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EXTENSION OF BILINEAR FUNCTIONALS AND BEST APPROXIMATION IN 2-NORMED SPACES

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Abstract. The paper investigates the relations between the extension properties of bounded bilinear functionals and the approximation properties in 2-normed spaces.

1. Introduction

In the sixties S.Gähler ([8] and [9]) introduced and studied the basic properties of 2-metric and 2-normed spaces. Since then these topics have been intensively studied and deve loped.The references given at the end of this paper are far from being complete, containing only the papers related to the problems treated here.

The aim of the present paper is to study the relations between the extension properties of bounded bilinear functionals and the approximation properties in 2-normed spaces. In the case of bounded linear functionals on normed linear spaces the problem was first considered by R.R.Phelps [19]. For other related results see I. Singer's book [20].

In the case of Banach spaces of Lipschitz functions similar results were obtained by the authors (see [1], [18]). The case of bilinear operators on 2-normed spaces has been considered in [2].

Throughout this paper all the linear spaces will be considered over the field $K = \mathbf{R}$ or $K = \mathbf{C}$. A 2-*norm* on a linear space X of algebraic dimension at least 2 , is a functional $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ verifying the axioms:

BN 1) $\|x,y\| = 0$ if and only if x, y are linearly dependent,

BN 2) $\|x,y\| = \|y,x\|$,

BN 3) $\|\lambda x,y\| = \|\lambda\| \cdot \|x,y\|$,

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BN 4) $\|x+y,z\| \le \|x,z\| + \|y,z\|$,

for all $x,y,z \in X$ and $\lambda \in K$ (see [9])

If $\|\cdot,\cdot\|$ is a 2-norm on the linear space X then the function $\rho : X^3 \to [0,\infty)$ defined by $\rho(x,y,z) = \|x-z, y-z\|$, $x,y,z \in X$ is a 2-metric on X, in the sense of S.Gähler [8], which is translation invariant, i.e. $\rho(x+a, y+a, z+a) = \rho(x,y,z)$ for all $x,y,z \in X$ and a fixed element $a \in X$.

For a fixed $b \in X$, the function $p_b(x) = \|x,b\|$, $x \in X$, is a seminorm on X and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X, called the *natural topology induced by the 2-norm* $\|\cdot,\cdot\|$.

A pair $(X, \|\cdot,\cdot\|)$ where X is a linear space and $\|\cdot,\cdot\|$ a 2-norm an X will be called a 2-*normed space.*

Remark 1. S.Gähler [10] considered only 2-normed space over the field $\mathbf{R}$ of real numbers, but his definition automatically extends to the complex scalars too.

2. Continuity and boundedness properties for bilinear functionals.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and X_1, X_2 two subspaces of X. A 2-*functional* is an application $f : X_1 \times X_2 \to K$. The 2-functional f is called *bilinear* if:

BL 1) $f(x+x', y+y') = f(x,y) + f(x,y') + f(x',y) + f(x',y')$

BL 2) $f(\alpha x, \beta y) = \alpha\beta f(x,y)$,

for all $(x,y), (x',y')$ in $X_1 \times X_2$ and all $\alpha, \beta \in K$.

A 2-functional $f : X_1 \times X_2 \to K$ is called *bounded* if there exists a real number $L \ge 0$ (called a *Lipschitz constant* for f) such that

$$|f(x,y)| \le L\|x,y\|, \tag{2.1}$$

for all $(x,y) \in X_1 \times X_2$.

This notion of boundedness was introduced by A.G.White Jr. [20] who defined also the *norm* of a bounded bilinear functional by:

$$\|f\| = \inf\{L \ge 0 : L \text{ is a Lipschitz constant for } f\} \tag{2.2}$$

Some immediate consequences of the definition are given in:

Proposition 2.1. *(A.G.White Jr. [21].) Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, X_1, X_2 two linear subspaces of X and $f : X_1 \times X_2 \to K$ a bounded bilinear functional. Then* *a)* $f(x,y) = 0$, *for any pair* $(x,y) \in X_1 \times X_2$ *of linear dependent elements;*

b) $f(y,x) = -f(x,y)$ *,i.e.* f *is an alternate bilinear functional;*

c) The norm $\|f\|$ *of* f *can be calculated also by the formulae:*

$$\begin{aligned} \|f\| &= \sup\{|f(x,y)| : (x,y) \in X_1 \times X_2, \|x,y\| \leq 1\} \\ &= \sup\{|f(x,y)| : (x,y) \in X_1 \times X_2, \|x,y\| = 1\} \\ &= \sup\{|f(x,y)|/\|x,y\| : (x,y) \in X_1 \times X_2, \|x,y\| > 0\}. \end{aligned} \tag{2.3}$$

A.G.White Jr. [21] defined a kind of continuity for 2-functionals, called subsequently 2-continuity by S.Gähler [11].

