STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLIII, Number 1, March 1998

OTHER CHARACTERIZATIONS OF THE ABELIAN GROUPS WITH THE DIRECT SUMMAND INTERSECTION PROPERTY

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. This work gives a series of characterizations, others than the previously known ones of the abelian groups with the direct summand intersection property, for short D.S.I.P., that is of those groups in which the intersection of any two direct summand is a direct summand as well. All through this paper by group we mean abelian group in additive notation.

1. The General Case

Definition. We say that a group A has the small direct summand intersection property (for short S.D.S.I.P.) if the intersection of any family of direct summands of A is again a direct summand in A.

Obviously, if a group has S.D.S.I.P., it also has D.S.I.P.. The converse is generally false.

Let A be a group and $Sd(A) = \{X \leq A | X \text{ is a direct summand in } A\}$. If A has S.D.S.I.P., then for any T, $S \in Sd(A)$, $T \cap S \in Sd(A)$ and according to [16,1.4.47.], $Sd(A)$ is a complete lattice.

Definition. A subgroup G of group A is called absolute direct summand (of A), if for any subgroup $H \leq A$, $H - G$-high in A, $A = G \oplus H$.

The absolute direct summands have been studied by Fuchs in [7]; there he demonstrated the following theorem:

Theorem 1.1. *A subgroup B of A is an absolute direct summand, in A, if and only if: B is divisible or A/B is a torsion group, whose p-component is annihilated by p^k, whenever B/pB contains an element of order p^k.*

1991 *Mathematics Subject Classification.* 20K25.

Key words and phrases. abelian groups, direct summands,intersection.

Let $Sda(A) = \{X \leq A | X$ is an absolute direct summand in $A\}$ be the set of absolute direct summands of A. Now we proof the following:

Theorem 1.2. *If the group A has D.S.I.P., then for any $T, S \in Sda(A)$, $T \cap S \in Sda(A)$.*

Proof. We are going to show that, together with T and S and $T \cap S$ satisfies (1.1.) as well. Let by $A = T \oplus T' = S \oplus S'$. According to the hypothesis $T \cap S$ is direct summand in S, so $S = T \cap S \oplus S''$.

Case 1. If T or S are divisible, then, according to [6,20.(E).], $T \cap S$ is divisible and $T \cap S \in Sda(A)$.

Case 2. If T and S are not divisible groups, then, according to (1.1.), A/T and A/S are torsion groups. So for any $a \in A$, there is a $n > 0$ so that $na \in T$ and there is a $m > 0$ so that $ma \in S$. Then $[m, n]a \in T \cap S$ ($[m, n]$ being the smallest common multiple of m and n). So $A/(T \cap S)$ is a torsion group. Let $A/T \cap S = \bigoplus_p (A/T \cap S)_p$ be, the direct decomposition of $A/(T \cap S)$ in its own p-subgroups, according to [6,8.4.]. We suppose that there is a $x \in T \cap S$ so that $p^k x \in p(S \cap T) \subset pS \cap pT$. So, there is $x \in S$ so that $p^k x \in pS$ and $x \in T$ so that $p^k x \in pT$. Now from (1.1.) it follows that $p^k(A/S)_p = S$ and $p^k(A/T)_p = T$. Then $p^k(A/T \cap S)_p = T \cap S$ too. So $T \cap S \in Sda(A)$.

Now we are going to present some other two necessary and sufficient conditions for a subgroup B of A to be a direct summand in A, if A has certain properties.

Theorem 1.3. *Let A be a group of finite rank, with property that the neat subgroups of A coincide with its direct summands. Then the following statements are equivalent for a subgroup B of group A:*

(a) B is a direct summand in A;

(b) for any prime number p, $r_p(A) = r_p(B) + r_p(A/B)_i$;

(c) there is a subgroup $C \leq A$, $C - B$-high in A so that for any prime number p, $p(A) \subseteq p(B) + C$.

Proof. (a) $\Rightarrow$ (b). If B is a direct summand in A, then $A \cong B \oplus A/B$ and $r(A) = r(B) + r(A/B)$ (see [5,2.2.5.]). Then: $r_0(A) + \sum_p r_p(A) = r_0(B) + \sum_p r_p(B) + r_0(A/B) + \sum_p r_p(A/B)$. But $r_0(A) = r_0(B) + r_0(A/B)$ ([5,2.2.(c)]). So $\sum_p r_p(A) = \sum_p r_p(B) + \sum_p r_p(A/B)$. For any prime number p, $r_p(A) = r(A_p) = r(S(A_p)) = \dim_{Z(p)} A[p]$, and if $A = B \oplus C$, then $A[p] = B[p] \oplus C[p]$ (there is immediate checking). So $\dim_{Z(p)} A[p] = \dim_{Z(p)} B[p] + \dim_{Z(p)}(A/B)[p]$.

(b) $\Rightarrow$ (a). If $r_p(A) = r_p(B) + r_p(A/B)$, is valid for any prime number p, then by summing up after all prime numbers and considering the relation: $r_0(A) = r_0(B) + r_0(A/B)$ (which occurs for any subgroup B of A), we obtain: $r(A) = r(B) + r(A/B)$. So $A \cong B \oplus A/B$, according to the hypothesis and to [6, p.132].

(a) $\Rightarrow$ (c). If B is direct summand in A, there is $C \leq A$ so that C is B-high in A, $A = B \oplus C$, and for any prime number p, $pA = pB + pC \subset pB + C$.

(c) $\Rightarrow$ (a). If (c) occurs, then, according to [5, consequence of 2.3.1.], $A = B \oplus C$.

Applying (1.3.) to groups with D.S.I.P. we obtain:

Corollary 1.4. *For an abelian group A, which satisfies (1).3), the following statements are equivalent:*

(a) A has D.S.I.P.;

(b) for any two direct summands T and S and for any p-prime number $r_p(A) = r_p(T \cap S) + r_p(A/T \cap S)$;

(c) for any two direct summands T and S, there is a subgroup $U \leq A$, $U - T \cap S$-high in A, so that for any p-prime number, $pA \subseteq p(T \cap S) + U$.

