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ASYMPTOTYC FORMULAE CONCERNING ARITHMETICAL FUNCTIONS DEFINED BY CROSS-CONVOLUTIONS , VII. DISTRIBUTION OF A-SEMI-k-FREE INTEGERS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. We define the A-semi-k-free integers as a common generalization of the k-free integers (i.e. integers not divisible by the k-th power of any prime) and of the semi-k-free integers (i.e. integers not divisible unitarily by the k-th power of any prime) in terms of Narkiewicz's regular A-convolutions. We establish asymptotic formulae for the number of A-semi-k-free integers $\leq x$ with and without assuming the Riemann hypothesis if A is a cross-convolution, investigated in our previous papers.

1. Introduction

Let A be a mapping from the set $\mathbb{N}$ of positive integers to the set of subsets of $\mathbb{N}$ such that $A(n) \subseteq D(n)$ for each n, $D(n)$ denoting the set of all (positive) divisors of n. The A-convolution of arithmetical functions f and g is given by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d). \tag{1}$$

W.Narkiewicz [Nar63] defined an A-convolution to be regular if

(α) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A-convolution,

(β) the A-convolution of multiplicative functions is multiplicative,

(γ) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, has an inverse μ_A with respect to the A-convolution and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power p^a.

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For example, the Dirichlet convolution D, where $D(n) = \{d \in \mathbb{N} : d|n\}$, and the unitary convolution U, where $U(n) = \{d \in \mathbb{N} : d||n\} = \{d \in \mathbb{N} : d|n, (d, n/d) = 1\}$, are regular.

It can be proved, see [Nar63], that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ for every $m, n \in \mathbb{N}, (m, n) = 1$,

(ii) for every prime power p^a there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$.

The elements of the set $A(n)$ are called the A-divisors of n. For other properties of regular convolutions see also P. J. McCarthy [McC86] and V. Sita Ramaiah [Sit78].

We say that A is a *cross-convolution* if for every prime p we have either $t_A(p^a) = 1$, i.e. $A(p^a) = \{1, p, p^2, ..., p^a\} \equiv D(p^a)$ for every $a \in \mathbb{N}$ or $t_A(p^a) = a$, i.e. $A(p^a) = \{1, p^a\} \equiv U(p^a)$ for every $a \in \mathbb{N}$. Let $P_A = P$ and $Q_A = Q$ be the sets of the primes of the first and second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $P = \mathbb{P}$ and $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ and $Q = \mathbb{P}$ we obtain the unitary convolution U.

Furthermore, let (P) and (Q) denote the multiplicative semigroups generated by $\{1\} \cup P$ and $\{1\} \cup Q$, respectively. Every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P), n_Q \in (Q)$.

If A is a cross-convolution, then

$$A(n) = \{d \in \mathbb{N} : d|n, (d, n/d) \in (P)\} \tag{2}$$

and (1) can be written in the form

$$(f *_A g)(n) = \sum_{\substack{d|n \\ (d,n/d) \in (P)}} f(d) g(n/d). \tag{3}$$

Asymptotic properties of arithmetical functions defined by cross-convolutions were investigated by us in [T97-i], [T-ii], [T-iii], [T-vi].

In this paper we define the A-semi-k-free integers as a common generalization of the k-free integers and of the semi-k-free integers (i.e. integers not divisible by the k-th power of any prime and not divisible unitarily by the k-th power of any prime, respectively) in terms of regular A-convolutions.

In case of a cross-convolution A we establish an asymptotic formula for the number of A-semi-k-free integers $\leq x$ (Theorem 4), in fact we deduce a slightly more general result concerning A-semi-k-S-free integers (Theorem 3), defined with the aid of an arbitrary subset S of $\mathbb{N}$, and we improve the order of the error term given by Theorem 4, using some known estimates regarding the Möbius function, with and without assuming the Riemann hypothesis.

Our results generalize and unify the corresponding known results concerning k-free integers and semi-k-free integers, see [W63], [SurSi73a], [SurSi73b].

2. A-semi-k-free integers

Let A be a regular convolution and let $k \in \mathbb{N}, k \geq 2$. We say that $n \in \mathbb{N}$ is A-semi-k-free if there exists no prime p such that $p^k \in A(n)$. The integer 1 is A-semi-k-free for every A and for every k.

