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ON THE LEVEL SETS OF (Γ,Ω)-QUASICONVEX FUNCTIONS

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. The aim of this paper is to show that the characteristic property of the real-valued quasiconvex functions to have convex level sets can be naturally extended in the class of (Γ,Ω)-quasiconvex functions, introduced by us in [5], which in particular contains the cone-quasiconvex vector-valued functions in the sense of Dinh The Luc [3].

1. Preliminaries

Quasiconvex functions play an important role in scalar and vector optimization, their characteristic property to have convex level sets being succesfully explored in order to derive optimality conditions or to study some topological properties of the efficient sets. Some fundamental properties concerning these topics can be found for instance in [2] or [3].

Our study here is based on the concept of (Γ,Ω)-quasiconvexity, introduced by us in [5] in order to describe some common properties of different classes of generalized quasiconvex functions in a unifying way.

For this aim we only need to endoved the domain of the (Γ,Ω)-quasiconvex functions with an abstract convexity induced by a set-valued mapping Γ, and to consider a binary relation Ω in the codomain. In the sequel we consider $\Gamma : E_1 \times E_1 \to 2^{E_1}$ and $\Omega : E_2 \to 2^{E_2}$, where E_1 and E_2 are two arbitrary nonempty sets.

We recall that a subset X of E_1 is said to be Γ-*convex* [5] iff

$$\Gamma(x^1,x^2) \subset X,\ \forall\, x^1,x^2 \in X.$$

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Obviously, the concept of Γ-convexity permits an unifying treatement of those notions of generalized convexity in which the line segments determined by two points are replaced by a continuous arc or by a discret subset of the domain.

On the other hand, when the codomain E_2 is not endoved with linear or topological structure, we shall need to replace the preordering induced by a pointed convex cone by an arbitrary binary relation Ω. Throughout the paper, this relation will be identified with the set-valued mapping $\Omega : E_2 \to 2^{E_2}$ defined by $\Omega y = \{y' \in E_2 \mid (y, y') \in \Omega\}$, $\forall\, y \in E_2$. We shall also associate to Ω the following relations: $\Omega^- y = \{y' \in E_2 \mid y \in \Omega y'\}$ and $\Omega^c y = E_2 \setminus (\Omega y)$, $\forall\, y \in E_2$.

Given a nonempty subset Y of E_2 we denote by $\Omega Y = \cup\{\Omega y \mid y \in Y\}$ the first order section of Y in the sense of J. Riguet [7] and by $[\Omega]Y = \cap\{\Omega y : \ y \in Y\}$ the second order section of Y, which is nowadays known as the *polar set* of Y.

By means of composite polarities, S. Dolecki and Ch. Malivert [1] have introduced the *cyrtological closure* operator $\mathrm{cl}_{\Omega^-} : 2^{E_2} \to 2^{E_2}$ defined by

$$\mathrm{cl}_{\Omega^-} Y = [\Omega][\Omega^-]Y, \ \forall\, Y \subset E_2.$$

As we shall see, the concepts of Γ-convexity and cyrtological closure are the key tools that we need to derive the main results of this work.

2. The characterization of the (Γ, Ω)-quasiconvexity by means of dominant level sets

Let us now recall [5] the definition of the (Γ, Ω)-quasiconvex functions:

Definition 2.1. Let $X \subset E_1$ be a nonempty and Γ-convex set. A function $f : X \to E_2$ is said to be (Γ, Ω)-*quasiconvex* on X if

$$f(\Gamma(x^1, x^2)) \subset \mathrm{cl}_{\Omega^-}\{f(x^1), f(x^2)\}, \ \forall\, x^1, x^2 \in X.$$

This definition calls for a few comments:

i) It is easy to see that f is (Γ, Ω)-*quasiconvex* on X if and only if

$$\forall\, x^1, x^2 \in X, \ \forall\, y \in E_2, \ f(\{x^1, x^2\}) \subset \Omega y \ \Rightarrow \ f(\Gamma(x^1, x^2)) \subset \Omega y. \quad (1)$$

ii) The terminology used in the above definition is relative; in fact, this definition concerns quasiconvexity as well as quasiconcavity, since we can interchange Ω with Ω^-.

For instance, if E_1 and E_2 are linear spaces and C is a convex cone in E_2, then for Γ and Ω defined by

$$\Gamma(x^1,x^2) = \mathrm{co}\{x^1,x^2\} = \{tx^1 + (1-t)x^2 |\ t \in [0,1]\},\ \forall\ x^1,x^2 \in E_1$$

and

$$\Omega y = y - C,\ \forall\ y \in E_2,$$

the (Γ,Ω)-quasiconvexity coincides with the cone-quasiconvexity in the sense of Dinh The Luc [3, 4].

