STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIII, Number 2, June 1998

# INVARIANT SETS IN MENGER SPACES

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**Abstract.** The purpose of the paper is to extend some results regarding the self-similar sets from the case of the ordinary metric spaces to the case of probabilistic metric spaces, introduced by K. Menger.

## 1. Introduction

In recent years the interest for sets having non-integer Hausdorff dimension is growing. There were named fractals by Mandelbrot. The most known fractals are invariant sets with respect to a system of contraction maps, especially the so called self-similar sets. In a famous work, Hutchinson [4] first studied the invariant sets systematically in a general framework. He proved among others the following:*Let X be a complete metric space and* $f_1, \dots, f_m : X \to X$ *be contraction maps. Then there exists a unique compact set* $K \subseteq X$ *such that* $K = \bigcup_{i=1}^{m} f_i(K)$. If the maps $f_i$ are similitudes, this invariant set $K$ is said to be *self-similar.*

Our aim in this work is to generalize the above result for probabilistic metric spaces introduced in 1942 by K. Menger [5] who generalized the theory of metric spaces, to the development of which he already brought a major contribution. He proposed to replace the distance $d(x, y)$ by a distribution function $F_{x,y}$ whose value $F_{x,y}(t)$, for any real number $t$, is interpreted as the probability that the distance between $x$ and $y$ is less than $t$. The theory of probabilistic metric spaces was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [8].

The study of contraction mappings for probability metric spaces was initiated by V.M.Sehgal [10],[11], H.Sherwood [13],[14], and A.T.Bharucha-Reid [1], [12]. For more recently papers dealing with generalizations and applications one can consult [2] and [6].

In section 2 we shall recall some fundamental notions from the theory of probabilistic metric spaces and prove some new results on the probabilistic Hausdorff-Pompeiu metric (Propositions 2.4 and 2.5). In section 3 we prove our main result (Theorem 3.1).

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1991 *Mathematics Subject Classification.* 60A10, 28A78, 28A80.

## 2. Preliminaries

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$. A mapping $F : \mathbf{R} \to [0,1]$ is called a ***distribution function*** if it is non-decreasing, left continuous with $\inf F = 0$. By $\Delta$ we shall denote the set of all distribution functions $F$. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x,y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbf{R}$ by $F_{x,y}(t)$, respectively. The ordered pair $(X, \mathcal{F})$ is a ***probabilistic metric space*** if $X$ is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbf{R}$;

2) $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3) if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s+t) = 1$.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following conditions are satisfied:

4) $T(a,1) = a$ for every $a \in [0,1]$;

5) $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$

6) if $a \geq c$ and $b \geq d$ then $T(a,b) \geq T(c,d)$;

7) $T(a, T(b,c)) = T(T(a,b), c)$ for every $a, b, c \in [0,1]$.

We list here the simplest:

$T_1(a,b) = max\{a+b-1, 0\}$,

$T_2(a,b) = ab$,

$T_3(a,b) = Min(a,b) = min\{a,b\}$,

A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, $T$ is a t-norm, and

8) $F_{x,y}(s+t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbf{R}_+$.

The $(t,\epsilon)$-topology in a Menger space was introduced in 1960 by B. Schweizer and A. Sklar [7]. The base for the neighbourhoods of an element $x \in X$ is given by

$$\{U_x(t,\epsilon) \subseteq X : t > 0, \epsilon \in ]0,1[\},$$

where

$$U_x(t,\epsilon) := \{y \in X : F_{x,y}(t) > 1 - \epsilon\}.$$

If t-norm T satisfies the condition

$$sup\{T(t,t) : t \in [0,1[\} = 1,$$

then the $(t,\epsilon)$ -topology is metrizable (see [9]).

In 1966, V.M. Sehgal [10] introduced the notion of a contraction mapping in probabilistic metric spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists a $r \in ]0,1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbf{R}_+$.

