SUFFICIENT CONDITIONS FOR STARLIKENESS II

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Abstract. In this paper we study a differential subordination of the form:

$$\alpha zp'(z) + \alpha p^2(z) + (\beta - \alpha)p(z) \prec h(z),$$

where

$$h(z) = \alpha nzq'(z) + \alpha q^2(z) + (\beta - \alpha)q(z),$$

with $\alpha > 0$, $\alpha + \beta > 0$, and the function q is convex with $q(0) = 1$, and

$$Re\,q(z) > \frac{\alpha - \beta}{2\alpha}.$$

Our results are obtained by using the method of differential subordinations developed in [1], [2] and [3]. For $\beta = 1$, $q(z) = 1 + \lambda z$ and $n = 1$ this problem was studied in [4].

1. Introduction and preliminaries

Let A_n denote the class of functions f of the form:

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, \ z \in U,$$

which are analytic in the unit disc U.

Let $A = A_1$ and let $S^* = \left\{f \in A, \ Re\frac{zf'(z)}{f(z)} > 0, \ z \in U\right\}$ be the class of starlike functions in U.

We will use the following notions and lemmas to prove our results.

If f and g are analytic functions in U,then we say that f is subordinate to g written $f \prec g$, of $f(z) \prec g(z)$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$. If g is univalent then $f \prec g$, if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma A. *([1], [2], [3]) Let q be univalent in $\overline{U}$ with $q'(\zeta) \neq 0$, $|\zeta| = 1$, $q(0) = a$ and let $p(z) = a + a_nz^n + \dots$ be analytic in U, with $p(z) \not\equiv a$, and $n \geq 1$.*

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If $p(z) \not\prec q(z)$ then there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ and there is $m \geq n$ such that:

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

The function $L(z,t)$, $z \in U$, $t \in I = [0, \infty)$ is called a Loewner chain or a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + a_3(t)z^3 + \ldots$ for $z \in U$ is analytic and univalent in U for any $t \in I$ and if $L(z,t_1) \prec L(z,t_2)$ when $0 \leq t_1 \leq t_2$.

Lemma B. *([7]) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in [0,1]$ and $M > 0$ such that:*

(i) $L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for each $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.

(ii) There is a function $p(z,t)$ analytic in U for any $t \geq 0$ measurable in $[0,\infty)$ for any $z \in U$, with $Re\, p(z,t) > 0$ for $z \in U$, $t \geq 0$ such that

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z}p(z,t), \text{ for } |z| < r \text{ and for almost any } t \geq 0.$$

2. Main results

Theorem 1. *Let q be a convex function in U, with $q(0) = 1$,*

$$Re\, q(z) > \frac{\alpha - \beta}{2\alpha}, \quad \alpha > 0, \quad \alpha + \beta > 0 \tag{1}$$

and let

$$h(z) = \alpha n z q'(z) + \alpha q^2(z) + (\beta - \alpha)q(z) \tag{2}$$

If the function $p(z) = 1 + p_n z^n + \ldots$ satisfies the condition:

$$\alpha z p'(z) + \alpha p^2(z) + (\beta - \alpha)p(z) \prec h(z) \tag{3}$$

where h is given by (2) then $p(z) \prec q(z)$ and q is the best dominant.

Proof. Let

$$L(z,t) = \alpha(n+t)zq'(z) + \alpha q^2(z) + (\beta - \alpha)q(z) = \psi(q(z), (n+t)zq'(z)) \tag{4}$$

If $t = 0$ we have $L(z,0) = \alpha n z q'(z) + \alpha q^2(z) + (\beta - \alpha)q(z) = h(z)$. We will show that condition (3) implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