Let the canonical prime power factorization of $n \in \mathbb{N}, n > 1$ be

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}. \tag{4}$$

The integer $n > 1$ is D-semi-k-free if it is k-free, i.e. $a_i < k$ for every $i \in \{1, 2, ..., r\}$. Furthermore, $n > 1$ is U-semi-k-free if it is semi-k-free, i.e. $a_i \neq k$ for every $i \in \{1, 2, ..., r\}$, this concept was introduced by D. SURYANARAYANA [Sur71].

From (i) and (ii) it follows that $n > 1$ is A-semi-k-free if $t_i \neq k, 2t_i \neq k, ..., s_i t_i = a_i \neq k$ for every $i \in \{1, 2, ..., r\}$, where $t_i = t_A(p_i^{a_i})$.

Let $q_{A,k}, q_{D,k} \equiv q_k$ and $q_{U,k} \equiv q_k^*$ denote the characteristic functions of the sets of A-semi-k-free integers, k-free integers and semi-k-free integers, respectively.

Remark 1. If A is a cross-convolution, then $q_{A,k}(n) = q_k(n_P) q_k^*(n_Q)$ for every $n \in \mathbb{N}$. Hence if A is a cross-convolution, then n is A-semi-k-free if and only if n_P is k-free and n_Q is semi-k-free.

Theorem 1. *If A is a cross-convolution and $k \in \mathbb{N}, k \geq 2$, then*

$$q_{A,k}(n) = \sum_{d^k \in A(n)} \mu(d) = \sum_{\substack{d^k e = n \\ (d,e) \in (P)}} \mu(d) \tag{5}$$

holds for every $n \in \mathbb{N}$, where μ is the Möbius function.

Proof. Taking into account the multiplicativity it is sufficient to prove (5) for $n = p^a$, a prime power. Let $F(n) = \sum_{d^k \in A(n)} \mu(d)$. If $p \in P$, then

$$F(n) = \sum_{d^k | p^a} \mu(d) = \begin{cases} \mu(1) = 1, \text{if } a < k \\ \mu(1) + \mu(p) = 0, \text{if } a \geq k \end{cases} = q_k(p^a) = q_{A,k}(p^a).$$

If $p \in Q$, then

$$F(n) = \sum_{d^k || p^a} \mu(d) = \begin{cases} \mu(1) = 1, \text{if } a \neq k \\ \mu(1) + \mu(p) = 0, \text{if } a = k \end{cases} = q_k^*(p^a) = q_{A,k}(p^a).$$

Hence $F(n) = q_{A,k}(n)$ for every $n \in \mathbb{N}$. Observe that, using (2), $d^k \in A(n)$ if and only if $d^k e = n$ and $(d, e) \in (P)$, which completes the proof. □

If S is an arbitrary subset of $\mathbb{N}$, we say that $n \in \mathbb{N}$ is semi-k-S-free if $n = 1$ or $n > 1$ and no exponent in (4) is of the form kb, where $b \in S$.

Furthermore, in case of a cross-convolution A we say that $n \in \mathbb{N}$ is A-semi-$k-S$-free if n_P is k-free and n_Q is semi-$k - S$-free.

Let $q_{A,k,S}$ denote the characteristic function of the A-semi-k-S-free integers and define the multiplicative function $\mu_{(A,S)}$ by

$$\mu_{(A,S)}(p^a) = \begin{cases} -1, & \text{if } p \in P, a = 1 \text{ or } p \in Q, a \in S, \\ 0, & \text{otherwise,} \end{cases}$$

for every prime power p^a.

The proof of the following formula is similar to the proof of Theorem 1.

Theorem 2. *If A is a cross-convolution, $k \in \mathbb{N}, k \geq 2$ and $S \subseteq \mathbb{N}$, then*

$$q_{A,k,S}(n) = \sum_{d^k \in A(n)} \mu_{(A,S)}(d) = \sum_{\substack{d^k e = n \\ (d,e) \in (P)}} \mu_{(A,S)}(d) \tag{6}$$

holds for every $n \in \mathbb{N}$.

Observe that for $S = \{1\}$ we reobtain the A-semi-k-free integers and $\mu_{(A,\{1\})} \equiv \mu$ for every A. For $S = \{1, 2, ..., r\}$ we obtain a direct generalization of the k-skew integers of rank r, cf. E. COHEN [Co61]. In case $S = \mathbb{N}$ we have the $A - k$-free integers, i.e. integers with no k-th power A-divisors > 1, introduced by us in [T-vi] and $\mu_{(A,\mathbb{N})} \equiv \mu_A$.

