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RATIONAL BÉZIER CURVES AND SURFACES WITH INDEPENDENT COORDINATE WEIGHTS

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Abstract. A generalization of the rational Bézier curves and surfaces was made in [3]. In this paper we make another extending of the possibilities for the modelling curves and surfaces by attaching different weights to each coordinate of the control Bézier points. Derivatives of high orders in the initial and final points of the curves are also deduced. Some figures show the increased flexibility of these partial or total rational Bézier curves and surfaces comparative with the polynomial and classical rational corresponding to the same control Bézier polygon. On observes that we do not always have the convex hull property (Fig.2) and the affine invariance (Fig.3).

1. Introduction

Rational Bézier curves and surfaces are represented by the equations (1) and (2) respectively

$$r(t) = \sum_{i=0}^{n} \frac{w_i b_{n,i}(t)}{\sum_{i=0}^{n} w_i b_{n,i}(t)} b_i, \quad t \in [0,1] \tag{1}$$

and

$$r(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w_{ij} b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij} b_{m,i}(u) b_{n,j}(v)} b_{ij}, \quad (u,v) \in [0,1]^2, \tag{2}$$

where $b_{n,i}(t) = \binom{n}{i}(1-t)^{n-i}t^i$, $i = \overline{0,n}$, $b_i \in R^3$ and $b_{ij} \in R^3$ are given points. The positive real numbers w_i, $i = \overline{0,n}$ from (1), called shape parameters, are used for the remodelling curve. So, as it is known, if one increases, say w_k, then the curve is pulled towards the points b_k and if w_k decreases then the contribution of b_k to the curve is diminished (see Figures 1 and 2; dotted curves and the dash curves are polynomial Bézier, corresponding to the same control polygon). We also mention the possibility to control the curvature and torsion at the points b_0 and b_n respectively, [1] p.180. Thus

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the flexibility of the rational Bézier curves is its characteristic property comparative with the polynomial Bézier. Similar remark is true regarding to the rational Bézier surfaces.

2. Partial and Total Coordinates Rational Bézier Curves

In the equation (1) the parameter w_i affects the all coordinates (x_i, y_i, z_i) of points b_i in the same measure. Next we attach to each coordinate (x_i, y_i, z_i) of the point b_i different shape parameters (independent weights) denoted by w_i^x, w_i^y and w_i^z respectively.

The equation of a rational Bézier curve with coordinate shape parameters is of the following form

$$R(t) = [x(t), y(t), z(t)]^T,$$

where

$$\begin{aligned} x(t) &= \sum_{i=0}^{n} \frac{w_i^x b_{n,i}(t)}{\sum_{i=0}^{n} w_i^x b_{n,i}(t)}, \\ y(t) &= \sum_{i=0}^{n} \frac{w_i^y b_{n,i}(t)}{\sum_{i=0}^{n} w_i^y b_{n,i}(t)}, \\ z(t) &= \sum_{i=0}^{n} \frac{w_i^z b_{n,i}(t)}{\sum_{i=0}^{n} w_i^z b_{n,i}(t)}, \qquad t \in [0,1]. \end{aligned} \tag{3}$$

Consider the folowing sets of positive real numbers

$$W = \{w_0, w_1, \ldots, w_n\}, \ U = \{1, 1, \ldots, 1\}$$

and

$$W^t = \{w_0^t, w_1^t, \ldots, w_n^t\}, \ t \in \{x, y, z\}.$$

Definitions.

1. We name W^t the set of t-coordinate shape parameters.

2. If $W^x = W$ and $W^y = W^z = U$ then we name (3) a **partial x-coordinate rational Bézier curve**. Analogously one defines another partial variable Bézier curve.

3. A curve is called **x,y-rational Bézier** if $W^t \neq U, t \in \{x, y\}$ and $W^z = U$.

4. A curve which is x,y,z-rational will be called a **total coordinates rational Bézier curve.**

Remarks:

1. If $W^t = U$, for any $t \in \{x, y, z\}$, then equations (3) represent a polynomial Bézier curve.

2. If $W^x = W^y = W^z = W$, then (3) are parametric equations of a classical rational Bézier curve.

In the Figures 1 and 2 are illustrated the effects of the coordinates shape parameters on a partial rational Bézier curve (continous curve). As witness curves we have taken the classical rational Bézier - dotted curve and the polynomial Bézier - reare dotted curve. The points $b_i, i = \overline{0,5}$ and the coordinate shape parameters are

$$b_0 = (-6,6), b_1 = (2,0), b_2 = (12,7), b_3 = (-5,18), b_4 = (3,23), b_5 = (6,18);$$

for Figure 1: $W^x = W = \{1,1,9,9,1,1\}$, $W^y = U$;
and for Figure 2: $W^x = W = \{1,1,2,4,1,1\}$, $W^y = \{1,3,1,1,5,1\}$.