Further on we are going to present two characterizations of the abelian groups with D.S.I.P. using the groups of extensions.

Theorem 1.5. *Being given an abelian group A, the following statements are equivalent:*

a) A has D.S.I.P.;

b) for every decomposition $A = B \oplus C$, and $\beta : B \to C$ an epimorphism, the induced map $\beta^* : Ext(B, G) \to Ext(C, G)$ is monomorphism, for any group G.

Proof. (a) $\Rightarrow$ (b) Being given A as a group with D.S.I.P., $A = \bigoplus C$ and $\beta : B \to C$ an epimorphism, then: (E) $0 \to ker\beta \to B \to C \to 0$ is an exact splitting sequence (according to [10, Proposition 1.4.]) and represents an element from $Ext(C, ker\beta)$. From [6,51.3.] we have the following exact sequence:

$$0 \to Hom(C,G) \to Hom(B,G) \to Hom(ker\beta, G) \xrightarrow{E^*} Ext(C,G) \xrightarrow{\beta^*}$$

$$\xrightarrow{\beta^*} Ext(B,G) \to Ext(ker\beta, G) \to 0.$$

Since (E) is splitting, for any $\eta : ker\beta \to G$, $E_*(\eta) = \eta E \in Ext(C,G)$ is a splitting extension according to [6, 51.1]. So $ImE^* = 0 = ker\beta^*$ and β^* is a monomorphism.

b) $\Rightarrow$ a). We consider the two exact sequences above mentioned. If β^* is monomorphism, then $ker\beta^* = 0 = ImE^*$, that is, for any $\eta : ker\beta \to G$, ηE is a splitting extension. If $G = ker\beta$ and $\eta = 1_{ker\beta}$, we get (E) a splitting extension and according to [10, Proposition 1.4.], A has D.S.I.P..

Theorem 1.6. (the dual of (1.5.)) *Let A be an abelian group.*

a) If A has D.S.I.P., then for any $B, C \in Sd(A)$ and $\alpha : B \to C$ a monomorphism, the induced map $\alpha_* : Ext(G, B) \to Ext(G, C)$ is a monomorphism, for any group G.

b) If for any $B \leq C \in Sd(A)$ and $\alpha : B \to C$ monomorphism, the induced map $\alpha_* : Ext(G, B) \to Ext(G, C)$ is a monomorphism for any group G, then A has D.S.I.P..

Proof. a) Being given $B, C \in Sd(A)$ and $\alpha : B \to C$ a monomorphism then $\alpha(B) \simeq B$ is a direct summand in C and $(E) : \ 0 \to B \overset{\alpha}{\to} C \to C/B \to 0$ is an exact splitting sequence. From [6,51.3.] we get the following exact sequence:

$$0 \to Hom(G, B) \to Hom(G, C) \to Hom(G, C/B) \overset{E_*}{\to}$$

$$\overset{E_*}{\to} Ext(G, B) \overset{\alpha_*}{\to} Ext(G, C) \to Ext(G, C/B) \to 0.$$

If $\eta : G \to C/B$ is some homomorphism then $E\eta = E_*(\eta) \in Ext(G, B)$ is a splitting extension, according to [6, 51.2]. So $ImE_* = ker\alpha_* = 0$ and α_* is a monomorphism. (It can be noticed that this implication is always valid; the condition that A should have D.S.I.P. hasn't been used anywehere).

b) We consider the two exact sequences from point a). If α_* is a monomorphism, then $ker\alpha_* = 0 = ImE_*$. So, for any $\eta : G \to C/B$, $E\eta$ is a splitting extension of B by G. If $G = C/B$ and $\eta = 1_{C/B}$, (E) is a splitting extension, that is B is a direct summand in C. Now, considering, H as another direct summand in A, noting $B = H \cap C$ and $\alpha : H \cap C \to C$ the inclusion map, we find that $H \cap C \in Sd(A)$, that is A has D.S.I.P..

We close this paragraph with the following result:

Theorem 1.7. *The group A has D.S.I.P., if and only if $Tor(A, C)$ has D.S.I.P., for any group C.*

Proof. A being a group with D.S.I.P., and $Tor(A, C) = Tor(T, C) \oplus Tor(T', C) = Tor(S, C) \oplus Tor(S', C)$ two direct decompositions of $Tor(A, C)$, then $Tor(A, C) \cong Tor(T \oplus T', C) \cong Tor(S \oplus S', C)$, according to [6,62.(E).].(*) But $Tor(A, C) \simeq Tor(B, C)$ has an exact place if $A \simeq B$, as the map $\varphi : (a, m, c) \mapsto (b, m, c)$ is an isomorphism between the generators of $Tor(A, C)$ and those of $Tor(B, C)$. This means that $A \cong$

$T \oplus T' \cong S \oplus S'$ and as A has D.S.I.P., $A \cong T \cap S \oplus T'' \oplus T'$. Then $Tor(A, C) \cong Tor(T \cap S \oplus T'' \oplus T', C) \cong Tor(T \cap S, C) \oplus Tor(T'', C) \oplus Tor(T', C)$ (**). Since $Tor(T \cap S, C) = Tor(T, C) \cap Tor(S, C)$, from relation (**) it follows that $Tor(A, C)$ has D.S.I.P..

Viceversa, we suppose that $Tor(A, C)$ has D.S.I.P. and let $A = T \oplus T' = S \oplus S'$ be two direct decompositions of A. Then $Tor(A, C) \cong Tor(T \oplus T', C) \cong Tor(T, C) \oplus Tor(T', C) \cong Tor(S \oplus S', C) \cong Tor(S, C) \oplus Tor(S', C) \cong Tor(T, C) \cap Tor(S, C) \oplus Tor(U, C) \cong Tor(T \cap S, C) \oplus Tor(U, C) \cong Tor((T \cap S) \oplus U, C)$, where $U \leq A$. Then $A \cong T \cap S \oplus U$, which means A has D.S.I.P..