A 2-functional $f : X_1 \times X_2 \to K$, where X_1, X_2 are linear subspaces of a 2-normed space $(X, \|\cdot,\cdot\|)$ is called *2-continuous* at $(x_0, y_0) \in X_1 \times X_2$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - f(x_0,y_0)| < \varepsilon$ whenever

$$\begin{aligned} &(i) \quad \|x, y - y_0\| < \delta \text{ and } \|x_0 - x, y\| < \delta, \text{ or} \\ &(ii) \quad \|x_0 - x, y\| < \delta \text{ and } \|x_0, y_0 - y\| < \delta \end{aligned} \tag{2.4}$$

A 2-functional f is called *2-continuous* on $X_1 \times X_2$ if it is 2-continuous at every point $(x,y) \in X_1 \times X_2$.

An example of 2-continuous 2-functional is given by:

Proposition 2.2. *(A.G.White Jr. [21, Th 2.2]) If* $(X, \|\cdot,\cdot\|)$ *is a 2-normed space then the 2-functional* $\|\cdot,\cdot\|$ *is 2-continuous on* $X \times X$.

It turns out that for bilinear functionals, boundedness and 2-continuity are equivalent and 2-continuity at $(0,0)$ implies 2-continuity on whole $X_1 \times X_2$:

Theorem 2.3. *(A.G.White Jr. [21, Theorems 2.3 and 2.4]) a) A bilinear functional* $f : X_1 \times X_2 \to K$ *is 2-continuous on* $X_1 \times X_2$ *if and only if it is bounded;*

b) A bilinear functional $f : X_1 \times X_2 \to K$ *which is 2-continuous at* $(0,0)$ *is continuous on* $X_1 \times X_2$.

S.Gähler [11] remarked that 2-continuity of a 2-functional f on $X \times X$ and its continuity with respect to the product topology on $X \times X$ are different notions. By proposition 2.2 a 2-norm is a 2-continuous functional on $X \times X$, but S.Gähler [11] exhibited an example of a 2-norm which is not continuous on $X \times X$ (with respect to the product topology) and gave conditions ensuring the continuity of a 2-norm on $X \times X$.

There are also examples of 2-functionals which are continuous on $X \times X$ with respect to the product topology but are not 2-continuous (see also S.Gähler [11]).

3. Extension theorems for bounded bilinear functionals.

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, X_1, X_2 two linear subspaces of X and $f: X_1 \times X_2 \to K$ a bounded bilinear functional. The extension problem for f consists in finding a bounded bilinear functional $F : X \times X \to K$ such that

$$\begin{aligned} &i)\ F(x,y) = f(x,y), \text{ for all } (x,y) \in X_1 \times X_2, \\ &ii)\ \|F\| = \|f\|. \end{aligned} \tag{3.1}$$

We agree to call such an F a *norm preserving extension* or a *Hahn-Banach extension* of f. As it was remarked by S.Gähler [11], p.345 Korollar zu S.5 und S.6, the norm preserving extension is not always possible. Some Hahn-Banach and Hahn type extension theorems for subspaces of the form $Y \times [b]$, where Y is a linear subspace of X, $b \in X$ and $[b]$ denotes the subspace of X spanned by b, were proved in the case of real 2-normed spaces by A.G.White Jr. [21], S.Mabizela [17] and I.Franić [7].

In the following we shall show that all these extension results can be derived directly from the classical Hahn-Banach theorem. This approach allows to consider simultaneously both the cases of real and complex scalars.

Our methods of proofs rely upon slight extensions of Hahn-Banach and Hahn theorems from normed to seminormed spaces.