3. Asymptotic formulae

In what follows all the constants implied by the O-symbols are independent of x and u.

For a cross-convolution A let

$$\zeta_P(s) \equiv \sum_{\substack{n=1 \\ n\in(P)}}^{\infty} \frac{1}{n^s} = \prod_{p\in P}(1-\frac{1}{p^s})^{-1}, \quad s>1.$$

First we prove the following formula.

Theorem 3. *If A is a cross-convolution, $k\in\mathbb{N}, k\geq 2$ and $S\subseteq\mathbb{N}$, then*

$$\sum_{n\leq x} q_{A,k,S}(n) = \alpha_{A,k,S}\,x + O(x^{1/k}),$$

where

$$\alpha_{A,k,S} = \frac{1}{\zeta_P(k)}\prod_{p\in Q}(1-(1-\frac{1}{p})\sum_{a\in S}\frac{1}{p^{ka}}).$$

Proof. Adopting the method of [Co64], using (6) and the estimate

$$(7) \qquad \sum_{\substack{n\leq x \\ (n,u)\in(P)}} 1 = \frac{\phi(u_Q)x}{u_Q} + O(x^{\varepsilon}\sigma^*_{-\varepsilon}(u)), \quad 0\leq\varepsilon<1,$$

where ϕ is Euler's function and $\sigma^*_r(u)$ denotes the sum of r-th powers of the unitary divisors of u, cf. [T97-i], Lemma 7, we have

$$\sum_{n\leq x} q_{A,k,S}(n) = \sum_{\substack{d^k e=n\leq x \\ (d,e)\in(P)}} \mu_{(A,S)}(d) = \sum_{d\leq\sqrt[k]{x}} \mu_{(A,S)}(d) \sum_{\substack{e\leq x/d^k \\ (e,d)\in(P)}} 1$$

$$= \sum_{d\leq\sqrt[k]{x}} \mu_{(A,S)}(d)\left(\frac{\phi(d_Q)x}{d_Q d^k} + O\left(\left(\frac{x}{d^k}\right)^{\varepsilon}\sigma^*_{-\varepsilon}(d)\right)\right)$$

$$= x\sum_{d\leq\sqrt[k]{x}} \frac{\mu_{(A,S)}(d)\phi(d_Q)}{d^k d_Q} + O\left(x^{\varepsilon}\sum_{d\leq\sqrt[k]{x}}\frac{\sigma^*_{-\varepsilon}(d)}{d^{k\varepsilon}}\right)$$

$$= x\sum_{n=1}^{\infty}\frac{\mu_{(A,S)}(n)\phi(n_Q)}{n^k n_Q} + O(x\sum_{d>\sqrt[k]{x}}\frac{1}{d^k}) + O\left(x^{\varepsilon}\sum_{d\leq\sqrt[k]{x}}\frac{\sigma^*_{-\varepsilon}(d)}{d^{k\varepsilon}}\right).$$

Here the series of the main term is absolutely convergent, its general term is multiplicative in n and applying Euler's product formula its sum is given by

$$\prod_{p\in P}\left(1+\sum_{i=1}^{\infty}\frac{\mu_{(A,S)}(p^i)\phi((p^i)_Q)}{p^{ik}(p^i)_Q}\right) = \prod_{p\in P}\left(1+\sum_{i=1}^{\infty}\frac{\mu(p^i)}{p^{ik}}\right)\prod_{p\in Q}\left(1+\sum_{i=1}^{\infty}\frac{\mu_{(A,S)}(p^i)\phi(p^i)}{p^{ik}p^i}\right)$$

$$= \prod_{p\in P}\left(1-\frac{1}{p^k}\right)\prod_{p\in Q}\left(1-(1-\frac{1}{p})\sum_{a\in S}\frac{1}{p^{ik}}\right) = \zeta_P(k)\prod_{p\in Q}\left(1-(1-\frac{1}{p})\sum_{a\in S}\frac{1}{p^{ik}}\right).$$