For example, if $(X,d)$ is a metric space and $G \in \Delta^+$, $G \not= H$, in [7] one defines

$$F_{x,y}(t) = G(\frac{t}{d(x,y)}) \quad if\ x \not= y,$$

and

$$F_{x,y}(t) = H(t) \quad if\ x = y,$$

where the distribution function $H$ is defined by $H(t) = 1$ if $t > 0$, and $H(t) = 0$ if $t \leq 0$. If $f : X \to X$ is a contraction with ratio $r$, then it is a contraction in Sehgal sence with the same ratio. Indeed, we have

$$F_{f(x),f(y)}(rt) = G(\frac{rt}{d(f(x),f(y))}) \geq G(\frac{rt}{rd(x,y)})\ if\ \ f(x) \not= f(y)\ and\ x \not= y,$$

$$F_{f(x),f(y)}(rt) = G(\frac{rt}{rd(x,y)}) \geq H(t)\ if\ \ x \not= y\ and\ f(x) = f(y),$$

$$F_{f(x),f(y)}(rt) = H(t) = F_{x,y}(t) \quad if\ x = y.$$

A sequence $(x_n)_{n\in\mathbf{N}}$ in $X$ is said to be *fundamental* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 0$. The element $x \in X$ is called *limit* of the sequence, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent. If $(X,d)$ is a metric space, then the metric $d$ induces a mapping $\mathcal{F} : X \times X \to \Delta^+$, where $\mathcal{F}(x,y) = F_{x,y}$ is defined by

$$F_{x,y}(t) = H(t - d(x,y)), \ t \in \mathbf{R}.$$

Moreover $(X, \mathcal{F}, Min)$ is a Menger space. It is complete if the metric $d$ is complete (see [12]). The space $(X, \mathcal{F}, Min)$ thus obtained is called the *induced Menger space.*

**Proposition 2.1.** *(V.M. Sehgal [10], see also [2]) Every contraction mapping $f : X \to X$ on a complete Menger space $(X, \mathcal{F}, Min)$ has a unique fixed point $x_0$. Moreover, $f^n(x) \to x_0$ for each $x \in X$.*

Let $(X, \mathcal{F}, T)$ be a Menger space with $T$ continuous and let $A$ be a nonempty subset of $X$. The function $D_A : \mathbf{R} \to [0,1]$ defined by

$$D_A(t) := \sup_{s<t} \inf_{x,y \in A} F_{x,y}(s)$$

is called the ***probabilistic diameter of** A*. It is a distribution function from $\Delta^+$. The set $A \subseteq X$ is ***probabilistic bounded*** if $\sup_{t>0} D_A(t) = 1$. If $B$ and $C$ are two subsets of $X$ with $B \cap C \neq \emptyset$, then

$$D_{B \cup C}(s+t) \geq T(D_B(s), D_C(t)); \; s, t \in \mathbf{R} \tag{1}$$

(see [3, Theorem 10]).

Set

$$\mathcal{D}^+ = \{F \in \Delta^+ : \sup_{t \in R} F(t) = 1\}.$$

In the following we suppose that $(X, \mathcal{F}, T)$ is a Menger space with $\mathcal{F} : X \times X \to \mathcal{D}^+$and T is continuous. In this case every set with two elements is probabilistic bounded.

**Proposition 2.2.** *If $A$ is a probabilistic bounded set in $(X, \mathcal{F}, T)$ and $b \in X$, then the set $A_1 = A \cup \{b\}$ is also bounded.*

*Proof.* Let $a \in A$. Then $A_1 = A \cup \{a, b\}$,hence by (1)

$$D_{A_1}(2t) \geq T(D_A(t), F_{a,b}(t)).$$

Since $\sup_{t \in R} D_A(t) = 1$ and $\sup_{t \in R} F_{a,b}(t) = 1$, we have $\sup_{t \in R} D_{A_1}(2t) = 1$. □

**Corollary 2.1.** *Every finit set in $(X, \mathcal{F}, T)$ is probabilistic bounded.*

**Corollary 2.2.** *If $A$ and $B$ are probabilistic bounded sets in $(X, \mathcal{F}, T)$, then $A \cup B$ is also probabilistic bounded.*

An example for probabilistic unbounded set is the following. Let $\mathcal{F} : \mathbf{R}\times\mathbf{R} \to D^+$ be defined by $F_{x,y}(t) = H(t - |x - y|)$ . Let N be the set of all natural numbers. Then $D_N(t) = 0$ for every t, hence N is probabilistic unbounded.

In a probabilistic metric space $(X, \mathcal{F})$, the set $A$ is said to be ***precompact*** if for every $t > 0$ and $\epsilon \in ]0,1[$ there exists a finite cover $\{C_i\}_{i\in I}$ of $A$ such that $D_{C_i}(t) > 1-\epsilon$ for all $i \in I$. A precompact set $A$ is ***totally bounded,*** i.e. for every $t > 0$ and $\epsilon \in ]0,1[$ there exists a finite subset $B \subseteq A$ such that, for each $x \in A$, there is an $y \in B$ with $F_{x,y}(t) > 1-\epsilon$ (see [2, Proposition 1.2.3.]). In a Menger space with a t-norm $T$ such that $\sup_{a<1} T(a,a) = 1$ the converse assertion also holds: *a set A* is precompact if and only if it is totally bounded (see [2, Theorem 1.2.1.]).