It is known that if the euclidean space $E_2 = \mathbb{R}^n$ is partially ordered by the positive cone $C = \mathbb{R}^n_+$ then a vector-valued function $f = (f_1,\ldots,f_n) : X \to \mathbb{R}^n$ is (Γ,Ω)-quasiconvex if and only if their scalar components $f_1,\ldots,f_n$ are quasiconvex in the usual sense. Obviously, if we replace C by $-C$ then f is (Γ,Ω)-quasiconvex if and only if $f_1,\ldots,f_n$ are quasiconcave in the usual sense.

Definition 2.2. Let $f : X \to E_2$ be a function defined on a nonempty subset X of E_1. Given $y \in E_2$, the set

$$L_f(y) = \{x \in X \mid f(x) \in \Omega y\} \tag{2}$$

is called the *level set* of f corresponding to the *level* y.

The (Γ,Ω)-quasiconvexity can be characterized by means of these level sets as follows:

Proposition 2.1. *If the function $f : X \to E_2$ is defined on a nonempty and Γ-convex set $X \subset E_1$, then the following assertions are equivalent:*

i) f is (Γ,Ω)-quasiconvex on X;

ii) $L_f(y)$ is a Γ-convex set, $\forall y \in E_2$.

Proof. The implication i) $\Rightarrow$ ii) follows directly from the above definitions.

To prove the converse implication, let $x^1,x^2 \in X$ and $y \in E_2$ be such that $f(\{x^1,x^2\}) \subset \Omega y$. Then x^1, $x^2 \in L_f(y)$ and by ii) we obtain $\Gamma(x^1,x^2) \subset L_f(y)$, i.e. $f(\Gamma(x^1,x^2)) \subset \Omega y$. Using the relation (1), we conclude that f is (Γ,Ω)-quasiconvex on X. □

In what follows we shall refine this result by taking account of the following categories of level sets:

Definition 2.3. Let $f : X \to E_2$ be a function defined on a nonempty subset X of E_1 and let $y \in E_2$. We shall say that $L_f(y)$ given by (2) is:

i) a *dominant level set*, if $y \in \Omega f(X)$;

ii) an *attainable level set*, if $y \in f(X)$.

Obviously, if Ω is reflexive then every attainable level set is a dominant one.

Remark 2.1. The above definition is motivated by the fact that real-valued quasiconvex functions have some special properties which cannot be extended in the general case of (Γ, Ω)-quasiconvex functions without strong assumptions on their codomain. Indeed, a real-valued function $f : X \to \mathbb{R}$ is quasiconvex on a convex nonempty subset X of a linear space if and only if all their attainable level sets are convex. As shown by Example 3.2 this property fails to be true even in the case of cone-quasiconvex vector functions.

The following result show that we can although characterize the (Γ, Ω)-quasiconvexity using only dominant level sets:

Corollary 2.1. *Let $X \subset E_1$ be a nonempty and Γ-convex set. A function $f : X \to E_2$ is (Γ, Ω)-quasiconvex on X if and only if their dominant level sets are Γ-convex.*

Proof. Follows immediately from Proposition 2.1, because for any point $y \in E_2 \setminus \Omega^- f(X)$ the corresponding level set $L_f(y)$ is empty. □

3. The characterization of the (Γ, Ω)-quasiconvexity by means of attainable level sets

In order to obtain some characterizations of the (Γ, Ω)-quasiconvex functions in terms of attainable level sets, the following preliminary result will be usefull:

Lemma 3.1. *Let $f : X \to E_2$, where X is a nonempty subset of E_1. If $\Omega \subset E_2 \times E_2$ is transitive, then the set-valued mapping $L_f : E_2 \to 2^{E_2}$ given by (2) is isotone, i.e.*

$$L_f(y_1) \subset L_f(y_2), \ \forall \, y_1, y_2 \in E_2, \ y_1 \in \Omega y_2.$$

Proof. Let $y_1, y_2 \in E_2$ such that $y_1 \in \Omega y_2$ and let $x \in L_f(y_1)$. Then, by definition of L_f we have $f(x) \in \Omega y_1$ and therefore $f(x) \in \Omega y_1$, since Ω is transitive. □

Remark 3.1. The assumption on the transitivity of Ω in Lemma 3.1 is essential. This is illustrated by the following example:

Example 3.1. Consider $E_1 = E_2 = \mathbb{R}^2$, $\Gamma(x^1, x^2) = \text{co}\{x^1, x^2\}$, $\forall x^1, x^2 \in \mathbb{R}^2$ and let Ω be given by $\Omega y = y + C$, $\forall y \in \mathbb{R}^2$, where $C = \mathbb{R}^2 \setminus (\mathbb{R}_+^*)^2$. Obviously, C is a closed non convex cone and therefore the induced relation Ω is reflexive but it is not transitive.