Since $Tor(A, C) \simeq Tor(C, A)$, we also have its symmetric of (1.7.).

Corollary 1.8. *The group C has D.S.I.P. if and only if $Tor(A, C)$ has this property, for any group A.*

2. Torsion groups

The following proposition presents a series of elementary properties of p-groups with D.S.I.P..

Proposition 2.1. A being a p-group with D.S.I.P. the following statements occur:

(a) A is a simply presented group;

(b) A has a nice system;

(c) A has a nice composition series;

(d) A has the projective property relative to all the balanced-exact sequence of p-groups;

(e) A is a direct summand of a direct sum of generalized Prüfer groups;

(f) A is totally projective;

(g) A is fully transitive;

(h) For any increasing sequence of ordinals and symbols ∞, $u = (\sigma_0, \dots, \sigma_n, \dots)$, $A(u)$ and $A/A(u)$ are totally projective.

Proof. If A is a p-group with A D.S.I.P., either is indecomposable or $A = B_p \oplus C_p$, where $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$ (see [12, Theorem 2.]).

(a) Since $Z(p^n)$, $n \in \mathbf{N}^*$ and $Z(p^\infty)$ are simply presented groups, and a direct sum of simply presented groups is again a simply presented group (see [6, §83.]), it follows that A is a simply presented group.

(b) By [6,83.2.], every simply presented p-group has a nice system.

(c), (d), (e), (f). The statements of points (b), (c), (d), (e) and (f) are equivalent, according to [6,81.9.] and [6,82.3.].

(g) Any totally projective p-group is fully transitive (see [11] or [6,81.4.]).

(h) Every totally projective p-group A has the enunciate property, according to [6,§83.].

The following result makes another connection between the torsion p-groups with D.S.I.P. and the torsion product.

Proposition 2.2. *A being A a p-group with D.S.I.P., E a pure subgroup in A with $Z(p^\infty) \subseteq E$, then $Tor(E,G)$ is a balanced subgroup of $Tor(A,G)$, for any group G.*

Proof. According to the hypothesis and to [12, Theorem 2.], A is the direct sum between a divisible group and a bounded group. Then any subgroup of A, that is for E as well, is a nice subgroup, as the equality from [6,79.2.] is clearly demonstrated (see [6, p.75]). If E is pure in A and $Z(p^\infty) \subseteq E$, then $pE = E \cap pA = E \cap Z(p^\infty) = Z(p^\infty)$ and $pA + E = Z(p^\infty) + E = E$. So $(A/E)^1 = 0$ and according to [6,80.(G).], E is an isotype in A. Then E becomes a balanced subgroup in A. According to [6,62.(F).,62.(D).], only the case in which G is a p-group is of some interest (the other ones being quite ordinary). From [6,63.2.], and from [6, p.75] we find that $Tor(E,G)$ is a nice subgroup in $Tor(A,G)$. We demonstrate that $Tor(E,G)$ is an izotype in $Tor(A,G)$. The equality $p^\sigma Tor(E,G) = Tor(E,G) \cap p^\sigma Tor(A,G)$ becomes, according to [6,64.2.]: $Tor(p^\sigma E, p^\sigma G) = Tor(E,G) \cap Tor(p^\sigma A, p^\sigma G)$, which is quite obvious as $p^\sigma E = E \cap p^\sigma A$. So $Tor(E,G)$ is balanced in $Tor(A,G)$.

Further on we are going to determine the ring $E(A)$ and the group $AutA$ of the endomorphisms, respectively of the automorphisms of a torsion group A, with D.S.I.P.. For the beginning we have the following basic remark:

Remark 2.3. If $A = \bigoplus_{i \in I} A_i$ is a direct decomposition of the group A in fully invariant subgroups, the ring $E(A)$ of its endomorphisms is the direct product of the rings of the endomorphisms of the groups A_i, $i \in I$, that is:

$$E(A) \cong \prod_{i\in I} E(A_i).$$

Proof. If $f \in EndA$, then $\forall\, i \in I$, $f(A_i) \subseteq A_i$. Noting $f|A_i \overset{not}{=} f_i$, $i \in I$, we obtain the map $f \mapsto (f_i)_{i\in I} \in \prod_{i\in I} E(A_i)$, which is an isomorphism from $E(A)$ to $\prod_{i\in I} E(A_i)$.

Lemma 2.4. *If A is a p-group with D.S.I.P., then:*

$$E(A) \cong A,$$

or

$$E(A) \cong M^{(f)}_{m_p\times m_p}(Z(p)) \oplus \left(\prod_{m_p} Z(p)\right) \oplus R_p \cong \left(\prod_{m_p^2} Z(p)\right) \oplus \left(\prod_{m_p} Z(p)\right) \oplus R_p,$$

where:

- *$m_p \in N$ or $m_p = \infty$;*
- *$M^{(f)}_{m_p\times m_p}(Z(p))$ is the ring of the square matrices of order m_p with elements from $Z(p)$ and the columns of which have a finite number of non-null elements;*
- *$R_p = 0$ or $R_p = Q_p^*$ - the completion, in p-adic topology of Q_p - the ring of p-adic integers.*