Let $A$ and $B$ nonempty subsets of $X$. The ***probabilistic Hausdorff-Pompeiu distance*** between $A$ and $B$ is the function $F_{A,B} : \mathbf{R} \to [0,1]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T(\inf_{x\in A}\sup_{y\in B} F_{x,y}(s), \inf_{y\in B}\sup_{x\in A} F_{x,y}(s)).$$

**Proposition 2.3.** *If $\mathcal{C}$ is a nonempty collection of nonempty closed bounded sets in $(X,\mathcal{F},T)$ , then $(\mathcal{C},\mathcal{F}_{\mathcal{C}},T)$ is also a Menger space, where $\mathcal{F}_{\mathcal{C}}$ is defined by $\mathcal{F}_{\mathcal{C}}(A,B) := F_{A,B}$ for all $A, B \in \mathcal{C}$ .*

*Proof.* We have, for all $A, B \in \mathcal{C}$ ,

$$F_{A,B}(x) \geq \sup_{t<x} T(\inf_{p\in A}\inf_{q\in B} F_{p,q}(t), \inf_{q\in B}\inf_{p\in A} F_{p,q}(t)) \geq$$

$$\geq T(D_{A\cup B}(t), D_{A\cup B}(t)).$$

Since by Corollary 2.2, the set $A\cup B$ is probabilistic bounded, it follows $\sup_{x\in\mathbf{R}} F_{A,B}(x) = 1$. Therefore, by [3, Theorem 18] $(\mathcal{C},\mathcal{F}_{\mathcal{C}},T)$ is a Menger space. □

In the following we suppose that $T = Min$.

**Proposition 2.4.** *If $(X,\mathcal{F},Min)$ is a complete Menger space and $\mathcal{C}$ is the collection of all nonempty closed bounded subsets of $X$ in $(t,\epsilon)-$ topology, then $(\mathcal{C},\mathcal{F}_{\mathcal{C}},Min)$ is also a complete Menger space.*

*Proof.* Let $(A_n)_{n\in\mathbf{N}}$ be a fundamental sequence in $\mathcal{C}$ and let

$$A = \{x \in X : \forall n \in \mathbf{N}, \exists x_n \in A_n, \forall t > 0, \lim_{n\to\infty} F_{x_n,x}(t) = 1\}. \quad (2)$$

Let $\overline{A}$ denote the closure of $A$. By [3, Theorem 15] we have $F_{A_n,A} = F_{A_n,\overline{A}}$, so we must show that (i) $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$, and (ii) $\overline{A} \in \mathcal{C}$.

(i) Let $\epsilon > 0$ and $t > 0$ be given. Then there exists $n_\epsilon(t) \in \mathbb{N}$ so that $n, m \geq n_\epsilon(t)$ implies $F_{A_n,A_m}(\frac{t}{2}) > 1-\epsilon$. Let $n > n_\epsilon(t)$. We claim that $F_{A_n,A}(t) \geq 1-\epsilon$.

If $x \in A$ then there is a sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in A_k$ and $\lim_{k\to\infty} F_{x_k,x}(\frac{t}{2}) = 1$. So, for large enough $k \geq n_\epsilon(t)$, we have $F_{x_k,x}(\frac{t}{2}) > 1-\epsilon$. Thus, since $F_{A_n,A_k}(\frac{t}{2}) > 1-\epsilon$, for $n \geq n_\epsilon(t)$, there exist $y \in A_n$ and $z \in A_k$ such that

$$Min(F_{x_k,y}(\frac{t}{2}), F_{z,y}(\frac{t}{2})) > 1-\epsilon,$$

hence $F_{x_k,y}(\frac{t}{2}) > 1-\epsilon$. By 8) we have $F_{x,y}(t) > 1-\epsilon$, hence

$$\sup_{s<t} \inf_{x\in A} \sup_{y\in A_n} F_{x,y}(s) > 1-\epsilon. \tag{3}$$

Now suppose $y \in A_n$. Choose integers $k_1 < k_2 < ... < k_i < ...$ so that $k_1 = n$ and