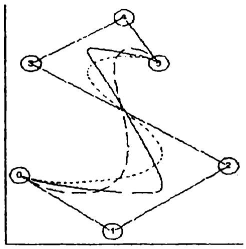

Fig.1

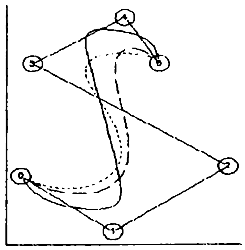

Fig.2.

Remarks:

1. If one increases say w_i^x then the curve is pulled towards the straight line (in R^3 to the plane) $x = x_i$, because the contribution of x_i, to the function $x(t)$ increases.

2. If $W^x = W$ then the partial x-coordinate rational and classical rational Bézier curves have the same function $x(t)$. From geometrical point of view these curves have comon tangent, perpendicular on the Ox axis, as can be seen from the above figures. Similar remarks are valueable if $W^y = W$ or $W^z = W$.

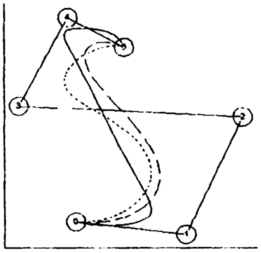

Fig.3.

3. In the case of total coordinates rational Bézier curve we have independent control of the scalar functions $x(t), y(t)$ and $z(t)$.

4. If we make a rotation of the coordinate system then, as it is known, the polynomial and classical rational Bézier curves are invariant [1], p.232, but a coordinate rational curve one modifies as we can see in fig.3. Fig.3 results from fig.2 if one does a rotation of the control polygon with angle $\alpha = \frac{\pi}{6}$.

5. Total rational coodonates Bézier curves do not always have conex hull property (see figure 2). Denoting

$$\min_{i=\overline{0,n}} x_i = a_1, 2cm \qquad \max_{i=\overline{0,n}} x_i = b_1,$$
$$\min_{i=\overline{0,n}} y_i = a_2, \qquad \max_{i=\overline{0,n}} y_i = b_2,$$
$$\min_{i=\overline{0,n}} z_i = a_3, \qquad \max_{i=\overline{0,n}} z_i = b_3,$$

and taking in view that $x(t), y(t)$ and $z(t)$ in (3) are weighted means, one can say that this type of curve lies in the interior of parallelepiped

$$D = \{(x, y, z) | a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3\}.$$

Concerning to the derivatives of $R(t)$ we have

$$R^{(k)}(t) = \left[x^{(k)}(t), y^{(k)}(t), z^{(k)}(t)\right]^T, t \in [0,1].$$

First we will give the expression of the $x^{(k)}(t)$, proceeding as in [1], p.236. From (3), denoting

$$p(t)=\sum_{i=0}^{n} b_{n,i}(t)w_i^x x_i \quad \text{and} \quad \mathrm{w}(t)=\sum_{i=0}^{n} \mathrm{b}_{n,i}(t)\mathrm{w}_i^x \tag{4}$$

results $p(t) = x(t)w(t)$. Further, using the Leibniz's formula for the computation of $[x(t)w(t)]^{(k)}$ on obtains

$$p^{(k)}(t)=\sum_{j=0}^{k}\binom{k}{j}x^{(k-j)}(t)w^{(j)}(t)=x^{(k)}(t)w(t)+\sum_{j=1}^{k}\binom{k}{j}x^{(k-j)}(t)w^{(j)}(t).$$

From here results the following recursive formula

$$x^{(k)}(t)=\frac{1}{w(t)}\left[p^{(k)}(t)-\sum_{j=1}^{n}\binom{k}{j}x^{(k-j)}(t)w^{(j)}(t)\right]. \tag{5}$$

