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ON SOME o-SCHUNCK CLASSES

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Abstract. In this paper, Ore's generalized theorems given in [4] are used to study some special o-Schunck classes. Thus we prove that: 1) the equivalence of D, A and B properties (given in [7] and [3]) on a o-Schunck class takes place; 2) the "composite" of two o-Schunck classes with the D property is in turn a o-Schunck class with the D property; 3) the class D of all o-Schunck classes with the D property, ordered by inclusion, forms respect to the operations of "composite" and intersection a complete lattice.

1. Preliminaries

All groups considered in the paper are finite. We denote by o an arbitrary set of primes and by o' the complement to o in the set of all primes.

Definition 1.1. a) A class $\underline{\mathbf{X}}$ of groups is a *homomorph* if $\underline{\mathbf{X}}$ is closed under homomorphisms.

b) A group G is *primitive* if G has a stabilizer, i.e. a maximal subgroup W with $\mathrm{core}_G W = 1$, where

$$\mathrm{core}_G W = 3\{W^g/g\chi G\}.$$

c) A homomorph $\underline{\mathbf{X}}$ is a *Schunck class* if $\underline{\mathbf{X}}$ is *primitively closed*, i.e. if any group G, all of whose primitive factor groups are in $\underline{\mathbf{X}}$, is itself in $\underline{\mathbf{X}}$.

Definition 1.2. Let $\underline{\mathbf{X}}$ be a class of groups, G a group and H a subgroup of G. We say that:

a) H is an $\underline{\mathbf{X}}$-*subgroup* of G if $H\chi\underline{\mathbf{X}}$;

b) H is an $\underline{\mathbf{X}}$-*maximal subgroup* of G if:

(1) $H\chi\underline{\mathbf{X}}$;

(2) from $H[H^*[G, H^*\chi\underline{\mathbf{X}}$ follows $H = H^*$.

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c) H is an $\underline{\mathbf{X}}$*-covering subgroup* of G if :

(1) $H\chi\underline{\mathbf{X}}$;

(2) $H[V[G, V_0 \leftrightarrow V, V/V_0\chi\underline{\mathbf{X}}$ imply $V = HV_0$.

Obviously we have:

Proposition 1.3. *Let $\underline{\mathbf{X}}$ be a homomorph, G a group and H a subgroup of G. If H is an $\underline{\mathbf{X}}$-covering subgroup of G, then H is $\underline{\mathbf{X}}$-maximal in G.*

The converse of 1.3. does not hold generally.

Definition 1.4. a) A group G is *σ-solvable* if any chief factor of G is either a solvable σ-group or a σ'-group. For σ the set of all primes we obtain the notion of "solvable group".

b) A class $\underline{\mathbf{X}}$ of groups is said to be *σ-closed* if:

$$G/O\pi'(G) \in \underline{\mathbf{X}} \Rightarrow G \in \underline{\mathbf{X}},$$

where $O\pi(G)$ denotes the largest normal π'-subgroup of G. We shall call *π-homomorph* a π-closed homomorph and *π-Schunck class* a π-closed Schunck class.

In our considerations we shall use the following result of R. Baer given in [1]:

Theorem 1.5. *A solvable minimal normal subgroup of a group is abelian.*

2. Ore's generalized theorems

In [4] we obtained a generalization on π-solvable groups of some of Ore's theorems given only for solvable groups. In this paper we shall use the following of them:

Theorem 2.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Theorem 2.2. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

Theorem 2.3. *Let G be a π-solvable group such that:*

(i) there is a minimal normal subgroup M of G which is a solvable π-group and $C_G(M) = M$;

(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a π'-group. Then G is primitive.

Theorem 2.4. *If G is a π-solvable group satisfying (i) and (ii) from 2.3., then any two stabilizers W_1 and W_2 of G are conjugate in G.*

3. Some special π-Schunck classes

Ore's generalized theorems are a powerful tool in the formation theory of π-solvable groups. This is proved by [5], which we complete here with new results. We first give a new proof, based on Ore's generalized theorems, for the equivalence of D, A and B properties (given in [7] and [3]) on a π-Schunck class.