$$F_{A_k,A_{k_i}}(\frac{t}{2^{i+1}}) > 1 - \frac{\epsilon}{2^{i-1}},$$

for all $k > k_i$. Hence we can choose $s < t$ such that $\inf_{z\in A_{k_i}} \sup_{x\in A_k} F_{x,z}(\frac{s}{2^{i+1}}) > 1-\frac{\epsilon}{2^{i-1}}$. Then define a sequence $(y_k)$ with $y_k \in A_k$ as follows: For $k < n$, choose $y_k \in A_k$ arbitrarily. Choose $y_n = y$. If $y_{k_i}$ has been chosen, and $k_i < k \leq k_{i+1}$, choose $y_k \in A_k$ with $F_{y_{k_i},y_k}(\frac{s}{2^{i+1}}) > 1-\frac{\epsilon}{2^{i-1}}$. Then, for $k_i < k \leq k_{i+1} < ... < k_j < l \leq k_{j+1}$, we have

$$F_{y_l,y_k}(\frac{s}{2^{i-1}}) \geq T(F_{y_k,y_{k_i}}(\frac{s}{2^i}), T(F_{y_{k_i},y_{k_{i+1}}}(\frac{s}{2^{i+1}}), \dots,$$

$$T(F_{y_{k_{j-1}},y_{k_j}}(\frac{s}{2^j}), F_{y_{k_j},y_l}(\frac{s}{2^j}))...) > 1 - \frac{\epsilon}{2^{i-1}}.$$

Let $r > 0$, $\eta > 0$, and choose $i$ so that $r > \frac{s}{2^{i-1}}$ and $\frac{\epsilon}{2^{i-1}} < \eta$. We have

$$F_{y_k,y_l}(r) \geq F_{y_k,y_l}(\frac{s}{2^{i-1}}) > 1 - \frac{\epsilon}{2^{i-1}} > 1-\eta.$$

Hence $(y_k)$ is a fundamental sequence, so it converges. Let $x$ be its limit. Then $x \in A$ and we have

$$F_{x,y}(t) \geq T(F_{x,y_k}(\frac{t}{2}), F_{y_k,y}(\frac{t}{2})).$$

We choose $k$ such that $F_{x,y_k}(\frac{t}{2}) > 1-\epsilon$. Since $F_{y,y_k}(\frac{t}{2}) > 1-\epsilon$, it follows $F_{x,y}(t) > 1-\epsilon$. Therefore we have

$$\sup_{s<t} \inf_{y\in A_n} \sup_{x\in A} F_{x,y}(s) > 1-\epsilon. \tag{4}$$

By (3) this shows that

$$F_{A_n,A}(t) = \sup_{s<t} T(\inf_{x\in A} \sup_{y\in A_n} F_{x,y}(s), \inf_{y\in A_n} \sup_{x\in A} F_{x,y}(s)) > 1-\epsilon.$$

So $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$.

(ii) Taking $\epsilon = 1$ in the last argument, we have proved that $A$ is nonempty.

We have to show that $A$ is bounded. Since $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $\epsilon > 0$ and $t_0 > 0$ we have $\inf_{x\in A} \sup_{w\in A_n} F_{x,w}(t_0) > 1-\epsilon$ and $\inf_{y\in A_n} \sup_{x\in A} F_{x,y}(t_0) > 1-\epsilon$. $A_n$ being probabilistic bounded, for all $\epsilon > 0$ there is $t_\epsilon > t_0$ such that $\inf_{u,v\in A_n} F_{u,v}(t_\epsilon) > 1-\epsilon$.

For $x, y \in A$ there exist $u, v \in A_n$ such that

$$F_{x,u}(t_0) > 1-\epsilon,\ F_{y,v}(t_0) > 1-\epsilon.$$

We have

$$F_{x,y}(3t_\epsilon) \geq T(F_{x,u}(t_\epsilon), F_{u,y}(2t_\epsilon)) \geq T(F_{x,u}(t_0), T(F_{u,v}(t_\epsilon), F_{v,y}(t_0))) > 1-\epsilon.$$

So $D_A(3t_\epsilon) \geq 1-\epsilon$, consequently $\sup_{t\in\mathbf{R}} D_A(t) = 1$. By [3] it follows that $D_A = D_{\overline{A}}$. Since $A$ is bounded, $\overline{A}$ is also bounded and closed, so $\overline{A} \in \mathcal{C}$. □

**Proposition 2.5.** *Let $\mathcal{K}$ be the collection of all nonempty compact sets in the complete Menger space $(X, \mathcal{F}, Min)$ and let $\mathcal{C}$ be the collection of all nonempty closed bounded subsets of $X$ in $(t,\epsilon)-$ topology. Then $(\mathcal{K}, \mathcal{F}_{\mathcal{K}}, Min)$ is a closed subspace of $(\mathcal{C}, \mathcal{F}_{\mathcal{C}}, Min)$.*