In what follows (X,p) will denote a seminormed space (over the field $K = \mathbf{R}$ or $\mathbf{C}$), with p a nontrivial seminorm on X (i.e. $p \neq 0$). It is well known that a linear functional x^* is continuous on X if and only if it is bounded (or Lipschitz) on X, i.e. there exists a number $L \geq 0$ such that

$$|x^*(x)| \leq L \cdot p(x), \text{ for all } x \in X. \tag{3.2}$$

A number $L \geq 0$ verifying (3.2) is called a *Lipschitz constant* for x^*.

Proposition 3.1. *Let (X,p) be a seminormed space, X^* its conjugate space and let $q : X^* \to [0,\infty)$ be defined by*

$$q(x^*) = \sup\{|x^*(x)| : x \in X, p(x) \leq 1\} \tag{3.3}$$

Then

a) $|x^*(x)| \le q(x^*) \cdot p(x)$, *for all* $x \in X$;

b) $q(x^*) = \inf\{L \ge 0 : L$ *is a Lipschitz constant for* $x^*\}$;

c) The functional q *is a norm on* X^* *and* (X^*, q) *is a Banach space.*

Proof. a) Since $x^* \in X^*$ there exists $L \ge 0$ such that (3.2) holds. Now, if $x \in X$ is such that $p(x) = 0$ then $x^*(x) = 0$ too, and the inequality a) is trivially verified. If $p(x) > 0$ then $p\left(\frac{1}{p(x)} \cdot x\right) = 1$ so that $|x^*\left(\frac{1}{p(x)} \cdot x\right)| \le q(x^*)$, which is equivalent to a).

b) If $L \ge 0$ verifies (3.2) then $|x^*(x)| \le L$, for all $x \in X$ with $p(x) \le 1$, implying $q(x^*) \le L$. Since $L \ge 0$ is an arbitrary Lipschitz constant it follows

$$q(x^*) \le \inf\{L \ge 0 : L \text{ is a Lipschitz constant for } x^*\}.$$

Because $q(x^*)$ is a Lipschitz constant for x^* it follows that

$$q(x^*) = \min\{L \ge 0 : L \text{ is a Lipschitz constant for } x^*\}$$

implying the equality b).

c) It is immediate from (3.3) that q is a seminorm on X^*. If $x^* \ne 0$ and $x_0 \in X$ is such that $x^*(x_0) \ne 0$ then by a)

$$0 < |x^*(x_0)| \le q(x^*) \cdot p(x_0)$$

implying $q(x^*) > 0$ and showing that q is a norm on X^*.

The proof that (X^*, q) is a Banach space is standard and we omit it. □

Theorem 3.2. *(Hahn-Banach Theorem). Let* (X, p) *be a seminormed space (over* $K =$ **R** *or* **C***) with* $p \ne 0$, Y *a linear subspace and* $y^* \in Y^*$ *a continuons linear functional on* Y. *Define* $q_1(y^*)$ *by*

$$q_1(y^*) = \sup\{|y^*(y)| : y \in Y, p(y) \le 1\}. \tag{3.4}$$

Then there exists a continuous linear functional x^* *on* X*such that*

$$\begin{aligned} &i)\ x^*|_Y = y^* \text{ and} \\ &ii)\ q(x^*) = q_1(y^*) \end{aligned} \tag{3.5}$$

where $q(x^*)$ *is defined by* (3.3).

Proof. The functional $p_1 : X \to [0,\infty)$ defined by $p_1(x) = q_1(y^*) \cdot p(x)$, $x \in X$ is a seminorm on X and $|x^*(y)| \le p_1(y)$ for all $y \in Y$, i.e. y^* is dominated by p_1. By the Hahn-Banach Theorem (see e.g. [6] or [14]) there exists $x^* \in X^*$ such that

$$\begin{aligned} &i) \quad x^* |_Y = y^* \\ &ii) \quad |x^*(x)| \le q_1(y^*) \cdot p(x), \text{ for all } x \in X. \end{aligned} \tag{3.6}$$

By (3.6) ii) and Proposition 3.1 b) we obtain $q(x^*) \le q_1(y^*)$. The reverse inequality follows from

$$\begin{aligned} q(x^*) &= \sup\{|x^*(x)| : x \in X, p(x) \le 1\} \\ &\ge \sup\{|x^*(y)| : y \in Y, p(y) \le 1\} \\ &= q_1(y^*). \end{aligned}$$

□

Hahn's theorem ([6, Lemma II. 3.12) can be transposed to the seminormed case too

Theorem 3.3. *(Hahn Theorem). Let (X,p) be a seminormed space, Y a linear subspace of X and $x_0 \in X \setminus \overline{Y}$. Then there exists a functional $x^* \in X^*$ such that*

$$\begin{aligned} &i) \quad x^*(x_0) = 1 \text{ and } x^*(Y) = \{0\}; \\ &ii) \quad q(x^*) = \delta^{-1} \end{aligned} \tag{3.7}$$

where $\delta = \inf\{p(x_0 - y) : y \in Y\}$.