Definition 3.1. ([7]; [3]) Let $\underline{\mathbf{X}}$ be a π-Schunck class. We say that $\underline{\mathbf{X}}$ has *the D property* if for any π-solvable group G, every $\underline{\mathbf{X}}$-subgroup H of G is contained in an $\underline{\mathbf{X}}$- covering subgroup E of G.

Remark 3.2. Definition 3.1. has sense because of the existence theorem of $\underline{\mathbf{X}}$- covering subgroups in finite π-solvable groups ([5]), where $\underline{\mathbf{X}}$ is a π-Schunck class. Furthermore, any two covering subgroups are conjugate.

Theorem 3.3. *Let $\underline{\mathbf{X}}$ be a π-Schunck class. $\underline{\mathbf{X}}$ has the D property if and only if in any π-solvable group G, every $\underline{\mathbf{X}}$-maximal subgroup is an $\underline{\mathbf{X}}$-covering subgroup.*

Proof. Suppose $\underline{\mathbf{X}}$ has the D property. Let G be a π-solvable group and H an $\underline{\mathbf{X}}$-maximal subgroup of G. Obviously $H \in \underline{\mathbf{X}}$. Applying the D property we obtain that $H \subseteq E$, where E is an $\underline{\mathbf{X}}$-covering subgroup of G. But H is $\underline{\mathbf{X}}$-maximal in G. It follows that $H = E$ and so H is an $\underline{\mathbf{X}}$-covering subgroup of G.

Conversely, suppose that in any π-solvable group G every $\underline{\mathbf{X}}$-maximal subgroup is an $\underline{\mathbf{X}}$-covering subgroup. Let G be a π-solvable group and H an $\underline{\mathbf{X}}$-subgroup of G. If H itself is $\underline{\mathbf{X}}$-maximal in G, we put $E = H$ and E is an $\underline{\mathbf{X}}$-covering subgroup of G. If H is not $\underline{\mathbf{X}}$-maximal in G, let E be an $\underline{\mathbf{X}}$-maximal subgroup of G such that $H \subseteq E$. Then $H \subseteq E$ and E is an $\underline{\mathbf{X}}$-covering subgroup of G. So $\underline{\mathbf{X}}$ has the D property. □

Definition 3.4. ([7];[3])

a) The π-Schunck class $\underline{\mathbf{X}}$ has the *A property* if for any π-solvable group G and any subgroup H of G with $core_G H \neq 1$, every $\underline{\mathbf{X}}$-covering subgroup of H is contained in an $\underline{\mathbf{X}}$-covering subgroup of G.

b) Let G be a group and S a subgroup of G. The subgroup S ***avoids*** the chief factor M/N of G if $S \cap M \subseteq N$. Particularly, if N is a minimal normal subgroup of G, S ***avoids*** N if $S \cap N = 1$.

c) The π-Schunck class $\underline{\mathbf{X}}$ has the B ***property*** if for any π-solvable group G and any minimal normal subgroup N of G, the existence of an $\underline{\mathbf{X}}$-covering subgroup of G which avoids N implies that every $\underline{\mathbf{X}}$-maximal subgroup of G avoids N.

Theorem 3.5. *Let $\underline{\mathbf{X}}$ be a π-Schunck class. The following statements are equivalent:*

(i) $\underline{\mathbf{X}}$ has the A property;

(ii) $\underline{\mathbf{X}}$ has the D property;

(iii) $\underline{\mathbf{X}}$ has the B property.

Proof. A proof of 3.5. is given in [3], using some of R. Baer's theorems from [1]. We consider the same proof like in [3] for (2) $\Rightarrow$ (3) and for (3) $\Rightarrow$ (1).

A new proof is given here for (1) $\Rightarrow$ (2). This proof is based on Ore's generalized theorems. Let $\underline{\mathbf{X}}$ be a π-Schunck class and suppose that $\underline{\mathbf{X}}$ has the A property. In order to prove that $\underline{\mathbf{X}}$ has the D property we use 3.3. Let G be a π-solvable group and H an $\underline{\mathbf{X}}$- maximal subgroup of G. Let now S be an $\underline{\mathbf{X}}$-covering subgroup of G (S exists by 3.2.). We shall prove by induction on $|G|$ that H and S are conjugate in G. Two cases are considered:

1) $G \in \underline{\mathbf{X}}$. Then $H = S = G$.