*Proof.* First we show that $\mathcal{K} \subseteq \mathcal{C}$. For this we have to show that any nonempty compact set $A$ is probabilistic bounded. Let $\epsilon \in ]0,1[$. For every $t > 0$, there exists a finite cover $\{C_i\}_{i\in I}$ of $A$ such that $D_{C_i}(t) > 1-\epsilon$ for all $i \in I$. Let $I = \{1, ..., m\}$ and set $C = \cup_{i=1}^m C_i$.

For every $i \in \{1, \cdots, m\}$ choose an element $c_i \in C_i$ and set $B = \{c_1, \cdots, c_m\}$. Then, for $C_i^* = C_i \cup B$, we have $C = \cup_{i=1}^m C_i^*$. Let $s > 0$ such that $D_B(s) > 1-\epsilon$. By (1) we have

$$\begin{aligned} D_A(m(t+s)) &\geq D_C(m(t+s)) \geq \\ &\geq T(D_{C_1^*}(t+s), T(D_{C_2^*}(t+s), ..., T(D_{C_{m-1}^*}(t+s), D_{C_m^*}(t+s))...) \\ &\geq T(D_{C_1}(t), T(D_{C_2}(t), \cdots, T(D_{C_{m-1}}(t), T(D_{C_m}(t), D_B(s))...) > 1-\epsilon, \end{aligned}$$

hence $A$ is probabilistic bounded.

Let $(A_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{K}$ converging to $A \in \mathcal{C}$. We shall show that $A$ is totally bounded. Let $\epsilon > 0$, $t > 0$, and choose $n$ so that $F_{A_n,A}(\frac{t}{3}) > 1-\epsilon$. The set $A_n$ being precompact in the $(t,\epsilon)$ -topology, there exist $x_1, ...x_m \in A_n$ such that

$$A_n \subseteq \bigcup_{i=1}^{m} U_{x_i}(\frac{t}{3}, \epsilon).$$

For each $x_i$ there is $y_i \in A$ with $F_{x_i,y_i}(\frac{t}{3}) > 1-\epsilon$. For $y \in A$ there exists $x \in A_n$ with $F_{x,y}(\frac{t}{3}) > 1-\epsilon$. Let $i \in \{1, ..., m\}$ such that $x \in U_{x_i}(t,\epsilon)$. Then

$$F_{y,y_i}(t) \geq T(F_{y,x}(\frac{t}{3}), T(F_{x,x_i}(\frac{t}{3}), F_{x_i,y_i}(\frac{t}{3})) > 1-\epsilon,$$

hence

$$A \subseteq \bigcup_{i=1}^{m} U_{y_i}(t,\epsilon).$$

Therefore $A$ is totally bounded. The $(t,\epsilon)$-topology being metrizable and $A$ being closed, it is compact. □

**Corollary 2.3.** *If $(X, \mathcal{F}, Min)$ is a complete Menger space and $\mathcal{K}$ is the collection of all nonempty compact subsets of $X$ in $(t,\epsilon)-$ topology, then $(\mathcal{K}, \mathcal{F}_{\mathcal{K}}, Min)$ is also a complete Menger space.*

## 3. Invariant sets

In this section we will generalize Hutchinson's theorem on invariant sets.

**Proposition 3.1.** *Let $(X, \mathcal{F}, Min)$ be a Menger space and $\mathcal{C}$ be the collection of all nonempty closed bounded sets in $X$. Let $f_1, ..., f_m : X \to X$ be contractions with ratios $r_1, ...r_m \in ]0,1[$ and let $\phi : \mathcal{C} \to \mathcal{C}$ be defined by*

$$\phi(E) := \cup_{i=1}^{p} f_i(E).$$

*Then $\phi$ is a contraction.*

*Proof.* Let $r = max\{r_i, 1 \leq i \leq m\}$ and $A, B \in \mathcal{C}$. We shall show that

$$F_{\phi(A),\phi(B)}(rt) \geq F_{A,B}(t), \tag{5}$$

for all $t > 0$.

