STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume XLIII, Number 1, March 1998

DATA DEPENDENCE OF THE FIXED POINTS SET OF WEAKLY PICARD OPERATORS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. Data dependence in case of the weakly Picard operators is given.

1. Introduction.

Let X be a nonempty set and $f : X \to X$ an operator. We will use the notation

$F_f = \{x \in X \mid f(x) = x\}$, the fixed points set of f;

$O_f(x;n) = \{x, f(x), f^2(x), ..., f^n(x)\}$

$O_f(x) = \{x, f(x), f^2(x), ..., f^n(x), ...\}$, the orbit of $x \in X$;

$P(X) = \{A \subseteq X \mid A \neq \emptyset\}$.

For a metric space (X, d) we have

$\delta(A) = \sup\{d(a,b) \mid a, b \in A\}$, the diameter of $A \in P(X)$;

$P_{b,cl}(X) = \{A \in P(X) \mid A$ is bounded and closed $\}$;

$H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}_+$, $H(A,B) = \max(\sup\limits_{a\in A} \inf\limits_{b\in B} d(a,b), \sup\limits_{b\in B} \inf\limits_{a\in A} d(a,b))$, the Hausdorff- Pompeiu distance on $P_{b,cl}(X)$ set.

$H : P_{cl}(X) \times P_{cl}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ - the generalized Hausdorff - Pompeiu distance.

$C(X) = \{f : X \to X \mid f$ is continous operators $\}$

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function.

Definition 1. *φ is a strict comparison function if φ satisfies the following:*

i) φ is continous;

1991 *Mathematics Subject Classification.* 47H10; 45G.

Key words and phrases. weakly Picard operator, strict comparison function.

ii) φ *is monotone increasing ;*

iii) $\varphi^n(t) \overset{n\to\infty}{\longrightarrow} 0$, *for all* $t > 0$;

iv) $t - \varphi(t) \overset{t\to\infty}{\longrightarrow} \infty$;

Let (X, d) be a metric space and $f : X \to X$ an operator.

Definition 2. *The operator* f *is called weakly Picard if the sequence* $(f^n(x))_{n\geq 1}$ *converges for all* $x \in X$ *to a fixed point of* f*, which will be denote by* $f^\infty(x)$.

For more details about the weakly Picard operators see [2], [3] [4].

Definition 3. *The operator* f *is called a strict* $\varphi-$ *contraction if :*

i) φ *is a strict comparison function;*

ii) $d(f(x), f(y)) \leq \varphi(d(x, y))$, *for all* $x, y \in X$.

About the strict φ - contractions we have the next

Theorem 1. *Let* (X, d) *be a metric space,* $f : X \to X$ *a strict* φ *- contraction and* $x \in X$*. Then*

i) $d(f^i(x), f^j(x)) \leq \varphi(\delta(O_f(x; n)))$, *for all* $i, j \in \{1, 2, ..., n\}$ *with* $i < j$;

ii) for each $n \in \mathbb{N}$ *exists* $p \in \mathbb{N}$, *such that* $\delta(O_f(x; n)) = d(x, f^p(x))$;

iii) $\delta(O_f(x; n)) \leq \tau_{d(x,f(x))}$ *for each* $n \in \mathbb{N}$,

where $\tau_{d(x,f(x))} = \sup\{t \mid t - \varphi(t) \leq d(x, f(x))\}$;

For more details and results see [1],[2].

The aim of this paper is to give an answer to the following

PROBLEM "Let (X, d), be a metric space and $f, g : X \to X$ two weakly Picard operators.If exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for any $x \in X$, estimate the "distance" between F_f and F_g."

2. Main results.

Lemma 2. *Let* (X, d) *be a metric space and* $f, g : X \to X$ *two weakly Picard operators.Then* $H(F_f, F_g) \leq \max(\sup_{x\in F_g} d(x, f^\infty(x)), \sup_{x\in F_f} d(x, g^\infty(x)))$.

Proof. We remark that $f^{\infty}(x) \in F_f$ and $g^{\infty}(x) \in F_g$. The proof follows from the definition of H.

Theorem 3. *Let (X, d) be a complete metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a strict comparison function and $f, g : X \to X$ two orbitaly continous operators. We suppose that:*

i) $d(f(x), f^2(x)) \leq \varphi(d(x, f(x))$, for any $x \in X$ and
$d(g(x), g^2(x)) \leq \varphi(d(x, g(x))$, for any $x \in X$;
ii) there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for any $x \in X$.
Then:
a) f and g are weakly Picard operators;
b) $H(F_f, F_g) \leq \tau_\eta$, where $\tau_\eta = \sup\{t \mid t - \varphi(t) \leq \eta\}$.

Proof. a) Let $x \in X$ and $i, j \in \mathbb{N}$ with $i < j$.

We have $d(f^i(x), f^j(x)) \leq \varphi(d(f^{i-1}(x), f^{j-1}(x))) \leq$
$... \leq \varphi^i(d(x, f^{j-i}(x)) \leq \varphi^i(\delta(O_f(x; j-i))) \leq \varphi^i(\tau_{d(x,f(x))})$.

Finaly, if we put $i = n$, $j = n + p$, we obtain
$d(f^n(x), f^{n+p}(x)) \leq \varphi^n(\tau_{d(x,f(x))}) \overset{n\to\infty}{\longrightarrow} 0$.

Hence $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and $f^{\infty}(x)$ will be the limit of it. Because f is orbitaly continous then $f^{\infty}(x) \in F_f$.

