ON THE CONTINUITY AND DIFFERENTIABILITY OF THE IMPLICIT FUNCTIONS FOR GENERALIZED EQUATIONS

DOMOKOS ANDRÁS

Abstract. The aim of this paper is to show that the existence, continuity and differentiabilty of the implicit functions can be proved at the same time, using one sequence of succesive approximations of a mapping of two variables. The proof from this paper unifies methods used in the study of local stability and sensitivity of the solutions of integral equations [7], variational inequalities and nonsmooth generalized equations [1, 2, 5, 6]. We will prove the continuous differentiability of the solution mapping mapping in a neighborhood of a fixed parameter λ_0.

Throughout this paper X will be a Banach space, Y, Z, Λ will be normed spaces. Let X_0 and Λ_0 be open neighborhoods of the fixed points $x_0 \in X$ and $\lambda_0 \in \Lambda$ respectively. We will study the behaviour of the solutions of the following generalized equation:

$$0 \in f(x,\lambda) + G(x) ,$$

where $f : X_0 \times \Lambda_0 \to Z$ is a single-valued mapping and $G : X_0 \to Z$ is a set-valued mapping.

This problem includes the variational inequalities from the papers [4, 5, 6]. Assumption (*iii*) of Theorem 2 appears in both nonsmooth [1] and smooth [2] cases and generalize the strong-regularity condition for variational inequalities [4, 5]. Theorem 2 also generalize the classical version of the implicit function theorem [3].

Let $M \subset X$, $N \subset Y$. We will denote by $\mathbf{L}(X,Y)$ the set of the linear and continuous mappings from X to Y, by $\mathbf{C}(M,N)$ the set of the continuous mappings from M to N, by $\mathbf{B}(M,N)$ the set of the bounded mappings from M to N, by $B(x_0,r)$ the closed ball with center at x_0 and radius r.
We will apply the following result:

Theorem 1. [7] *Let (S,d) and (U,ρ) be complete metric spaces and let $A: S\times U \to S\times U$ be a mapping with the following properties:*

(i) A is continuous;

(ii) there exist the mappings $P : S \to S$, $Q : S\times U \to U$ such that $A(s,u) = (P(s), Q(s,u))$ and

- *P is a contraction,*
- *there exists $l \in [0,1)$ such that*

$$\rho(Q(s,u_1), Q(s,u_2)) \le l\,\rho(u_1,u_2)$$

for all $s\in S$, $u_1,u_2\in U$.

Then for all $(s,u)\in S\times U$ the succesive approximations $A^n(s,u)$ converge to a unique $(\overline{s},\overline{u})\in S\times U$, where $\overline{s}$ is the unique fixed point for P.

Theorem 2. *Let us suppose that Λ is finite dimensional and :*

(i) $0\in f(x_0,\lambda_0) + G(x_0)$;

(ii) f is continuous Fréchet differentiable on $X_0\times\Lambda_0$:

(iii) there exist an open neighborhood Z_0 of 0_Z and a mapping $g : Z_0 \to X$ which is continuous differentiable on Z_0, $g(0) = 0$ and for all $z\in Z_0$

$$g(z) \in \left(f(x_0,\lambda_0) + \nabla_x f(x_0,\lambda_0)(\cdot - x_0) + G(\cdot)\right)^{-1}(z)\,.$$

Then there exists an open neighborhood Λ_0' of λ_0 and a mapping $x:\Lambda_0'\to X_0$ such that x is continuous differentiable on Λ_0', $x(\lambda_0) = x_0$ and $0\in f(x(\lambda),\lambda) + G(x(\lambda))$, for all $\lambda\in\Lambda_0'$.

Proof. We can suppose ∇f bounded on $X_0\times\Lambda_0$, and the mean value inequality for Fréchet differentiable mappings implies that f is Lipschitz continuous on $X_0\times\Lambda_0$. The same is true for g and let us denote by γ the Lipschitz constant of g.

