's property with respect to $F \widehat{\otimes}_\pi E$, then every ε-Pettis-integrable function f such that $f \otimes m$ has values in $F \widehat{\otimes}_\pi E$ is π-Pettis-integrable.*

In particular if F et E are separable and if F has the metric approximation property, then every ε-Pettis-integrable function f such that $f \otimes m$ is $(F \widehat{\otimes}_\pi E)$-valued is π-Pettis-integrable.

Proof. The first asssertion follows from Proposition 8.1..

For the second assertion call to [PB] (théorème XI.37): the metric approximation implies:

$$\forall X \in F \widehat{\otimes}_\pi E \qquad \|X\|_\pi = \sup\{\langle X, B\rangle \mid B \in E' \otimes F', \|B\|_\varepsilon \le 1\}$$

By [TH] (théorème II.3 de l'appendice II), this implies that $F' \otimes E'$ has the Orlicz-Pettis's property for $F \widehat{\otimes}_\pi E$. The conclusion is now immediate. □

Proposition 8.13. *If $f \in_{F}$, then*

$$\left\| \int f \otimes m \right\|_\pi \le \int \|f(.)\| \operatorname{var}(m)$$

Proof. It is a particular case of Proposition 5.2.. □

Corollary 8.14. *The contextual semi-variation $\underline{w}$ attached to the integration with values in $F \widehat{\otimes}_\pi E$ is majorized by* $\operatorname{var}(m)$. *In particular if* $\operatorname{var}(m)$ *is σ-finite then the same is true for $\underline{w}$.*

Theorem 8.15. *Suppose $\underline{w}$ σ-finite. Any F-valued π-Pettis-integrable function is integrable in the sense of Definition 7.3. and conversely.*

Proof. (1) Suppose f is π-Pettis-integrable. If is Bochner-integrable, consider a sequence $\{g_n\}$ in $_F$ such that $f = \lim_n g_n$ in $\mathcal{L}^1_F(\underline{w})$. By extracting a subsequence we may suppose g_n converges to f ma.e.. Then f is integrable in the sense of definition 7.3.. Consider now the general case. For every π-Pettis-integrable function f let us put $p(f) = (f \otimes m)^\bullet_\pi(T)$ Let us take a countable $\mathcal{T}$-partition $\{T_k\}$ such that que $\mathbf{1}_{T_k} f \in \mathcal{L}^1_F(\underline{w})$ for all k. For every k, there exists a sequence $\{g_{n,k} \mid n \in \mathbb{N}\}$ in $_F(m)$ such that $g_{n,k}$ vanishes outside T_k, $g_{n,k}$ converges m-a.e. to $f\mathbf{1}_{T_k}$ and $p(f\mathbf{1}_{T_k} - g_{n,k}) \le \frac{1}{n}2^{-k}$. Let us put $S_k = \cup_{h \le k} T_h$ and $R_k = T \setminus S_k$.

For all n, let $K(n)$ be such that $p(f\mathbf{1}_{R_{K(n)}}) \le \frac{1}{n}$. Put $g_n = \sum_{k \le K(n)} g_{n,k}$. Then g_n converges m-a.e. to f and we have:

$$\begin{aligned} p(f - g_n) &\le p(f\mathbf{1}_{R_{K(n)}} - g_n) + p(f\mathbf{1}_{R_{K(n)}}) \\ &\le \sum_{k \le K(n)} p(f\mathbf{1}_{T_k} - g_{n,k}) + p(f\mathbf{1}_{R_{K(n)}}) \\ &\le 2/n \end{aligned}$$

As the sequence $\{g_n\}$ converges m-a.e. and $\lim_n p(f - g_n) = 0$, f is integrable in the sense of Definition 7.3..

(2) Suppose now f is integrable in the sense of Definition 7.3.. Then it is ε-Pettis-integrable by Theorem 8.8.. Furthermore $f \otimes m$ is a strong measure with values in $F \widehat{\otimes}_\pi E$. Hence f is π-Pettis-integrable. □

References

[1] R. Bartle *A general bilinear integral*, Studia Math., **XV**, pp. 337-350
[2] I. Dobrakov, *On Integration in Banach spaces I.*, Czech. Math.Journal, **20** (1970), pp. 511-536
[3] I. Dobrakov, *On Integration in Banach spaces II.*, Czech. Math.Journal, **20** (1970), pp. 680-695
[4] N. Dinculeanu, *Vector Measures*, ,Pergamon Press, 1967
[5] N. Dunford, J. Schwartz, *Linear Operators I.*, Interscience, 1958
[6] J. Diestel, JR. Uhl, *Vector Measures*, A.M.S. Math Surveys 15, 1977
[7] A. Grothendieck, *Produits tensoriels topologiques et Espaces nucléaires*, Memoirs of the A.M.S.,**16** (1955)
[8] A. Grothendieck, *Sur les applications faiblement compactes d'espaces de type C(K)*, Canadian J. Math, **5** (1953)
[9] M. Guessous, *Intégration de fonctions vectorielles et de multi-applications par rapport à une multi-mesure* Thèse, Université de Rabat, 1988
[10] B. Jefferies, *Evolution Processes and the Feynman-Kac Formule* Kluwer, 1995
[11] B. Jefferies, S.Okada, *Bilinear Integration*, Rocky Mountain Math. Journal, to appear
[12] P.R. Halmos, *Measure Theory*, Van Nostrand, 1967
[13] N.J. Kalton, *Topologies on Riesz groups ans applications to Measure Theory*, Proc. London Math. Soc., **28 (3)** (1974), pp. 253-273
[14] I. Kluvanek, G. Knowles, *Vector Measures and Control Systems*, North Holland, 1976
[15] P.W. Lewis, *Some regularity conditions on vector measures with finite semivariation*, Rev. Roumaine Math. Pures Appli., **15** (1970)
[16] T. Panchapagesan, *On the Distinguishing Features of the Dobrakov Integral*, Divulgaciones Matemáticas, **3 (1)** (1995), 79-114
[17] R. Pallu de La Barrièrre, *Intégration* Editions Ellipses, Paris, 1997
[18] C. Swartz, *Integrability for the Dobrakov integral*, Szec. Math J, **30** (1980), pp. 640-646
[19] C. Swartz, *Integrating bounded functions for the Dobrakov integral*, Math. Slovaka, **33** (1983), pp. 141-144
[20] C. Swartz, *Integrating bounded functions for the Dobrakov integral*, Correction, Math. Slovaka, **35** (1985), pp. 98-98
[21] G. Thomas, *L'intégration par rapport à une mesure de Radon vectorielle*, Ann. Institut Fourier, **XX (2)** (1970)
[22] M.M. Wilhelm, *Integration of functions with values in a normed group* Bulletin de l'Académie Polonaise des Sciences, **XX (11)** (1972), pp. 911-916
[23] M.M. Wilhelm, *Real integrable spaces*, Colloquium Mathematicum, **XXXII (2)** (1975), pp. 233-248

INSTITUT DE MATHÉMATIQUES, UNIVERSITÉ PIERRE ET MARIE CURIE, PLACE JUSSIEU, 75005 PARIS

E-mail address: `rpbmath.jussieu.fr` . URL: http://www.math.jussieu.fr/~rpb