In the inequality $d(f^n(x), f^{n+p}(x)) \leq \varphi^n(\tau_{d(x,f(x))})$ if we take $\lim\limits_{p\to\infty}$ we obtain that $d(f^n(x), f^{\infty}(x)) \leq \varphi^n(\tau_{d(x,f(x))})$, for any $n \in \mathbb{N}$.

Similarly, for any $x \in X$, we have the convergence of $(g^n(x))_{n\in\mathbb{N}}$ and $g^{\infty}(x)$, the limit of this sequence, has two properties:

$g^{\infty}(x) \in F_g$ and $d(g^n(x), g^{\infty}(x)) \leq \varphi^n(\tau_{d(x,g(x))})$, for any $n \in \mathbb{N}$.

b) From the estimation $d(f^n(x), f^{\infty}(x)) \leq \varphi^n(\tau_{d(x,f(x))})$, which is true for any $x \in X$ and $n \in \mathbb{N}$, we obtain for $n = 0$,that $d(x, f^{\infty}(x)) \leq \tau_{d(x,f(x))}$ (*).

By a similar argument we have that $d(x, g^{\infty}(x)) \leq \tau_{d(x,g(x))}$ (**)

From (*), (**) and ii) it follows

$$d(x, f^{\infty}(x)) \leq \tau_\eta, \text{ for any } x \in F_g \text{ and}$$
$$d(x, g^{\infty}(x)) \leq \tau_\eta, \text{ for any } x \in F_f\ ,$$

we apply Lemma 2.1. ∎

As a consequence of the Theorem 2.2 we have

Theorem 4. *Let (X,d) be a complete metric space, $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ a strict comparison function and $f_n, f : X \longrightarrow X, n \in \mathbb{N}$ orbitaly continous operators.*

We suppose that:

i) $d(f(x), f^2(x)) \leq \varphi(d(x, f(x))$, *for any* $x \in X$;

ii) $d(f_n(x), f_n^2(x)) \leq \varphi(d(x, f_n(x))$, *for any* $x \in X$ *and* $n \in \mathbb{N}$;

iii) $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to* f.

Then

a) f *and* f_n, $n \in \mathbb{N}$, *are weakly Picard operators;*

b) $H(F_{f_n}, F_f) \overset{n\to\infty}{\longrightarrow} 0$;

Remark 1. *,If we take* $\varphi(t) = at$, *with* $a \in [0,1[$, *from the Theorem 2.2 we have*

Theorem 5. *Let (X,d) be a complete metric space and $f,g : X \to X$ two orbitaly continous operators. We suppose that*

i) $d(f^2(x), f(x)) \leq ad(x, f(x))$, *for any* $x \in X$ *and*

$d(g^2(x), g(x)) \leq ad(x, g(x))$, *for any* $x \in X$;

ii) there exists $\eta > 0$ *such that* $d(f(x), g(x)) \leq \eta$, *for any* $x \in X$.

Then $H(F_f, F_g) \leq \frac{\eta}{1-a}$.

3. Applications.

Let $K_1, K_2 \in C([a,b] \times [a,b] \times \mathbb{R})$. We consider the following integral equations with deviating argument:

$$x(t) = x(a) + \int_a^b K_1(t, s, x(s))ds,\ t \in [a,b]\ (1)$$

$$x(t) = x(a) + \int_a^b K_2(t, s, x(s))ds,\ t \in [a,b]\ (2)$$

By the theorem 2.3 we have

Theorem 6. *We suppose that:*

i) $K_i(a, s, u) = 0$, *for any* $x \in [a,b]$, $s, u \in \mathbb{R}$; $(i = 1,2)$

ii) there exists $\eta > 0$ *such that*

$| K_1(t,s,u) - K_2(t,s,u) | \leq \eta$, *for all* $t, s \in [a,b]$ *and* $u \in \mathbb{R}$;

iii) there exists $L > 0$ *such that*

$| K(t,s,u) - K(t,s,v) | \leq L | u - v |$, *for all* $t, s \in [a,b]$ *and* $u, v \in \mathbb{R}$, $(i = 1,2)$

iv) $L(b-a) < 1$;

Let S_{K_i} be the solutions set of the equations (i) in $C[a,b]$ such that

$x(a) \in [\alpha, \beta]$ $(i = 1, 2)$.

Then

a) $S_{K_i} \neq \emptyset$, $i = 1, 2$;

b) $H_{\|.\|}(S_{K_1}, S_{K_2}) \leq \frac{\eta(b-a)+\beta-\alpha}{1-L(b-a)}$, *where by $H_{\|.\|}$ we denote the Hausdorff - Pompeiu metric with respect to Cebyshev norm on $C[a,b]$.*

Proof. We consider the operators $f, g : C[a,b] \to C[a,b]$ defined by

$$f(x)(t) = x(a) + \int_a^b K_1(t, s, x(s))ds \text{ and}$$

$$g(x)(t) = x(a) + \int_a^b K_2(t, s, x(s))ds.$$

It is clear that we have

$$\|f(x) - f^2(x)\| \leq L(b-a)\|x - f(x)\|, \text{ for all } x \in C[a,b] \text{ ,}$$

$$\|g(x) - g^2(x)\| \leq L(b-a)\|x - g(x)\|, \text{ for all } x \in C[a,b]$$

and

$$\|f(x) - g(x)\| \leq \beta - \alpha + (b-a)\eta,$$

we apply now the theorem 2.3. ∎

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