Proof. Observe that $x_0 \in X \setminus \overline{Y}$ implies $\delta > 0$. Let $Z = Y \dot{+} Kx_0$ and let $z^* : Z \to K$ be defined by $z^*(y + \alpha x_0) = \alpha$, for $y \in Y$ and $\alpha \in K$. Obviously that z^* is linear and, for $\alpha \ne 0$,

$$|z^*(y + \alpha x_0)| = |\alpha| \le |\alpha| \cdot \delta^{-1} \cdot p\left(\alpha^{-1} y + x_0\right) = \delta^{-1} \cdot p(y + \alpha x_0)$$

Since, for $\alpha = 0$, $|z^*(y)| = 0 \le \delta^{-1} \cdot p(y)$ it follows the continuity of z^* and $q_1(z^*) \le \delta^{-1}$, where $q_1(z^*) = \sup\{|z^*(z)| : z \in Z, p(z) \le 1\}$. Taking a minimizing sequence $(y_n) \subseteq Y$ (i.e. $p(x_0 - y_n) \to \delta$, for $n \to \infty$), we obtain

$$1 = z^*(x_0 - y_n) = |z^*(x_0 - y_n)| \le q_1(z^*) \cdot p(x_0 - y_n),$$

which for $n \to \infty$ gives $q_1(z^*) \ge \delta^{-1}$, implying $q_1(z^*) = \delta^{-1}$.

Now Theorem 3.3 follows from Theorem 3.2 applied to Z and z^*. □

Remark 2. The functional $x_1^* \in X^*$, $x_1^* = \delta \cdot x^*$, verifies the conditions:

$$\begin{aligned} &i)\ x_1^*(x_0) = \delta \text{ and } x_1^*(Y) = \{0\} \\ &ii)\ q(x_1^*) = 1 \end{aligned} \tag{3.8}$$

Pass now to the extension theorems for bounded bilinear functionals. The reduction to Hahn-Banach and Hahn's theorems for bounded linear functionals on seminormed linear spaces will be based on the following result:

Proposition 3.4. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space (over $K = \mathbf{R}$ or $\mathbf{C}$), Z a subspace of X, $b \in X \setminus \{0\}$ and let $[b]$ be the subspace of X spanned by b. Denote by p_b the seminorm on Z given by*

$$p_b(z) = \|z, b\|,\ z \in Z,$$

and let q_b be its conjugate norm on Z^*, in the sense of Proposition 3.1. Then

a) If $f : Z \times [b] \to K$ is a bounded bilinear functional then the functional $z^* : Z \to K$ defined by $z^*(z) = f(z, b)$, $z \in Z$ is a continuous linear functional on Z and

$$q_b(z^*) = \|f\|.$$

b) Conversely, if z^* is a bounded linear functional on Z, then the 2-functional $f : Z \times [b] \to K$ defined by $f(z, \alpha b) = \alpha z^*(z)$, for $(z, \alpha) \in Z \times K$, is a bounded bilinear functional and

$$\|f\| = q_b(z^*).$$

Proof. a) Obviously that, for a given bounded bilinear functional $f : Z \times [b] \to K$, the functional $z^* : Z \to K$ defined by $z^*(z) = f(z, b)$, $z \in Z$, is a linear functional on Z and

$$|z^*(z)| = |f(z, b)| \le \|f\| \cdot \|z, b\| = \|f\| \cdot p_b(z),$$

for all $z \in Z$, implying that z^* is a continuous linear functional on the seminormed space (Z, p_b) and

$$q_b(z^*) \le \|f\|.$$

On the other hand

$$|f(z, \alpha b)| = |f(\alpha z, b)| = |z^*(\alpha z)| \le q_b(z^*) \cdot p_b(\alpha z) = q_b(z^*) \cdot \|\alpha z, b\| = q_b(z^*) \cdot \|z, \alpha b\|$$

implying that $q_b(z^*)$ is a Lipschitz constant for f, so that $||f|| \le q_b(z^*)$ and, therefore, $||f|| = q_b(z^*)$.