From (4) we easily deduce:

$$\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} = (n+t)\left[1+\frac{zq''(z)}{q'(z)}\right] + 2q(z) + \frac{\beta-\alpha}{\alpha}$$

and by using the convexity of q and condition (1) we obtain:

$$Re\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \geq 0$$

. Hence by Lemma B we deduce that $L(z,t)$ is a subordination chain. In particular, the function $h(z) = L(z,0)$ is univalent and $h(z) \prec L(z,t)$, for $t \geq 0$. If we suppose that $p(z)$ is not subordinate to $q(z)$, then, from Lemma A, there exist $z_0 \in U$, and $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ with $|\zeta_0| = 1$, and $z_0 p'(z_0) = (n+t)\zeta_0 q'(\zeta_0)$, with $t \geq 0$. Hence

$$\psi_0 = \psi(p(z_0), z_0 p'(z_0)) = \psi(q(\zeta_0), (n+t)\zeta_0 q'(\zeta_0)) = L(\zeta_0, t),\ t \geq 0,$$

Since $h(z_0) = L(z_0, 0)$, we deduce that $\psi_0 \notin h(U)$, which contradicts condition (3). Therefore, we have $p(z) \prec q(z)$ and $q(z)$ is the best dominant. □

If we let $p(z) = \frac{zf'(z)}{f(z)}$, where $f \in A$, then Theorem 1 can be written in the following equivalent form:

Theorem 1'. *Let q be a convex function with $q(0) = 1$, and*

$$Re\, q(z) > \frac{\alpha-\beta}{2\alpha},\ \alpha > 0,\ \alpha+\beta > 0.$$

If $f \in A_n$, with $\frac{f(z)}{z} \neq 0$, $z \in U$, satisfies the condition:

$$\frac{\alpha z^2 f''(z)}{f(z)} + \beta\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in U$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

an q is the best dominant.

3. Particular cases

1) If we let $\alpha = 1$, $\beta \geq 1$, and $q(z) = \frac{1+z}{1-z}$, then

$$h(z) = \frac{2nz}{(1-z)^2} + \left(\frac{1+z}{1-z}\right)^2 + \gamma\frac{1+z}{1-z} \quad \text{with} \quad \gamma = \beta - 1 \geq 0.$$

If $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, then

$$h(e^{i\theta}) = \frac{-n}{2\sin^2\frac{\theta}{2}} - \cot^2\frac{\theta}{2} + \gamma\, i\cot\frac{\theta}{2} = u + iv$$

and the domain *D=h(U)* is the exterior of the parabola $u = -\frac{n}{2} - \frac{n+2}{2\gamma^2}v^2$. If $\gamma = 0$, then D is the complex plane slit along the half-line $v = 0$ and $u \leq -\frac{n}{2}$.

Using Theorema 1' we deduce the following criterion for starlikeness:

$$\text{If } f \in A_n \text{ and } \frac{z^2 f''(z)}{f(z)} + \beta\frac{zf'(z)}{f(z)} \in D \text{ then } f \in S^*.$$

2) If we let $\alpha = 1$, $\beta = 0$, and $q(z) = \frac{1}{1-z}$, then, $h(z) = \frac{z(n+1)}{(1-z)^2}$ and *h(U)* is the complex plane slit along the half-line *v=0* and $u \leq -\frac{n+1}{4}$. Using Theorema 1' we deduce the following criterion for starlikeness of order $\frac{1}{2}$: If $f \in A_n$ with $\frac{f(z)}{z} \neq 0$ satisfy the condition:

$$\frac{z^2 f''(z)}{f(z)} \prec \frac{(n+1)z}{(1-z)^2} \quad \text{then} \quad Re\frac{zf'(z)}{f(z)} > \frac{1}{2}.$$

In particular:

$$Re\left[\frac{z^2 f''(z)}{f(z)}\right] > -\frac{n+1}{4} \quad \Rightarrow \quad Re\frac{zf'(z)}{f(z)} > \frac{1}{2} \tag{5}$$

Example 1.

$$\text{If } f(z) = \frac{1}{\lambda}\sin\lambda z \text{ then } f \in A_2, \text{ and } \frac{f''(z)}{f(z)} = -\lambda^2,$$

and by using (5) we deduce that $f_\lambda \in S^*\left(\frac{1}{2}\right)$ for $|\lambda| \leq \frac{\sqrt{3}}{2}$.

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