Taking in view (5) and formula 4.19 from [1] p.44, we have

$$p^{(k)}(t)=\frac{n!}{(n-k)!}\sum_{i=0}^{n-k} b_{n-k,i}(t)\Delta^k(w_i^x x_i) \tag{6}$$

and

$$w^{(j)}(t)=\frac{n!}{(n-j)!}\sum_{i=0}^{n-j} b_{n-j,i}(t)\Delta^j(w_i^x) \tag{7}$$

where the forward difference of order m has the expression

$$\Delta^m y_p=\sum_{q=0}^{m}(-1)^{m-q}\binom{m}{q}y_{p+q}. \tag{8}$$

Taking into account (6),(7) and (8), the formula (5) has the following final form

$$\begin{aligned} x^{(k)}(t)= & \frac{n!}{w(t)}\left[\frac{1}{(n-k)!}\sum_{i=0}^{n-k} b_{n-k,i}(t)\sum_{q=0}^{k}(-1)^{k-q}\binom{k}{q}w_{i+q}^x x_{i+q}-\right.\\ & \left.-\sum_{j=1}^{k}\binom{k}{j}\frac{x^{(k-j)}(t)}{(n-j)!}\sum_{j=0}^{n-j} b_{n-j,i}(t)\sum_{q=0}^{j}(-1)^{j-q}\binom{j}{q}w_{i+q}^x\right]. \end{aligned} \tag{9}$$

For the particular cases $t=0$ and $t=1$ this formula becomes

$$\begin{aligned} x^{(k)}(0)= & \frac{n!}{w_0^x}\left[\frac{1}{(n-k)!}\sum_{q=0}^{k}(-1)^{k-q}\binom{k}{q}w_q^x x_q-\right.\\ & \left.-\sum_{j=1}^{k}\binom{k}{j}\frac{x^{(k-j)}(0)}{(n-j)!}\sum_{q=0}^{j}(-1)^{j-q}\binom{j}{q}w_q^x\right] \end{aligned} \tag{10}$$

and

$$x^{(k)}(1) = \frac{n!}{w_n^x}\left[\frac{1}{(n-k)!}\sum_{q=0}^{k}(-1)^{k-q}\binom{k}{q}w_{n-k+q}^x x_{n-k+q} - \right. \\ \left. -\sum_{j=1}^{k}\binom{k}{j}\frac{x^{(k-j)}(1)}{(n-j)!}\sum_{q=0}^{j}(-1)^{j-q}\binom{j}{q}w_{n-j+q}^x\right]. \tag{11}$$

Similar formulas we have for $y^{(k)}(t)$, $y^{(k)}(0)$, $y^{(k)}(1)$, $z^{(k)}(t)$, $z^{(k)}(0)$ and $z^{(k)}(1)$.

Special interest present the derivatives $\dot{R}(t)$ and $\ddot{R}(t)$ for $t = 0$ and $t = 1$, respectively.

From (10) and (11) results

$$\dot{R}(0) = n\left[\frac{w_1^x}{w_0^x}\Delta x_0, \frac{w_1^y}{w_0^y}\Delta y_0, \frac{w_1^z}{w_0^z}\Delta z_0\right]^T, \\ \dot{R}(1) = n\left[\frac{w_{n-1}^x}{w_n^x}\Delta x_{n-1}, \frac{w_{n-1}^y}{w_n^y}\Delta y_{n-1}, \frac{w_{n-1}^z}{w_n^z}\Delta z_{n-1}\right]^T \tag{12}$$

and

$$\ddot{R}(0) = \left[\ddot{x}(0), \ddot{y}(0), \ddot{z}(0)\right]^T, \ddot{R}(1) = \left[\ddot{x}(1), \ddot{y}(1), \ddot{z}(1)\right]^T .$$

where

$$\ddot{x}(0) = \frac{n}{w_0^x}\left\{(n-1)\left[\Delta^2\left(w_0^x x_0\right) - x_0\Delta^2 w_0^x\right] - 2n\frac{w_1^x}{w_0^x}\Delta x_0 \Delta w_0^x\right\} \\ \ddot{x}(1) = \frac{n}{w_n^x}\left\{(n-1)\left[\Delta^2\left(w_{n-2}^x x_{n-2}\right) - x_n\Delta^2 w_{n-2}^x\right] - 2n\frac{w_{n-1}^x}{w_n^x}\Delta w_{n-1}^x\right\} \tag{13}$$

Analogous formulas we have for $\ddot{y}(0), \ddot{y}(1), \ddot{z}(0)$ and $\ddot{z}(1)$.

Remarks.

1. Denoting by $m(t)$ and $m_T(t)$ the slopes of the polynomial (or classical rational) and total coordinates rational Bézier curves, respectively, in virtue of (12), we have

$$m_T(0) = \frac{w_0^x w_1^y}{w_1^x w_0^y}m(0) \text{ and } \mathrm{m_T}(1) = \frac{\mathrm{w_n^x w_{n-1}^y}}{\mathrm{w_{n-1}^x w_n^y}}\mathrm{m}(1), \tag{14}$$

so we can control the slopes in b_0 and b_n by means of the coordinate shape parameters. As a consequence, a total coordinates rational Bézier curve do not always have convex hull property (see Figure 2).