b) Suppose now that z^* is a given continuous linear functional on the seminormed space (Z, p_b) and define $f : Z \times [b] \to K$ by $f(z, \alpha b) = \alpha \cdot z^*(z)$, $(z, \alpha) \in Z \times K$. Obviously that f is a bilinear functional and

$$|f(z,\alpha b)| = |\alpha z^*(z)| = |z^*(\alpha z)| \le q_b(z^*) \cdot p_b(\alpha z) = \\ = q_b(z^*) \cdot ||\alpha z, b|| = q_b(z^*) \cdot ||z, \alpha b||,$$

for all $(z, \alpha) \in Z \times K$, showing that f is a bounded bilinear functional and that $||f|| \le q_b(z^*)$.

Taking into account the fact that $p_b(z) = ||z, b||$ we obtain

$$q_b(z^*) = \sup\{|z^*(z)| : z \in Z, ||z,b|| \le 1\} = \sup\{|f(z,b)| : z \in Z, ||z,b|| \le 1\} \le \\ \le \sup\{|f(z,\alpha b)| : (z,\alpha) \in Z \times K, ||z, \alpha b|| \le 1\} = ||f||$$

Again the equality $||f|| = q_b(z^*)$ holds. □

Now we are in position to prove the promised extension theorem.

Theorem 3.5. *(Hahn-Banach Extension Theorem, A.G.White Jr. [21, Th.2.7]) Let $(X, ||\cdot,\cdot||)$ be a 2-normed space (over $K = \mathbf{R}$ or $\mathbf{C}$), Y a subspace of X, $b \in X$ and let $[b]$ be the subspace of X spanned by b. If $f : Y \times [b] \to K$ is a bounded bilinear functional then there exists a bounded bilinear functional $F : X \times [b] \to K$ such that*

$$\begin{aligned} &i) \quad F|_{Y \times [b]} = f, \text{ and} \\ &ii) \quad ||F|| = ||f||. \end{aligned} \tag{3.9}$$

Proof. Let $p_b : X \to [0, \infty)$ be the seminorm defined by $p_b(x) = ||x, b||$, $x \in X$, and let $y^* : Y \to K$ be given by $y^*(y) = f(y, b)$. Then by Proposition 3.4 a), y^* is a continuous linear functional on Y and $q_b'(y^*) = ||f||$, where

$$q_b'(y^*) = \sup\{|y^*(y)| : y \in Y, p_b(y) \le 1\}. \tag{3.10}$$

By Theorem 3.2 there exists a bounded linear functional $x^* \in X^*$ such that $x^*|_Y = y^*$ and $q_b(x^*) = q_b'(y^*)$, where

$$q_b(x^*) = \sup\{|x^*(x)| : x \in X, p_b(x) \le 1\}. \tag{3.11}$$

Defining now $F : X \times [b] \to K$ by $F(x, \alpha b) = \alpha \cdot x^*(x)$, for $(x, \alpha) \in X \times K$ and applying Proposition 3.4 b) it follows that the bilinear functional F fulfils all the requierements of the Theorem. □

The analogue of Hahn's·theorem for bilinear functionals is:

Theorem 3.6. *(S.Mabizela [17, Th.2]) Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space over $K = \mathbf{R}$ or $\mathbf{C}$, Y a linear subspace of X, $b \in X$ and [b] the subspace of X spanned by b. If $x_0 \in X$ is such that $\delta > 0$, where*

$$\delta = \inf\{\|x_0 - y, b\| : y \in Y\} \tag{3.12}$$

then there exists a bounded bilinear functional $F : X \times [b] \to K$ such that

$$\begin{aligned} &i)\ F(x_0, b) = 1, F(y, b = 0) \text{ for all } y \in Y, \text{ and} \\ &ii)\ \|F\| = \delta^{-1} \end{aligned} \tag{3.13}$$

Proof. Consider again the seminormed space (X, p_b), where $p_b(x) = \|x, b\|, x \in X$, and apply Theorem 3.3 to obtain a bounded linear functional x^* on X such that

$$\begin{aligned} &i)\ x^*(x_0) = 1 \text{ and } x^*(Y) = \{0\}, \text{ and} \\ &ii)\ q_b(x^*) = \delta^{-1}, \cdot \end{aligned} \tag{3.14}$$

where $q_b(x^*)$ is given by (3.11).