The first and the second O-terms are both $O(x^{1/k})$, using the estimates

$$\sum_{n>x} n^{-r} = O(x^{1-r}), \quad r>1 \quad \text{and} \quad \sum_{n\leq \sqrt[k]{x}} \frac{\sigma^*_{-\varepsilon}(n)}{n^{k\varepsilon}} = O(x^{1/k-\varepsilon}), \quad 0<\varepsilon<1/k.k$$

□

Theorem 4. *If A is a cross-convolution and $k\in\mathbb{N}, k\geq 2$, then*

$$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(x^{1/k}), \tag{8}$$

where

$$\alpha_{A,k} = \frac{1}{\zeta_P(k)}\prod_{p\in Q}(1-\frac{1}{p^k}+\frac{1}{p^{k+1}}). \tag{9}$$

Proof. Apply Theorem 3 for $S=\{1\}$. □

Corollary. *If A is a cross-convolution, $k\in\mathbb{N}, k\geq 2$ and $S\subseteq\mathbb{N}$, then the asymptotic densities of the A-semi-k-S-free integers and of the A-semi-k-free integers are $\alpha_{A,k,S}$ and $\alpha_{A,k}$, respectively.*

Remark 2. From this result we reobtain, among others, the asymptotic densities of the k-skew integers of rank r, see [Co61] and of the $A-k$-free integers, see [T-vi].

Now we improve the order of the error term of formula (8) using the method of [SurSi73b] based on the following estimates regarding the Möbius function.

Lemma 1. ([SurSiv73], Lemma 3.5 and Lemma 5.2) *Let $x\geq 3, u\in\mathbb{N}$ and $\varepsilon>0$. Then*

(1) $$M_u(x) \equiv \sum_{\substack{n\leq x\\(n,u)=1}} \mu(n) = O(\sigma^*_{-1+\varepsilon}(u)x\delta(x)), 0$$

where

(1) $$\delta(x) = \exp(-A(\log x)^{3/5}(\log\log x)^{-1/5}), 1$$

A being a positive constant.

If the Riemann hypothesis (R.H.) is true, then

(1) $$M_u(x) \equiv \sum_{\substack{n\leq x\\(n,u)=1}} \mu(n) = O(\sigma^*_{-1/2+\varepsilon}(u)x^{1/2}\omega(x)), 2$$

where

$$\omega(x) = \exp(A(\log x)(\log\log x)^{-1}), 3 \tag{1}$$

A being a positive constant.

Lemma 2. *If A is a cross-convolution and $x \geq 3$, then*

$$N_A(x) \equiv \sum_{n\leq x} \mu(n)\frac{\phi(n_Q)}{n_Q} = O(x\delta(x)).4 \tag{1}$$

If the R.H. is true, then

$$N_A(x) \equiv \sum_{n\leq x} \mu(n)\frac{\phi(n_Q)}{n_Q} = O(x^{1/2}\omega(x)).5 \tag{1}$$

Proof. We have

$$\frac{\phi(n_Q)}{n_Q} = \sum_{\substack{d|n \\ d\in(Q)}} \frac{\mu(d)}{d},$$

cf. [T-iii], Lemma 4. Therefore

$$N_A(x) = \sum_{n\leq x} \mu(n)\frac{\phi(n_Q)}{n_Q} = \sum_{\substack{de=n\leq x \\ d\in(Q)}} \mu(de)\frac{\mu(d)}{d}$$

$$= \sum_{\substack{de\leq x \\ (d,e)=1 \\ d\in(Q)}} \mu(d)\mu(e)\frac{\mu(d)}{d} = \sum_{\substack{d\leq x \\ d\in(Q)}} \frac{\mu^2(d)}{d} \sum_{\substack{e\leq x/d \\ (e,d)=1}} \mu(e) = \sum_{\substack{d\leq x \\ d\in(Q)}} \frac{\mu^2(d)}{d} M_d(x/d).$$

Now using (10) with $\varepsilon < 1$ we get

$$N_A(x) = \sum_{\substack{d\leq x \\ d\in(Q)}} \frac{\mu^2(d)}{d} O(\sigma^*_{-1+\varepsilon}(d)\frac{x}{d}\delta(\frac{x}{d}))$$

$$= O(x\sum_{d\leq x} \frac{\tau(d)}{d^2}\delta(\frac{x}{d})) = O(x^{1-\varepsilon}\sum_{d\leq x} \frac{\tau(d)}{d^{2-\varepsilon}}(\frac{x}{d})^{\varepsilon}\delta(\frac{x}{d})),$$

where $\tau(m)$ stands for the number of divisors of m.