We can suppose $x_0 = 0$ and we denote by $h(x) = f(0,\lambda_0) + \nabla_x f(0,\lambda_0)(x)$. The continuity of $\nabla_x f$ at $(0,\lambda_0)$ implies that h strongly approximates f at $(0,\lambda_0)$ [6], i. e. for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|f(x_1,\lambda) - h(x_1) - (f(x_2,\lambda) - h(x_2))\| \le \varepsilon\|x_1 - x_2\|$$

for all $x_1,x_2\in B(0,\delta)$, $\lambda\in B(\lambda_0,\delta)$.

Let us choose the constants $\varepsilon,\delta,s,r > 0$ such that:

- $\varepsilon \cdot \gamma < 1$;
- $B(0,\delta) \subset X_0$, $B(\lambda_0,\delta) \subset \Lambda_0$;
- $h(x) - f(x,\lambda) \in Z_0$, for all $x \in B(0,\delta)$, $\lambda \in B(\lambda_0,\delta)$;
- $s \leq \delta$, $r \leq \delta$;
- $\gamma\|f(0,\lambda) - f(0,\lambda_0)\| \leq (1-\gamma\varepsilon)r$, for all $\lambda \in B(\lambda_0,s)$;
- $\|\nabla g\,(h(x) - f(x,\lambda)) \circ (\nabla_x f(0,\lambda_0) - \nabla_x f(x,\lambda))\| \leq a < 1$, for all $x \in B(0,r)$, $\lambda \in B(\lambda_0,s)$.

Let us define the mapping $F : B(0,r) \times B(\lambda_0,s) \to X$ by

$$F(x,\lambda) = g(\,h(x) - f(x,\lambda)\,)\,.$$

For all $x \in B(0,r)$, $\lambda \in B(\lambda_0,s)$ we have

$$\|F(x,\lambda)\| = \|g(h(x) - f(x,\lambda))\| = \|g(h(x) - f(x,\lambda)) - g(h(0) - f(0,\lambda_0))\| \leq$$

$$\leq \gamma\|h(x) - f(x,\lambda) - h(0) + f(0,\lambda_0))\| \leq$$

$$\leq \gamma\|h(x) - f(x,\lambda) - h(0) + f(0,\lambda)\| + \gamma\|f(0,\lambda_0) - f(0,\lambda)\| \leq$$

$$\leq \gamma\varepsilon r + (1-\gamma\varepsilon)r = r\,.$$

Hence $F\,(B(0,r) \times B(\lambda_0,s)) \subset B(0,r)$.

We can define now the mapping

$$P : \mathbf{C}\,(B(\lambda_0,s)\,,\,B(0,r)) \to \mathbf{C}\,(B(\lambda_0,s)\,,\,B(0,r))$$

by $P(x)(\lambda) = F(x(\lambda),\lambda)$.

Let $x_1, x_2 \in \mathbf{C}(B(\lambda_0,s)\,,\,B(0,r))$. Then

$$\|P(x_1) - P(x_2)\| = \sup_{\lambda\in B(\lambda_0,s)} \|P(x_1)(\lambda) - P(x_2)(\lambda)\| =$$

$$= \sup_{\lambda\in B(\lambda_0,s)} \|g\,(h(x_1(\lambda)) - f(x_1(\lambda),\lambda)) - g\,(h(x_2(\lambda)) - f(x_2(\lambda),\lambda))\,\| \leq$$

$$\leq \gamma\varepsilon \sup_{\lambda\in B(\lambda_0,s)} \|x_1(\lambda) - x_2(\lambda)\| = \gamma\varepsilon\|x_1 - x_2\|\,.$$

For $x \in B(0,r)$ and $\lambda \in B(\lambda_0,s)$ we have

$$\|\nabla_x F(x,\lambda)\| = \|\nabla g\,(h(x) - f(x,\lambda)) \circ (\nabla_x f(0,\lambda_0) - \nabla_x f(x,\lambda))\,\| \leq a\,.$$