Defining $F : X \times [b] \to K$ by $F(x, \alpha b) = \alpha \cdot x^*(x)$, $(x, \alpha) \in X \times K$, and applying Proposition 3.4 b), it follows that the bounded bilinear functional F verifies the conditions (3.13) of the Theorem. □

Remark 3. S.Mabizela [17, Th.2] requieres for x_0 and b to be linearly independent. Observe that if x_0, b are linearly dependent then, by the axiom BN 1) in Section 1, $\|x_0, b\| = 0$ and *a fortiori* $\delta = 0$, because

$$0 \le \delta \le \|x_0 - 0, b\| = \|x_0, b\| = 0$$

Therefore the hypothesis $\delta > 0$ forces x_0 and b to be linearly independent and $x_0 \in X \backslash \overline{Y}$, where $\overline{Y}$ denotes the closure of Y in the seminormed space (X, p_b).

An immediate consequence of Theorem 3.6 is the following result, known also as Hahn's Theorem:

Theorem 3.7. *If $(X, \|\cdot,\cdot\|)$ is a 2-normed space and x_0, b are linearly independent elements in X then there exists a bounded bilinear functional $F : X \times [b] \to K$ such that:*

$$\begin{aligned} &i)\ F(x_0, b) = \|x_0, b\|, \text{ and} \\ &ii)\ \|F\| = 1. \end{aligned} \tag{3.15}$$

Proof. Putting $Y = \{0\}$ in Theorem 3.6 and taking into account the linear independence of x_0 and b, one obtains $\delta = \|x_0, b\| > 0$.

By Theorem 3.6, it follows the existence of a bounded bilinear functional $G : X \times [b] \to K$ such that $G(x_0, b) = 1$ and $\|G\| = \delta^{-1}$. Then $F = \delta \cdot G$ satisfies the conditions (3.15) of the theorem. □

4. Unique extension of bounded bilinear functionals and unique best approximation

For a 2-normed space $(X, \|\cdot,\cdot\|)$, a subspace Y of X and $b \in X$ denote by $Y_b^\sharp$ the linear space of all bounded bilinear functionals on $Y \times [b]$. Equipped with the norm (2.2), $Y_b^\sharp$ is a Banach space (see A.G.White Jr.[20]) The Banach space $X_b^\sharp$ is defined similarly.

For $f \in Y_b^\sharp$ denote by $E(f)$ the set of all norm-preserving extensions of f to $X \times [b]$,i.e.

$$E(f) = \{F \in X_b^\sharp : F|_{Y\times[b]} = f \, and \, \|F\| = \|f\|\} \tag{4.1}$$

By Theorem 3.5, $E(f) \neq \phi$ and $E(f)$ is a convex subset of the unit sphere $S(0, \|f\|) = \{G \in X_b^\sharp : \|G\| = \|f\|\}$. Indeed, for $F_1, F_2 \in E(f)$ and $\lambda \in [0,1]$,

$$(\lambda F_1 + (1-\lambda) F_2)|_{Y\times[b]} = f$$

and

$$\|\lambda F_1 + (1-x) F_2\| \leq \lambda\|F_1\| + (1-\lambda)\|F_2\| = \lambda\|f\| + (1-\lambda)\|f\| = \|f\|.$$

Denoting $G = \lambda F_1 + (1-\lambda) F_2$ it follows $G|_{Y\times[b]} = f$ and

$$\|G\| = \sup\{|G(x,\alpha b)| : (y,\alpha) \in X \times K, \|x,\alpha b\| \leq 1\} \geq$$
$$\geq \sup\{|G(y,\alpha b)| : (y,\alpha) \in Y \times K, \|y,\alpha b\| \leq 1\} = \|f\|$$

For a subspace Y of a 2-normed space $(X, \|\cdot,\cdot\|)$ let

$$Y_b^{\perp} = \{G \in X_b^{\sharp} : G(Y \times [b]) = \{0\}\} \tag{4.2}$$

be the *annihilator* of Y in $X_b^{\sharp}$.