Since $x^{\varepsilon}\delta(x)$ is monotonic increasing,

$$(\frac{x}{d})^{\varepsilon}\delta(\frac{x}{d}) \leq x^{\varepsilon}\delta(x),$$

and using that $\tau(m) = O(m^{\varepsilon}), \varepsilon > 0$ we obtain $N_A(x) = O(x\delta(x))$.

If the R.H. is true, then using the estimate (12) instead of (10),

$$N_A(x) = \sum_{\substack{d\leq x \\ d\in(Q)}} \frac{\mu^2(d)}{d} O(\sigma^*_{-1/2+\varepsilon}(d)(\frac{x}{d})^{1/2}\omega(\frac{x}{d})) = O(x^{1/2}\sum_{d\leq x} \frac{\tau(d)}{d^{3/2}}\omega(\frac{x}{d})).$$

Since $\omega(x)$ is monotonic increasing, we obtain that $N_A(x) = O(x^{1/2}\omega(x))$. □

Lemma 3. *If A is a cross-convolution, $x \geq 3$ and $s > 1$, then*

(1) $$\sum_{n>x} \frac{\mu(n)\phi(n_Q)}{n^s n_Q} = O(\frac{\delta(x)}{x^{s-1}}).6$$

If the R.H. is true, then

(1) $$\sum_{n>x} \frac{\mu(n)\phi(n_Q)}{n^s n_Q} = O(\frac{\omega(x)}{x^{s-1/2}}).7$$

Proof. Using (14) and (15), these results follow by partial summation, cf. [SurSi73b], proof of Lemma 2.5. □

We also need the following result, cf. [SurSi73b], eq. (2.3).

Lemma 4. *If $\varepsilon > 0$ and $0 \leq s < 1$, then*

(1) $$\sum_{n\leq x} \frac{\sigma^*_{-\varepsilon}(n)}{n^s} = O(x^{1-s}).8$$

Theorem 5. *If A is a cross-convolution, $k \in \mathbb{N}, k \geq 2$ and $x \geq 3$, then*

(1) $$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(x^{1/k}\delta(x)),9$$

where $\alpha_{A,k}$ and $\delta(x)$ are given by (9) and (11), respectively.

If the R.H. is true, then

(2) $$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(x^{1/k}\omega(x)),0$$

where $\omega(x)$ is defined by (13).

Proof. From (5) we have

$$\sum_{n\leq x} q_{A,k}(n) = \sum_{\substack{d^k e\leq x\\ (d,e)\in(P)}} \mu(d).$$

Let $z = x^{1/k}$ and $0 < \rho = \rho(x) < 1$, where $\rho(x)$ will be chosen suitably later. If $d^k e \leq x$, then both $d > \rho z$ and $e > \rho^{-k}$ cannot simultaneously hold good, therefore

$$\sum_{n\leq x} q_{A,k}(n) = \sum_{\substack{d^k e\leq x\\ d\leq \rho z\\ (d,e)\in(P)}} \mu(d) + \sum_{\substack{d^k e\leq x\\ e\leq \rho^{-k}\\ (d,e)\in(P)}} \mu(d) - \sum_{\substack{d\leq \rho z\\ e\leq \rho^{-k}\\ (d,e)\in(P)}} \mu(d) = S_1 + S_2 - S_3,$$

say. We consider each of these sums separately.

By (7) we have

$$S_1 = \sum_{d\le\rho z}\mu(d)\sum_{\substack{e\le x/d^k\\(e,d)\in(P)}} 1 = \sum_{d\le\rho z}\mu(d)\left(\frac{\phi(d_Q)x}{d^k d_Q}+O(\tau(d))\right)$$

$$= x\sum_{d\le\rho z}\frac{\mu(d)\phi(d_Q)}{d^k d_Q} + O(\sum_{d\le\rho z}\tau(d))$$

$$= x\sum_{n=1}^{\infty}\frac{\mu(n)\phi(n_Q)}{n^k n_Q} + O\left(x\sum_{n>\rho z}\frac{\mu(n)\phi(n_Q)}{n^k n_Q}\right) + O(\rho z\log(\rho z))$$