2. Similar remark is valueable relative to the direction cosines of a total coordinates rational Bézier curve in b_0 and b_n, respectively.

3. Partial and Total Coordinates Rational Bézier Surfaces

As for curves, we generalize the rational Bézier surfaces by introduceing independent coordinate shape parameters. The vectorial equation of this generalized surfaces is

$$R(u,v) = [x(u,v),\ y(u,v),\ z(u,v)]^T,$$

where

$$\begin{aligned} x(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w_{ij}^x b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}^x b_{m,i}(u) b_{n,j}(v)} x_{ij}, \\ y(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w_{ij}^y b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}^y b_{m,i}(u) b_{n,j}(v)} y_{ij}, \\ z(u,v) &= \sum_{i=0}^{m}\sum_{j=0}^{n} \frac{w_{ij}^z b_{m,i}(u) b_{n,j}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{ij}^z b_{m,i}(u) b_{n,j}(v)} z_{ij}, \\ &(u,v) \in [0,1]\text{x}[0,1]. \end{aligned} \tag{15}$$

We denote the matrices of the shape parameters and the coordinate shape parameters, respectively as follows

$$W = [w_{ij}],\ U = [1],\ W^t = [w_{ij}^t],\ t \in \{x,y,z\},\ i = \overline{0,m},\ j = \overline{0,n}.$$

We observe that if $W^x = W^y = W^z = U$ then equations (15) represent a polynomial Bézier surface and for $W^x = W^y = W^z = W$ results a classical rational Bézier surface. The definition 1-4, from curves, one extends to coordinates rational Bézier surfaces.

In Figure 4 is represented the polynomial Bézier surface corresponding to the following control points:

	j=0	j=1	j=2	j=3	j=4	j=5
i=0	(-4,-2,5)	(-3,-1,6)	(-2,1,5)	(-1,3,5)	(-1,5,6)	(0,6,7)
i=1	(-2,-2,10)	(-2,-1,8)	(-1,1,7)	(-2,3,8)	(-3,5,4)	(-2,7,3)
i=2	(2,-2,9)	(3,-1,7)	(2,1,6)	(-1,2,7)	(-6,3,9)	(-5,4,10)
i=3	(6,-2,2)	(5,-1,6)	(4,1,8)	(3,3,9)	(2,5,11)	(3,6,9)
i=4	(12,-2,4)	(10,-1,4)	(8,2,3)	(7,4,3)	(6,5,1)	(6,6,1)
i=5	(8,-3,5)	(9,0,7)	(10,2,6)	(8,4,5)	(10,5,6)	(11,6,7)

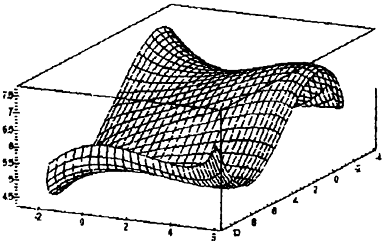

Fig.4.

The coordinate rational surfaces from Fig.5 and Fig.6 have the same control net as the surface presented in Figure 4 and the coordinate shape parameters $\left(w_{ij}^{x},\ w_{ij}^{y},\ w_{ij}^{z}\right)$ are specified for each figure.

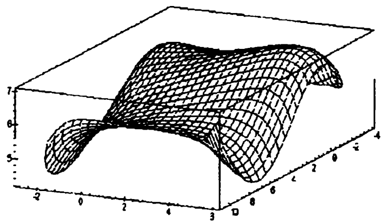

Fig.5.

$$W^x = W^y = U \text{and} \mathrm{W}^z = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0.1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 20 & 1 & 1 & 1 \end{pmatrix}$$

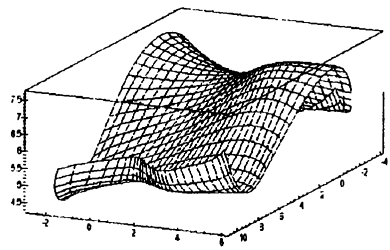

Fig.6.

$$W^z = U \text{and} \mathrm{W}^x = \mathrm{W}^y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0.1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0.1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 10 & 1 & 1 & 1 \end{pmatrix}$$

It evidently is that we have more possibilities to control the shape of a coordinate rational Bézier surface.

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