For a nonvoid subset Z of $X_b^{\sharp}$ the *distance* of an element $F \in X_b^{\sharp}$ to Z is defined by

$$d(F,Z) = \inf\{\|F - G\| : G \in Z\}. \tag{4.3}$$

An element $G_0 \in Z$ such that $\|F - G_0\| = d(F,Z)$ is called an *element of best approximation* (or a *nearest point*) for F in Z.

Let

$$P_Z(F) = \{G \in Z : \|F - G\| = d(F,Z)\} \tag{4.4}$$

denote the set of all elements of best approximation for F in Z. The set Z is called *proximinal* if $P_Z(F) \neq \emptyset$ for all $F \in X_b^{\sharp}$, *Chebyshev* provided $P_Z(F)$ is a singleton for all $F \in X_b^{\sharp}$ and *semi-Chebyshev* if $\text{card}P_Z(F) \leq 1$, for all $F \in X_b^{\sharp}$.

A subspace of the form $Y_b^{\perp}$ of $X_b^{\sharp}$ is always proximinal and we have simple formulae for the distance of an element $F \in X_b^{\sharp}$ to $Y_b^{\perp}$ and for the set of nearest points.

Theorem 4.1. *If $(X, \|\cdot,\cdot\|)$ is a 2-normed space, Y a subspace of X, $b \in X$ and $F \in X_b^{\sharp}$ then*

$$d(F, Y_b^{\perp}) = \|F|_{Y\times[b]}\| \tag{4.5}$$

Moreover, $Y_b^{\perp}$ is a proximinal subspace of $X_b^{\sharp}$ and

$$P_{Y_b^{\perp}}(F) = F - E(F|_{Y\times[b]}) = \{F - H : H \in E(F|_{Y\times[b]})\} \tag{4.6}$$

Proof. Since $(F-G)|_{Y\times[b]} = F|_{Y\times[b]}$, for any $G \in Y_b^{\perp}$ it follows

$$\|F|_{Y\times[b]}\| = \|(F-G)|_{Y\times[b]}\| \leq \|F - G\|,$$

so that

$$\|F|_{Y\times[b]}\| \leq d(F, Y_b^{\perp}).$$

To prove the reverse inequality observe that $f = F\mid_{Y\times[b]} \in Y_b^{\sharp}$. Now if H is a norm-preserving extension of f to $X \times [b]$ then $F - H \in Y_b^{\perp}$ and

$$\|F\mid_{Y\times[b]}\| = \|H\| = \|F - (F-H)\| \geq d\left(F, Y_b^{\perp}\right),$$

proving the formula (4.5).

For $H \in E\left(F\mid_{Y\times[b]}\right)$ we have $F - H \in Y_b^{\perp}$ and $\|F - (F-H)\| = \|H\| = \|F\mid_{Y\times[b]}\| = d\left(F, Y_b^{\perp}\right)$, showing that $F - H$ is a nearest point to F in $Y^{\perp}$.

Conversely, if G is a nearest point to F in $Y_b^{\perp}$ then $(F-G)\mid_{Y\times[b]} = F\mid_{Y\times[b]}$ and, denoting $H = F - G$, it follows $G = F - H$ and

$$\|H\| = \|F - G\| = d\left(F, Y_b^{\perp}\right) = \|F\mid_{Y\times[b]}\|$$

showing that H is a norm preserving extension for $F\mid_{Y\times[b]}$. The equality (4.6) is proved and since, by Theorem 3.5, $E\left(F\mid_{Y\times[b]}\right) \neq \emptyset$, for all $F \in X_b^{\sharp}$, it follows the proximinality of the subspace $Y_b^{\perp}$ in $X_b^{\sharp}$. □

Now we are in position to state and prove the duality theorem relating the uniqueness of extension and of best approximation. Recall that for normed linear spaces and bounded linear functionals a similar result was first proved by R.R.Phelps [18].

Theorem 4.2. *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, Y a subspace of X and $b \in X$. Then the following assertions are equivalent:*

1^0 Every $f \in Y_b^{\sharp}$ has a unique norm preserving extension to $X \times [b]$;

2^0 $Y_b^{\perp}$ is a Chebyshev subspace of the Banach space $X_b^{\sharp}$.

Proof. The Theorem is an immediate consequence of the formula (4.6) from Theorem 4.1. □

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