2) $G \notin \underline{\mathbf{X}}$. Let N be a minimal normal subgroup of G. Applying the induction on G/N, we deduce that $HN = S^g N$, where $g \in G$. Hence $H \subseteq S^g N$. Again two cases are considered:

   a) $S^g N \subset G$. Applying the induction on $S^g N$, we obtain that H and S^g are conjugate in $S^g N$. Hence H and S are conjugate in G.

   b) $S^g N = G$. It follows that $G = (SN)^g$, hence $S^g N = G = SN$. If $\mathrm{core}_G S \neq 1$, the induction on $G/\mathrm{core}_G S$ leads to $H^x \mathrm{core}_G S = S$, where $x \in G$. Then $H^x \subseteq S$. So $H^x = S$, which means that H and S are conjugate in G. Let now $\mathrm{core}_G S = 1$. G being π-solvable, N is either a solvable π-group or a π'-group. Supposing that N is a π'-group we have $N \leq O\pi'(G)$ and

$$G/O\pi'(G)\varphi(G/N)/(O\pi'(G)/N),$$

where

$$G/N = SN/N\varphi S/S3N\chi\underline{\mathbf{X}}.$$

So $G/O\pi'(G) \in \underline{\mathbf{X}}$, which implies by the π-closure of $\underline{\mathbf{X}}$ that $G \in \underline{\mathbf{X}}$, a contradiction. It follows that N is a solvable π-group, hence by 1.5., N is abelian. This and $G = SN$ lead to $S \cap N = 1$ and S is a maximal subgroup of G. From $H \in \underline{\mathbf{X}}$ and $G \notin \underline{\mathbf{X}}$ we have $H \subset G$. Let M be a maximal subgroup of G such that $H \subseteq M$. Applying the induction on M it follows that H is an $\underline{\mathbf{X}}$-covering subgroup of M. We consider now two possibilities:

b.1) $core_G M \neq 1$. Applying the A property on G, $M < G$, $\text{core}_G M \neq 1$, the $\underline{\mathbf{X}}$-covering subgroup H of M and the $\underline{\mathbf{X}}$-covering subgroup S of G, we obtain $H \subseteq S^x$, where $x \in G$. Hence $H = S^x$. So H and S are conjugate in G.

b.2) $core_G M = 1$. Then S and M are two stabilizers of G. Hence G is primitive. We prove now that G satisfies (i) and (ii) from 2.3.:

(i) There is a minimal normal subgroup M of G which is a solvable π-group and $C_G(M) = M$. Indeed, we put $M = N$. We proved that N is a solvable π-group and by 2.2. we have $C_G(N) = N$.

(ii) There is a minimal normal subgroup L/N of G/N such that L/N is a π'-group. Suppose the contrary, i.e. any minimal normal subgroup L/N of G/N is a solvable π-group. Since N is also a solvable π-group, it follows that L is a solvable π-group. By 2.1., N is the only minimal normal subgroup of G. If L is a minimal normal subgroup of G, obviously follows that $L = N$ and $L/N = 1$, in contradiction with L/N minimal normal subgroup of G/N. If L is not a minimal normal subgroup of G, we have $N \subset L$ and again a contradiction is obtained by $G = SN \subset SL = G$.

So G satisfies (i) and (ii) from 2.3. Then by 2.4., S and M are conjugate in G, i.e. $M = S^x$, where $x \in G$. But $H \subseteq M$, hence $H \subseteq S^x$, where $S^x \in \underline{\mathbf{X}}$. H being $\underline{\mathbf{X}}$-maximal, it follows that $H = S^x$.

□

4. The "composite" of two π-Schunck classes

Let us note by $\underline{\mathbf{D}}$ the class of all π-Schunck classes with the D property.

Definition 4.1. ([3]) If $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are two π-Schunck classes, we define the "composite" $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ as the class of all π-solvable groups G such that $G =< S, T >$, where S is an $\underline{\mathbf{X}}$-covering subgroup of G and T is an $\underline{\mathbf{Y}}$-covering subgroup of G.