$$= \alpha_{A,k}x + O(x\delta(\rho z)(\rho z)^{1-k}) + O(\rho z\log(\rho z))$$

$$= \alpha_{A,k}x + O(\rho^{1-k}z\delta(\rho z)) + O(\rho z\log z)),$$

by (16) and by the well-known estimate $\sum_{n\le x}\tau(n) = O(x\log x)$. From (10) we obtain

$$S_2 = \sum_{e\le\rho^{-k}}\sum_{\substack{d\le\sqrt[k]{x/e}\\(d,e)\in(P)}}\mu(d) = \sum_{e\le\rho^{-k}}\sum_{\substack{d\le\sqrt[k]{x/e}\\(d,e_Q)=1}}\mu(d) = \sum_{e\le\rho^{-k}}M_{e_Q}(\sqrt[k]{x/e})$$

$$= \sum_{e\le\rho^{-k}} O\left(\sigma^*_{-1+\varepsilon}(e_Q)(\frac{x}{e})^{1/k}\delta((\frac{x}{e})^{1/k})\right)$$

$$= \sum_{e\le\rho^{-k}} O\left(\sigma^*_{-1+\varepsilon}(e)(\frac{x}{e})^{1/k}\delta((\frac{x}{e})^{1/k})\right).$$

Since $\delta(x)$ is monotonic decreasing and $\rho z \le (\frac{x}{e})^{1/k}$, we have $\delta((\frac{x}{e})^{1/k}) \le \delta(\rho z)$. Therefore,

$$S_2 = O\left(x^{1/k}\delta(\rho z)\sum_{e\le\rho^{-k}}\frac{\sigma^*_{-1+\varepsilon}(e)}{e^{1/k}}\right) = O(\rho^{1-k}z\delta(\rho z)),$$

from (18).

$$S_3 = \sum_{e\le\rho^{-k}}\sum_{\substack{d\le\rho z\\(d,e)\in(P)}}\mu(d) = \sum_{e\le\rho^{-k}}\sum_{\substack{d\le\rho z\\(d,e_Q)=1}}\mu(d) = \sum_{e\le\rho^{-k}}M_{e_Q}(\rho z)$$

$$= \sum_{e\le\rho^{-k}} O(\sigma^*_{-1+\varepsilon}(e_Q)\rho z\delta(\rho z)) = O(\rho z\delta(\rho z)\sum_{e\le\rho^{-k}}\sigma^*_{-1+\varepsilon}(e)) = O(\rho^{1-k}z\delta(\rho z)),$$

using (10) and (18). Therefore

$$\sum_{n\le x} q_{A,k}(n) = \alpha_{A,k}x + O(\rho^{1-k}z\delta(\rho z)) + O(\rho z\log z).$$

Choosing

$$\rho = \rho(x) = (\delta(x^{1/(2k)})^{1/k},$$

and following the proof of [SurSi73b], Theorem 3.1, see also [SurSiv73], we get formula (19).

If the R.H. is true, then applying (12) and (17) instead of (10) and (16), writing $x^{1/2}\omega(x) = x(x^{-1/2}\omega(x))$, where $x^{-1/2}\omega(x)$ is monotonic decreasing, and using the above arguments with $\delta(x)$ replaced by $x^{-1/2}\omega(x)$ we obtain

$$\sum_{n\leq x} q_{A,k}(n) = \alpha_{A,k}x + O(\rho^{1-k}z(\rho z)^{-1/2}\omega(\rho z)) + O(\rho z \log z).$$

Choosing $\rho = z^{-1/(2k+1)} < 1$ we have $\rho^{1/2-k}z^{1/2} = \rho z = x^{2/(2k+1)}$. Since $\omega(x)$ is monotonic increasing, we get $\omega(\rho z) \leq \omega(z) \leq \omega(x)$. We also have $\log z = O(\omega(x))$, and obtain the estimate (20). □

For $A = D$, i.e. for k-free integers and for $A = U$, i.e. for semi-k-free integers the result of Theorem 5 is due to A. WALFISZ [W63], Satz 1, p. 192 and to D. SURYANARAYANA and R. SITA RAMA CHANDRA RAO [SurSi73a], Corollary 3.2.1 ($n = 1$), [SurSi73b], Theorems 3.1 and 3.2.

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