Proof. A being a p-group with D.S.I.P., if A is indecomposable, then there is $n \in N^*$ so that $A = Z(p^n)$. In this case $E(A) = End(Z(p^n)) \cong Z(p^n) = A$ (see [6,§43]). If A is decomposable, then, according to [12, Theorem 2.], $A = B_p \oplus C_p$, where $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$. So $E(A) = Hom(A, A) = Hom(B_p \oplus C_p, B_p \oplus C_p) \cong Hom(B_p, B_p) \oplus Hom(B_p, C_p) \oplus Hom(C_p, B_p) \oplus Hom(C_p, C_p) = EndB_p \oplus Hom(B_p, C_p) \oplus EndC_p$ (*) (according to [6,43.1., 43.2., 43.(A).(iii)]). The group $EndB_p = End\left(\bigoplus_{m_p} Z(p)\right)$ is isomorphic to the ring of the square matrices $(\alpha_{ij})_{i,j=\overline{1,m_p}}$, of the type $m_p \times m_p$, where $\alpha_{ij} \in End(Z(p)) \cong Z(p)$, and for which the sum of elements on each column exists in the finite topology of $E(B_p)$ (see [6,106.1.]). Noting the $M^{(f)}_{m_p\times m_p}(Z(p))$ - ring of the square matrices of order $m_p \times m_p$, having elements from $Z(p)$ with its columns having a finite number of non-null elements, we find that $E(B_p) \cong M^{(f)}_{m_p\times m_p}(Z(p))$. $Hom(B_p, C_p) = 0$ or $Hom(B_p, C_p) = Hom\left(\bigoplus_{m_p} Z(p), Z(p^\infty)\right) \cong$

$\prod_{m_p} Hom(Z(p), Z(p^\infty)) \cong \prod_{m_p} Z(p^\infty)[p] \cong \prod_{m_p} Z(p)$ (see [6,§43]). Finally, $R_p = EndC_p = 0$ or $R_p = EndC_p = End(Z(p^\infty)) \cong Q_p^*$, according to [6,§43]. Replacing in the relation (*) we obtain the first isomorphism of the statement.

The second isomorphism can be stressed out in the following way:

$$EndB_p = Hom(B_p, B_p) = Hom\left(\bigoplus_{m_p} Z(p), \bigoplus_{m_p} Z(p)\right) \cong$$

$$\cong \prod_{m_p} Hom\left(Z(p), \bigoplus_{m_p} Z(p)\right) \cong \prod_{m_p}\prod_{m_p} Hom(Z(p), Z(p)) \cong \prod_{m_p^2} EndZ(p) \cong \prod_{m_p^2} Z(p).$$

Theorem 2.5. *If A is a torsion group with D.S.I.P., then:*

$$E(A) \cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0} M^{(f)}_{m_p\times m_p}(Z(p))\right) \oplus \left(\prod_{p\in P\setminus P_0}\prod_{m_p}(Z(p))\right) \oplus \left(\prod_{p\in P\setminus P_0} R_p\right) \cong$$

$$\cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0}\left(\prod_{m_p^2} Z(p)\right)\right) \oplus \left(\prod_{p\in P\setminus P_0}\prod_{m_p} Z(p)\right) \oplus \left(\prod_{p\in P\setminus P_0} R_p\right),$$

where:

i) P is the set of all prime numbers and $P_0 \subseteq P$;

ii) A_p is an indecomposable p-group, for any $p \in P_0$;

iii) $M^{(f)}_{m_p\times m_p}(Z(p))$ and R_p have the same meaning like (2.4.), for any $p \in P\setminus P_0$.

Proof. According to [18,3.3.], a torsion group has D.S.I.P., if and only if it takes the form: $A = \left(\bigoplus_{p\in P_0} A_p\right) \oplus \left(\bigoplus_{p\in P\setminus P_0} A_p\right)$, where A_p is an indecomposable p-group, for any $p \in P_0$, and for any $p \in P \setminus P_0$, $A_p = B_p \oplus C_p$, with $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$. Since the two direct summands of the decomposition of A are fully invariant (because if A_i, $i \in I$, are fully invariant subgroups of group A, then $\sum_{i\in I} A_i$ has the same property - see [6,§2.]), it follows that

$$E(A) \cong E\left(\bigoplus_{p\in P_0} A_p\right) \oplus \left(\bigoplus_{p\in P\setminus P_0} A_p\right) \cong$$

$$\cong \left(\prod_{p\in P_0} E(A_p)\right) \oplus \left(\prod_{p\in P\setminus P_0} E(B_p \oplus C_p)\right) \cong$$

$$\cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0} M^{(f)}_{m_p\times m_p}(Z(p))\right) \oplus \left(\prod_{p\in P\setminus P_0}\prod_{m_p} Z(p)\right) \oplus \prod_{p\in P\setminus P_0} R_p \cong$$

$$\cong \left(\prod_{p\in P_0} A_p\right) \oplus \left(\prod_{p\in P\setminus P_0} \prod_{m_p^2} Z(p)\right) \oplus \left(\prod_{p\in P\setminus P_0} \prod_{m_p} Z(p)\right) \oplus \left(\prod_{p\in P\setminus P_0} R_p\right),$$

according to (2.4.).

Lemma 2.6. *If A is a p-group with D.S.I.P., then there is a $n_p \in N^*$ so that the group $AutA$ is isomorphic to the multiplicative group $U(Z(p^{n_p}))$, of the units of the ring $(Z(p^{n_p}),+,\cdot)$, or:*

$$AutA \cong U(M^{(f)}_{m_p\times m_p}(Z(p))) \oplus \left(\prod_{m_p} Z(p-1)\right) \oplus U(R_p) \cong$$

$$\cong \left(\prod_{m_p^2} Z(p-1)\right) \oplus \left(\prod_{m_p} Z(p-1)\right) \oplus U(R_p),$$

where:

- *$U(M^{(f)}_{m_p\times m_p}(Z(p)))$ is the multiplicative group of the units of the ring $M^{(f)}_{m_p\times m_p}(Z(p))$;*
- *$U(R_p) = 0$ or $U(R_p) \cong Z(p-1) \times J_p$ (J_p being the additive group of Q_p^*).*

Proof. A being a p-group with D.S.I.P., if A is indecomposable, then there is a $n \in N^*$, such that $A = Z(p^n)$. In this case $AutA = Aut(Z(p^n))$ is isomorphic to the group $U(Z(p^n))$, presented in the statement, as an automorphism of A is an unit of $E(A)$. If $A = B_p \oplus C_p$, with $B_p = \bigoplus_{m_p} Z(p)$, $C_p = 0$ or $C_p = Z(p^\infty)$, then an automorphism of A is a inversable element of the direct product (of the direct sum) of the rings of (2.4.). Considering that $U(Z(p)) \cong Z(p-1)$ and $U(Q_p^*) \cong Z(p-1)\times J_p$ (according to [6,127.1.]), we obtain the isomorphisms of the statement.