In [3] we proved the following result:

Theorem 4.2. *If $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are two π-Schunck classes, then $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ is also a π- Schunck class.*

Using Ore's generalized theorems we can prove now:

Theorem 4.3. *If $\underline{\mathbf{X}} \in \underline{\mathbf{D}}$ and $\underline{\mathbf{Y}} \in \underline{\mathbf{D}}$, then $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle \in \underline{\mathbf{D}}$.*

Proof. By 4.2., $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ is a π-Schunck class. Let us prove that $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$ has the D property using 3.3. Let G be a π-solvable group and H an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-maximal subgroup of G. We prove by induction on $|G|$ that H is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G. We consider two cases:

1) $G \in \langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$. Then $H = G$ is its own $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup.
2) $G \notin \langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$. Applying 3.2., there is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup P of G. We shall prove that $H = P^x$, where $x \in G$.

Let N be a minimal normal subgroup of G. By the induction on G/N, if we take $HN/N\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-maximal in G/N and $PN/N\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G/N, we have $HN/N \subseteq P^gN/N$ for some $g \in G$. Hence $H \subseteq P^gN$. Now two possibilities:

a) $P^gN \subset G$. Applying the induction on P^gN, for H $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-maximal in P^gN and P^g an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of P^gN, it follows that $H = (P^g)^{g'} = P^{gg'}$, where $g' \in P^gN$. So $H = P^{gg'}$ is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G.
b) $P^gN = G$. Then $G = PN$. Again two cases:
   b.1) $\mathrm{core}_G P \neq 1$. By the induction on $G/\mathrm{core}_G P$, we have $H = P^x$, where $x \in G$. So H is an $\langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$-covering subgroup of G.
   b.2) $\mathrm{core}_G P = 1$. First N is a solvable π-group, for if we suppose that N is a π'-group, we have $N \subseteq O\pi'(G)$ and

$$G/O\pi'(G)\varphi(G/N)/(O\pi'(G)/N);$$

$$G/N = PN/N\varphi P/P \cap N \in \langle\underline{\mathbf{X}},\underline{\mathbf{Y}}\rangle$$

imply $G/O\pi'(G) \in \langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$, hence $G \in \langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$, a contradiction. By 1.5., N is abelian. From $G = PN$ and N abelian, we deduce that $P \cap N = 1$, hence P is a maximal subgroup of G. So P is a stabilizer of G and G is primitive. Then, by 2.1., we obtain that N is the only minimal normal subgroup of G and by 2.2. that $C_G(N) = N$. It is easy to notice that $HN = G$ and so, like for P, we have $H \cap N = 1$ and H is a maximal subgroup of G. Now we consider two possibilities:

b.2.1) $\mathrm{core}_G H \neq 1$. Applying the induction on $G/\mathrm{core}_G H$, we obtain that $H = P^x$ $(x \in G)$ is an $\langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$-covering subgroup of G.

b.2.2) $\mathrm{core}_G H = 1$. Then H is a stabilizer of G. Let us notice that we are in the hypotheses of theorem 2.4. Indeed, (i) is true, because N is a minimal normal subgroup of G which is a solvable π-group and $C_G(N) = N$. Further, (ii) is also true, for if we suppose the contrary, we obtain that any minimal normal subgroup L/N of G/N is a solvable π-group and in each of the two cases given below we get a contradiction:

(#): If L is a minimal normal subgroup of G, obviously $L = N$ and $L/N = 1$, in contradiction with L/N minimal normal subgroup of G/N.

(##): If L is not a minimal normal subgroup of G, then $N \subset L$ and $G = HN \subset HL = G$, a contradiction.

So we are in the hypotheses of theorem 2.4. It follows that the two stabilizers P and H of G are conjugate in G, i.e. there is $x \in G$ such that $H = P^x$. But this means that H is an $\langle \underline{\mathbf{X}}, \underline{\mathbf{Y}} \rangle$-covering subgroup of G.

□

An immediate consequence of theorem 4.3. is the following:

Theorem 4.4. *The class D, ordered by inclusion, forms respect to the operations of "composite" and intersection a complete lattice.*

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