We define now the mapping

$$Q\,:\,\mathbf{C}\,(B(\lambda_0,s)\,,\,B(0,r))\,\times \mathbf{B}\,(B(\lambda_0,s)\,,\,\mathbf{L}(\Lambda,X)) \to \mathbf{B}\,(B(\lambda_0,s)\,,\,\mathbf{L}(\Lambda,X))$$

by

$$Q(x,y)(\lambda) = \nabla_x F(x(\lambda),\lambda) \circ y(\lambda) + \nabla_\lambda F(x(\lambda),\lambda) .$$

Let $y_1, y_2 \in \mathbf{B}(B(\lambda_0, s), \mathbf{L}(\Lambda, X))$. Then

$$\|Q(x,y_1) - Q(x,y_2)\| = \sup_{\lambda \in B(\lambda_0,s)} \|Q(x,y_1)(\lambda) - Q(x,y_2)(\lambda)\| =$$
$$= \sup_{\lambda \in B(\lambda_0,s)} \|\nabla_x F(x(\lambda),\lambda) \circ (y_1(\lambda) - y_2(\lambda))\| \leq$$
$$\leq \sup_{\lambda \in B(\lambda_0,s)} \|\nabla_x F(x(\lambda),\lambda)\| \cdot \|y_1(\lambda) - y_2(\lambda)\| \leq a\|y_1 - y_2\| .$$

Using the continuity of ∇f and the compactness of $B(\lambda_0, s)$ we deduce that for $x \in \mathbf{C}(B(\lambda_0, s), B(0,r))$, the mappings $\nabla_x F(x(\cdot),\cdot)$ and $\nabla_\lambda F(x(\cdot),\cdot)$ are uniformly continuous on $B(\lambda_0, s)$, which implies the continuity of $Q(\cdot, y)$.

We apply now Theorem 1, to the mapping $A = (P, Q)$ and hence

$$A^n(x,y) \to (\overline{x}, \overline{y})$$

for all $x \in \mathbf{C}(B(\lambda_0, s), B(0,r))$ and $y \in \mathbf{B}(B(\lambda_0, s), \mathbf{L}(\Lambda, X))$. Let us choose $x \equiv 0$, $y \equiv 0$. Then

$$x_1(\lambda) = P(0)(\lambda) = F(0,\lambda)$$

and

$$y_1(\lambda) = Q(0,0)(\lambda) = \nabla_\lambda F(0,\lambda) = \nabla x_1(\lambda) .$$

If $y_n = \nabla x_n$, then

$$x_{n+1}(\lambda) = P(x_n)(\lambda) = F(x_n(\lambda), \lambda)$$

and

$$y_{n+1}(\lambda) = \nabla_x F(x_n(\lambda), \lambda) \circ \nabla x_n(\lambda) + \nabla_\lambda F(x_n(\lambda), \lambda) = \nabla x_{n+1}(\lambda) .$$

Hence $y_n = \nabla x_n$ for all $n \in \mathbf{N}$,

$$x_n \to \overline{x} \text{ in } \mathbf{B}(B(\lambda_0, s), B(0,r))$$

and

$$\nabla x_n \to \overline{y} \text{ in } \mathbf{B}(B(\lambda_0, s), \mathbf{L}(\Lambda, X)) .$$

This means that ∇x_n converges uniformly to $\overline{y}$ in $B(\lambda_0, s)$, so $\overline{x}$ is differentiable on $intB(\lambda_0, s)$ and $\nabla \overline{x} = \overline{y}$. Being the limit of a uniformly convergent sequence of continuous functions, $\overline{y}$ is also continuous.

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Babeş-Bolyai University, Faculty of Mathematics and Computer Science, Str. M. Kogălniceanu 1, RO-3400 Cluj-Napoca, Romania.