Similar to the proof of (2.5), but using (2.6.) the following result can be demonstrated:

Theorem 2.7. *If A is a torsion group with D.S.I.P., then:*

$$AutA \cong \left(\prod_{p\in P_0} U(Z(p^{n_p}))\right) \oplus \left(\prod_{p\in P\setminus P_0} U(M^{(f)}_{m_p\times m_p}(Z(p)))\right) \oplus \left(\prod_{p\in P\setminus P_0} \left(\prod_{m_p} Z(p-1)\right)\right) \oplus$$

$$\oplus \left(\prod_{p\in P\setminus P_0} U(R_p)\right) \cong \left(\prod_{p\in P_0} U(Z(p^{n_p}))\right) \oplus \left(\prod_{p\in P\setminus P_0} \left(\prod_{m_p^2} Z(p-1)\right)\right) \oplus$$

$$\oplus \left(\prod_{p\in P\setminus P_0} \left(\prod_{m_p} Z(p-1)\right)\right) \oplus \left(\prod_{p\in P\setminus P_0} U(R_p)\right),$$

where the notations have the same meaning like in (2.6.).

Remark 2.8. Since the groups B_p, $p \in P$, are elementary p-groups of rank m_p, these are vectorial spaces over the field $Z(p)$ (of characteristic p), and $\dim B_p = m_p$, and any automorphism of B_p is a linear transformation of this space, it follows that $M^{(f)}_{m_p \times m_p}(Z(p))$ is isomorphic to the general linear group $GL(m_p, p)$.

3. Torsion-free groups

In [18,4.1.] we have demonstrated that any torsion-free divisible group has D.S.I.P.. Using this we are going to demonstrate a few interesting results.

Theorem 3.1. *Let A be a torsion-free group with the property that for any epimorphism $\beta : B \to C$ (B and C being arbitrary groups), the induced map $\beta^* : Ext(C,A) \to Ext(B,A)$ is a monomorphism. Then A has D.S.I.P..*

Proof. If $\beta : B \to C$ is an epimorphism, then (E) $0 \to ker\beta \to B \to C \to 0$ is an exact sequence. From [6,51.3.], we obtain the following exact sequence: $0 \to Hom(C,A) \to Hom(B,A) \to Hom(ker\beta, A) \xrightarrow{E^*} Ext(C,A) \xrightarrow{\beta^*} Ext(B,A) \to Ext(ker\beta, A) \to 0$. Since β^* is monic, it follows that $ImE^* = 0$, that is for any $\eta : ker\beta \to A$, $E^*(\eta) = \eta E$ is splitting. Now considering $B = D$ - the divisible hull of A, $C = D/A$, $\beta = \pi_A$ - the canonic projection of D on D/A and $\eta = 1_A$, we find that $1_A E \cong E$ is a splitting extension, that is A is a direct summand in D. Then [6,20.(E).], shows that A is divisible. Now [18,4.1.] completes the proof.

Remark 3.2. It can be easily demonstrate that the converse of (3.1.) occurs for any divisible group.

Further on we are going to see what conditions the groups A and C have to satisfy so that $Hom(A,C)$ may have D.S.I.P..

Proposition 3.3. 1) A and C being two abelian groups $Hom(A,C)$ has D.S.I.P. in any of the following situations:

a) A is torsion-free and divisible;

b) C is torsion-free and divisible;

c) A is torsion-free indecomposable, C is divisible and $A \oplus C$ has D.S.I.P.;

d) A is torsion-free with D.S.I.P. and C is torsion-free of rank 1;

e) A is torsion-free of rank 1 and C is torsion-free with D.S.I.P.;

2) If A and C are torsion-free of rank 1, with $t(A) \leq t(C)$, then for any index set I, the group $H = \bigoplus_I Hom(A,C)$ has D.S.I.P.. In particular $E = \bigoplus_I End A$ has D.S.I.P., for any torsion-free group A of rank 1.

Proof. 1) a) If A is torsion-free and divisible, then for any group C, $Hom(A,C)$ is torsion-free and divisible ([6,43.(G).]). Now we apply [18,4.1.].

b) If C is torsion-free and divisible, then for any group A, $Hom(A,C)$ is torsion-free and divisible ([6,43(D)]. We apply [18,4.1.] once again.

c) If $A \oplus C$ has D.S.I.P., then, according to [10,3.4.1.], for any $\alpha \in Hon(A,C)$, $ker\alpha$ is a direct summand in A. Since A is indecomposable, any morphism $\alpha : A \to C$ is either null or injective. Let $0 \neq \beta \in Hom(A,C)$ be a morphism for which $n\beta = 0$, for a certain $n \in N^*$. Then for any $a \in A$, $n\beta(a) = \beta(na) = 0$. So $na = 0$, as β is injective. Since A is torsion-free, it follows that $n = 0$ - contradiction with the choice of n. So $Hom(A,C)$ is torsion-free. Now we are going to demonstrate that $Hom(A,C)$ is divisible. Since the group C is divisible, it follows that for any $\alpha \in Hom(A,C)$, with any $x \in A$ and any $n \in N^*$, there is $y \in C$ so that $\alpha(x) = uy$.