For all $A, B \in \mathcal{C}$ and $s < t$, we have

$$F_{\cup_{i=1}^{m} f_i(A), \cup_{i=1}^{m} f_i(B)}(rt) \geq T(\inf_{u \in \cup_{i=1}^{m} f_i(A)} \sup_{v \in \cup_{i=1}^{m} f_i(B)} F_{u,v}(rs), \inf_{v \in \cup_{i=1}^{m} f_i(B)} \sup_{y \in \cup_{i=1}^{m} f_i(A)} F_{u,v}(rs)).$$

Let $i_0$ and $j_0$ be such that

$$\inf_{u\in\cup_{i=1}^m f_i(A)} \sup_{v\in\cup_{i=1}^m f_i(B)} F_{u,v}(rs) = \inf_{u\in f_{i_0}(A)} \sup_{v\in\cup_{i=1}^m f_i(B)} F_{u,v}(rs) \geq \inf_{u\in f_{i_0}(A)} \sup_{v\in f_{i_0}(B)} F_{u,v}(rs),$$

$$\inf_{v\in\cup_{i=1}^m f_i(B)} \sup_{u\in\cup_{i=1}^m f_i(A)} F_{u,v}(rs) = \inf_{y\in f_{j_0}(B)} \sup_{u\in\cup_{i=1}^m f_i(A)} F_{u,v}(rs) \geq \inf_{v\in f_{j_0}(B)} \sup_{u\in f_{j_0}(A)} F_{u,v}(rs).$$

Hence

$$F_{\cup_{i=1}^m f_i(A),\cup_{i=1}^m f_i(B)}(rt) \geq T(\inf_{u\in f_{i_0}(A)} \sup_{v\in f_{i_0}(B)} F_{u,v}(rs), \inf_{v\in f_{j_0}(B)} \sup_{u\in f_{j_0}(A)} F_{u,v}(rs))$$

$$\geq T(\inf_{u\in f_{l_0}(A)} \sup_{v\in f_{l_0}(B)} F_{u,v}(rs), \inf_{v\in f_{l_0}(B)} \sup_{u\in f_{l_0}(A)} F_{u,v}(rs)) =$$

$$= T(\inf_{x\in A} \sup_{y\in B} F_{f_{l_0}(x),f_{l_0}(y)}(rs), \inf_{y\in B} \sup_{x\in A} F_{f_{l_0}(x),f_{l_0}(y)}(rs)) \geq$$

$$\geq T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)),$$

where $l_0 = i_0$ if $\inf_{u\in f_{i_0}(A)} \sup_{v\in f_{i_0}(B)} F_{u,v}(rs) \leq \inf_{v\in f_{j_0}(B)} \sup_{u\in f_{j_0}(A)} F_{u,v}(rs)$, and $l_0 = j_0$ else. Therefore we have (5). □

**Theorem 3.1.** *Let $(X, \mathcal{F}, Min)$ be a complete Menger space and let $f_1, ..., f_m : X \to X$ be contractions with ratios $r_1, ... r_m \in ]0,1[$, respectively. Then there exists a nonempty compact subset $K$ of $X$ such that*

$$f_1(K) \cup ... \cup f_m(K) = K.$$

*Moreover, the set $K$ with this property is unique in the space of all nonempty closed bounded sets in $X$.*

*Proof.* By Proposition 3.1 the function $\phi : \mathcal{C} \to \mathcal{C}$ defined by

$$\phi(E) = \cup_{i=1}^p f_i(E)$$

is a contraction, and by Proposition 2.4 $(\mathcal{C}, \mathcal{F}_\mathcal{C}, Min)$ is a complete Menger space. Then, by Proposition 2.1. there is a unique set $K$ in $\mathcal{C}$ such that $\phi(K) = K$. Moreover, we have $\lim_{n\to+\infty} \phi^n(K_0) = K$ for any $K_0 \in \mathcal{K}$. Thus, by Proposition 2.5 the set $K$ must be in $\mathcal{K}$. □

**Corollary 3.1.** *(Hutchinson [4]) Let $(X, d)$ be a complete metric space and $f_1, \ldots, f_m : X \to X$ be contraction maps with ratios $r_1, \ldots, r_m$, respectively. Then there exists a unique nonempty compact set $K \subseteq X$ such that $K = \bigcup_{i=1}^m f_i(K)$.*

*Proof.* Let $(X, \mathcal{F}, Min)$ be the induced Menger space by the metric $d$. Since, for each $t > 0$ and $i \in \{1, ..., m\}$,

$$F_{f_i(x), f_i(y)}(r_i t) = H(r_i t - d(f_i(x), f_i(y))) \geq H(r_i t - r_i d(x, y)) = H(t - d(x, y)) = F_{x,y}(t),$$

the conclusion follows from Theorem 3.1. □

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