We define $\gamma : A \to C$ by: for any $x \in A$, $\gamma(x) = y$, where $y \in C$ is the solution of equation $\alpha(x) = ut$. Then $\gamma \in Hom(A,C)$ and $\alpha(x) = n\gamma(x)$, for any $x \in A$. So $Hom(A,C)$ is divisible. Now [18,4.1.] completes the proof.

d) Let A be a torsion-free group with D.S.I.P. and C torsion-free of rank 1. Then $A = D \oplus E$, with D - divisible and E - reduced, completely decomposable homogeneous of finite rank ([18,5.16.]). So there is an $n \in N$ so that $E = \bigoplus_n B$, where B is reduced torsion-free of rank 1. Then there will be $Hom(A,C) = Hom(D \oplus E, C) \cong Hom(D,C) \oplus Hom(E,C) \cong Hom(D,C) \oplus \left(Hom\left(\bigoplus_n B, C\right)\right) \cong Hom(D,C) \oplus \left(\bigoplus_n Hom(B,C)\right)$, according to [6.43.1., 43.2.]. The group $Hom(D,C)$ has D.S.I.P., according to a). From [6,85.4.] we find that $Hom(B,C)$ is either 0 (if $t(B) > t(C)$) or a torsion-free group of rank 1 and of the type $t(C) : t(A)$, (if $t(A) \leq t(C)$). If $Hom(B,C) = 0$, then $Hom(A,C) \cong Hom(D,C)$ and the proof is ready for this case. If $Hom(B,C) \neq 0$, then $\bigoplus_n Hom(B,C)$ is either torsion-free divisible or reduced homogeneous completely decomposable group of finite rank. In the former case $Hom(A,C)$ will be torsion-free divisible so it has D.S.I.P., and in the latter $Hom(E,C)$ is reduced homogeneous completely decomposable group of finite rank having D.S.I.P., according to [12, Theorem 5.]. Now [18,5.12.] completes the proof.

The proof from point e) will be similar to the one from point d).

2) If A and C are the same as in the statement, then $Hom(A,C)$ is according to [6,85.4.], torsion-free group of rank 1. Now we can apply [10, Proposition 3.4.].

From (3.3.)b) and [6,§43.] (or 18,4.1.]) will have:

Corollary 3.4. *For any abelian group A and any $m \in N^*$, the group $Hom\left(A, \bigoplus_m Q\right) \cong \prod_{r_0(A)} \left[\bigoplus_m Q\right]$ has D.S.I.P..*

For any abelian group A, the group of characters of A is $CarA = Hom(A, Q/Z)$. From (3.3.a) we find that if A is torsion-free and divisible, then $CarA$ has D.S.I.P.. The next theorem will improve this result.

Theorem 3.5. *If A is a torsion-free group with D.S.I.P., then $CarA$ has the same property.*

Proof. Let A be a torsion-free group with D.S.I.P.. According to [18,5.16.], $A = \left(\bigoplus_m Q\right) \oplus \left(\bigoplus_n C\right)$, where $m, n \in N$ or $m = \infty$, and C is reduced, of rank 1. Then according to [6,43.1., 43.2.],

$$CarA = Hom(A, Q/Z) = Hom\left(\left(\bigoplus_m Q\right) \oplus \left(\bigoplus_n C\right), \bigoplus_p Z(p^\infty)\right) \cong$$

$$\cong Hom\left(\bigoplus_m Q, \bigoplus_p Z(p^\infty)\right) \oplus Hom\left(\bigoplus_n C, \bigoplus_p Z(p^\infty)\right) \cong$$

$$= \left[\prod_m \prod_p Hom(Q, Z(p^\infty))\right] \oplus \left[\prod_n \prod_p Hom(C, Z(p^\infty))\right].$$

According to [6,43.(G).], $Hom(Q, Z(p^\infty))$ is a torsion-free and divisible group. But then $\prod_n \prod_p Hom(Q, Z(p^\infty))$ is divisible and torsion-free too (see [6,20.(E).]). Since the p-basic subgroup of C is null, according to [6,47.1], $Hom(C, Z(p^\infty))$ is divisible and torsion-free; so the groups $\prod_n \prod_p Hom(C, Z(p^\infty))$ and $CarA$ have the same property. From [18,4.1.] it follows that, $CarA$ has D.S.I.P.. ∎

Now, we are going to study the ring $E(A) = EndA$ and the group $AutA$, when A is a torsion free group with D.S.I.P..

Theorem 3.6. *If A is torsion-free with D.S.I.P., then:*

$$E(A) \cong \left(\prod_m \left(\bigoplus_m Q\right]\right) \oplus \left(\bigoplus_n \left[\bigoplus_n Q\right]\right) \oplus \left(\bigoplus_n \left(\bigoplus_n EndC\right]\right),$$

where:

- *m and n are natural numbers or $m = \infty$;*

- *C is a reduced torsion-free group of rank 1.*

Proof. Let $A = D \oplus B$ be, with $D = \bigoplus_m Q$ (divisible) and $B = \bigoplus_n C$ - reduced completely decomposable homogeneous of finite rank (C being a reduced torsion-free group of rank 1), torsion-free group with D.S.I.P., according to [10,3.3.]. Then $E(A) = Hom(A, A) = Hom(D \oplus B, D \oplus B) = Hom(D, D) \oplus Hom(D, B) \oplus Hom(B, D) \oplus Hom(B, B) = EndD \oplus Hom(B, D) \oplus EndB$, according to [6.43.1, 43.2., 43.(A).(iii).]. But $EndD = Hom\left(\bigoplus_m Q, \bigoplus_m Q\right) = \prod_m \left[\bigoplus_m Q\right]$, (see [6,§43.]), $Hom(B, D) = Hom\left(\bigoplus_n C, \bigoplus_m Q\right)$! $\prod_n Hom\left(C, \bigoplus_m Q\right) \cong \bigoplus_n \left[\bigoplus_m Q\right]$, and $EndC = Hom\left(\bigoplus_n C, \bigoplus_n C\right) \cong \bigoplus_n \left[\bigoplus_n EndC\right]$.

Making a demonstration analogous to (2.7.) we obtain:

Theorem 3.7. *If A is a torsion-free group with D.S.I.P., then:*

$$AutA \cong \left(\prod_n \left[\bigoplus_m Q^*\right]\right) \oplus \left(\bigoplus_n \left[\bigoplus_m Q^*\right]\right) \oplus \left(\bigoplus_n \left[\bigoplus_n AutC\right]\right).$$

We'll close this paragraph with some other two condition necessary for a torsion-free group to have D.S.I.P..

Theorem 3.8. *Let A be a torsion-free group. In any of the following cases, A has D.S.I.P..*

(a) The group A has the following property: if A is an endomorphic image of a group B, then B contains a direct summand isomorphic to A.

(b) There is a prime number p so that the p basic subgroup B of A, is an endomorphic image of A and A/B is divisible.

Proof. (a) If A is like in the statement, then, according to [11, Theorem 1.], $A = D \oplus F$, where D is divisible and F - free. The groups D and F from the decomposition of A have D.S.I.P., because of [18.4.1.] and respectively [18.2.2.]. Now [18, 5.12.] completes the proof.

(b) Let A be a torsion-free group and $B = B_0 \oplus B_1 \oplus \cdots \oplus B_n \oplus \ldots$ its p-basic subgroup ($B_0 = \oplus Z$ and $B_n = \oplus Z(p^n)$, $n = 1, 2, \ldots$). This leads to the conclusion that $B = \oplus Z$, so it is a free group. If $f \in EndA$ and $f(A) = B$, then $A/kerf \cong B$, according to the first theorem of isomorphism. Since B is a free group, is an exactly splitting sequence: $0 \to kerf \to A \to B \to 0$ ([6,14.4.]). So $A \cong kerf \oplus B = D \oplus C \oplus B$, where D is the maximal divisible subgroup of A, and C is a reduced group. Since A/B is a divisible group it follows that $C = 0$. So $A = D \oplus B$, with D divisible and C - free. Now we are going to judge the same as at point a).

4. Mixed groups

In [18,4.4.] we have seen that a divisible group with D.S.I.P. cannot be mixed. Because of this, according to what was demonstrated in the former paragraphs and in [6,§32,§106,§113, §127, §128], we find:

Theorem 4.1. *Let A be a divisible group with D.S.I.P.*

(a) If A is a torsion group, then:

$$E(A) \cong \prod_p Q_p^*,$$

and

$$Aut(A) \cong \prod_p [Z(p-1) \times J_p].$$

(b) If A is a torsion-free group, then:

$$E(A) \cong M^{(f)}_{r_0 \times r_0}(Q) \cong \prod_{r_0} \left[\bigoplus_{r_0} Q \right],$$

and

$$Aut(A) \cong U(M^{(f)}_{r_0 \times r_0}(Q)) \cong \prod_{r_0} \left[\bigoplus_{r_0} Q^* \right] \cong \prod_{r_0} \left[\bigoplus_{r_0} \left(Z(2) \times \underset{\chi_0}{\times} Z \right) \right],$$

where the notations are the ones presented above.

Using (2.4.)-(2.8.), (3.6.)-(3.7.), (4.1.) and [18.6.4.], the problem of the determination of $EndA$ ($AutA$), for a splitting mixed group A with D.S.I.P., will be reduced to the determination of $Hom(B, Z(p^n))$, where $n \in N^*$ and B is a reduced torsion-free of rank 1 direct summand of A.

The following results present sufficient conditions for $T(A)$ and $A/T(A)$ (A being a mixed group), to have D.S.I.P., by using the ring $E(A)$.

Proposition 4.2. Let A be a mixed group with the property that any endomorphic image of A is a direct summand in A.

(a) If $T(A)$ is bounded, then A, $T(A)$ and $A/T(A)$ have D.S.I.P..

(b) If $T(A)$ is not bounded, it may have D.S.I.P., $A/T(A)$ has (always) D.S.I.P., but A does not have this property anymore.

Proof. Let A be a mixed group with the property presented in the statement. From [15,3.1., 4.2., 5.3], we find that each p-component of A is an elementary or divisible group, and $A/T(A)$ is divisible.

(a) If $T(A)$ is bounded, then each p-component is a elementary p-group and has D.S.I.P., according to [18, 3.3.]. This means that $T(A)$ is an elementary group and has D.S.I.P. ([12, Lemma 1.]). The group $A/T(A)$ has D.S.I.P., due to [18, 4.1.]. According to the hypothesis A it is splitting and as $T(A)$ and $A/T(A)$ are, in this case, fully invariant, we can apply [12, Lemma 1.].

(b) If $T(A)$ is not bounded, then it has a divisible direct summand. From [18,4.4.] we find that $T(A)$ can have D.S.I.P., if it takes the form $\bigoplus_p Z(p^\infty)$. In this case A has a mixed divisible direct summand and, according to [19, proposition 6.], doesn't have D.S.I.P..

Proposition 4.3. Let A be a mixed group. In any of the following situations $T(A)$ and $A/T(A)$ have D.S.I.P.:

(a) The kernels and the images of the endomorphisms of A are pure subgroups in A.

(b) The ring of the endomorphisms of A is regular.

Proof. (a) If A has the property given in the statement, according to [17,5. Proposition 3.], $T(A)$ is elementary and $A/T(A)$ is divisible. Now [18,3.3.] and [18,4.1.] completes the proof.

(b) We suppose that $E(A)$ is regular. If A is not reduced, then, according to [6,112.7], A is splitting, $T(A)$ is elementary, and $A/T(A)$ is divisible. So A, $T(A)$ and $A/T(A)$ has D.S.I.P.. If A is of torsion, then $A = T(A)$ is an elementary group, so it has D.S.I.P.. Finally, if A is reduced, then, again, $T(A)$ is elementary and $A/T(A)$ is divisible.

Corollary 4.4. *Any splitting group which satisfies the conditions from (4.3.)(a) has D.S.I.P..*

In the end we present other properties of the mixed groups with D.S.I.P..

Theorem 4.5. *Let A be a splitting mixed group, with D.S.I.P. with $T = T(A)$ and $\hat{T}$ - the completion of T in the Z-adic topology. Then:*

a) for any divisible group G, $Ext(G, \hat{T})$ is isomorphic to a direct summand of a direct product of groups of the form $A/p^n A$;

b) $\hat{T}$ is isomorphic to a direct summand of a direct product of groups of the form A/p^nA;

c) if C is a reduced torsion-free summand, of rank 1, from some decomposition of A, and $(Ext(Q/Z,C))_0 = 0$, then the pure-injective hull of T and the first subgroup Ulm of the cotorsion hull of T, are isomorphic to $(Ext(Q/Z,A))_0$ (so $(Ext(Q/Z,A))_0$ is a reduced algebraically compact group).

Proof. a) Ig A is a splitting mixed group with D.S.I.P., according to [12, Theorem 4], $T^1 = 0$. From [6,39.5] we find that: $0 \to T \to \hat{T} \to \hat{T}/T \to 0$ is an exact sequence. Now [6,53.7.] implies the exactness of the sequence: $0 = Hom(G,\hat{T}) \to Hom(G,\hat{T}/T) \to Pext(G,T) \to Pext(G,\hat{T}) = 0$; the last equality occurs because $\hat{T}$ is algebraically compact group ([6,39.1.]). This leads to $Hom(G,\hat{T}/T) \cong Pext(G,T) = (Ext(G,T))^1$. From [6,51.3.] we get exactness of the sequence: $0 = Hom(G,\hat{T}) \to Hom(G,\hat{T}/T) \to Ext(G,T) \to Ext(G,\hat{T}) \to Ext(G,\hat{T}/T) = 0$ (the last equality occurs because of [6,39.5.]). So $Ext(G,T)/Hom(G,\hat{T}/T) \cong Ext(G,T)/Pext(G,T) \cong Ext(G/T)/(Ext(G,T))^1 = (Ext(G,T))_0 \cong Ext(G,\hat{T})$. Since $A = T \oplus A/T$, it follows that $(Ext(G,A))_0 = (Ext(G,T))_0 \oplus (Ext(G,A/T))_0$ (see [6,37.5.]). So $Ext(G,\hat{T})$ is a direct summand in $(Ext(G,A))_0$. According to [6,30.1.], there is a direct sum of cyclic groups $X = \bigoplus_{i\in I}\langle x_i\rangle$ and an epimorphism $\eta : X \to G$ so that $ker\eta$ is a pure subgroup in X, that is, there is the following pure-exact sequence: $0 \to ker\eta \to X \to G \to 0$. From [6,57.1.] it follows that $(Ext(X,A))_0 = (Ext(G,A))_0 \oplus (Ext(ker\eta,A))_0$. But $(Ext(X,A))_0 = \left(Ext\left(\bigoplus_{i\in I}\langle x_i\rangle, A\right)\right)_0 \cong \left(\prod_{i\in I} Ext(\langle x_i\rangle, A)\right)_0 \cong \left(\prod_p A/p^nA\right)_0 \cong \prod_p A/p^nA$, according to [6,52.2,52.(D).,37.5.].

b) By [6,56.6.] it follows that $\hat{T} \cong (Ext(Q/Z,T))_0$, which is, see [6.57.1.], a direct summand in $(Ext(Q/Z,A))_0$. Since Q/Z is divisible, the statement follows from the proof of the point a) of this theorem.

c) Let $A = T \oplus D \oplus \left(\bigoplus_n C\right)$ be a splitting mixed group, with D.S.I.P., according to [18,6.4.] (so T is the torsion part of A, D is a torsion-free divisible group, and C is a torsion-free reduced group, of rank 1). From the hypothesis and from [6,37.5.,56.6.,52.(B).] we find that $(Ext(Q/Z,A))_0 = (Ext(Q/Z,T))_0 \cong \hat{T}$, which is a reduced algebraically compact group (see [6,39.1]). If $\hat{T}$ is the pure-injective hull of T, from [6,41.9.] and [12, Theorem 4.], it follows that $\tilde{T} \cong \hat{T}$.

Corollary 4.6. *Let A be a splitting mixed group with D.S.I.P. and B its reduced homogeneous completely decomposable summand of finite rank, according to [18,6.4.], E the divisible hull and G the pure-injective hull of B. If* $Hom(Q/Z, E/B) \cong Hom(Q/Z, G/B)$*, then* $(Ext(Q/Z, A))_0$ *is a reduced algebraically compact group.*

Proof. From [6,52.3.] we find that $Ext(Q/Z, B) \cong Hom(Q/Z, E/B)$. If G is the pure-injective hull of B, then $0 \to B \to G \to G/B \to 0$ is a pure-exact sequence. According to [6,53.7.]: $0 = Hom(Q/Z, G) \to Hom(Q/Z, G/B) \to Pext(Q/Z, B) \to Pext(Q/Z, G) = 0$ is an exact sequence; the two equalities are due to [6.43.(A).(iii).] and, respectively to [6,41.5.]. This means that $Pext(Q/Z, B) \cong Hom(Q/Z, G/B)$. Then, according to the hypothesis and to [6;53.3.], we get: $(Ext(Q/Z, B))_0 = Ext(Q/Z, B)/Pext(Q/Z, B)$ $Hom(Q/Z, E/B)/Hom(Q/Z, G/B) = 0$. Now, (4.5.c) completes the proof.

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