If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the identic function on $\mathbb{R}^2$, i.e. $f(x) = x$, $\forall\, x \in \mathbb{R}^2$, it is easy to see that the mapping L_f is given by

$$L_f(y) = \{x \in \mathbb{R}^2 \mid x \in \Omega y\} = y + C, \ \forall y \in \mathbb{R}^2$$

and it is not isotone. In fact, for $y^1 = (0,1), y^2 = (1,0)$ and we have $y^1 \in \Omega y^2$, but $y^0 = (2,1) \in L_f(y_1) \setminus L_f(y_2)$.

Proposition 3.1. *Let X be a nonempty and Γ-convex subset of E_1 and $f : X \to E_2$. If Ω is a complete preordering in E_2, i.e. $(\Omega \cup \Omega^-)(y) = E_2$, $\forall\, y \in E_2$, then the following assertions are equivalent:*

i) f is (Γ, Ω)-quasiconvex on X;

ii) $L_f(f(x))$ is Γ-convex, $\forall\, x \in X$.

Proof. The implication i) $\Rightarrow$ ii) is a simple consequence of Proposition 2.1.

To prove the converse implication, suppose that ii) is true and consider some arbitrary points $y \in E_2$ and $x^1, x^2 \in L_f(y)$. Since Ω is a complete relation we can suppose, without loss of generality, that $f(x^1) \in \Omega f(x^2)$. Moreover, Ω being reflexive, we have $x^1, x^2 \in L_f(f(x^2))$ and therefore $\Gamma(x^1, x^2) \subset L_f(f(x^2))$ according to assumption ii).

On the other hand, we have $f(x^2) \in \Omega y$. Since Ω is transitive, by Lemma 3.1 we can conclude that $L_f(f(x^2)) \subset L_f(y)$ and consequently $\Gamma(x^1, x^2) \subset L_f(y)$. The assertion i) follows than again from Proposition 2.1. □

Remark 3.2. Without the assumption on the completeness of Ω, the implication ii) $\Rightarrow$ i) in Proposition 3.1 fails to be true even in the class of cone-quasiconvex vector-valued functions, as shown by the following example:

Example 3.2. Consider $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}^2$, $\Gamma(x^1, x^2) = \text{co}\{x^1, x^2\}$, $\forall x^1, x^2 \in \mathbb{R}$ and $\Omega y = y - C$, $\forall y \in \mathbb{R}^2$, where $C = \mathbb{R}_+^2$. Remark that in this case Ω is reflexive and transitive, but it is not complete in $\mathbb{R}^2$.

It is easy to see that the function $f : X = [0,1] \to \mathbb{R}^2$, defined by

$$f(x) = \begin{cases} (x, 1-x) & \text{if} \quad x \in]0,1] \setminus \{1/2\} \\ (1/2, 1/2) & \text{if} \quad x = 0 \\ (0,1) & \text{if} \quad x = 1/2. \end{cases}$$

satisfies the condition ii) in Proposition 3.1, since for any point $x \in X$, $L_f(f(x)) = \{x\}$ is a convex set, but f is not (Γ, Ω)-quasiconvex on X, because for $y = (3/4, 3/4)$, the level set $L_f(y) = \{0\} \cup [1/4, 3/4] \setminus \{1/2\}$ is not convex.

Theorem 3.1. *Let $X \subset E_1$ be a nonempty and Γ-convex set and $f : X \to E_2$. If Ω is a complete preordering in E_2 then the following assertions are equivalent:*

i) f is (Γ, Ω)-quasiconvex on X;

ii) The set-valued mapping $L_f \circ f : X \to 2^X$ is $(\Gamma, \supset)$-quasiconvex on X.

Proof. We first notice that statement ii) can be rewritten as follows:

$$L_f(f(\Gamma(x^1, x^2))) \subset L_f(f(\{x^1, x^2\})), \ \forall x^1, x^2 \in X. \tag{3}$$

Indeed, by (1) the function $L_f \circ f$ is $(\Gamma, \supset)$-quasiconvex on X if and only if

$$\forall\, x^1, x^2 \in X, \ \forall\, Y \in 2^X, Y \supset L_f(f(\{x^1, x^2\})) \Longrightarrow Y \supset L_f(f(\Gamma(x^1, x^2))).$$

Suppose now that f is (Γ, Ω)-quasiconvex on X and consider two arbitrary points $x^1, x^2 \in X$. By the completeness of Ω we can suppose, without loss of generality, that $f(x^1) \in \Omega f(x^2)$ i.e. $x^1 \in L_f(f(x^2))$. On the other hand, since Ω is reflexive, we have also $x^2 \in f(\Gamma(x^1, x^2)) \in \Omega f(x^2)$ and hence $\Gamma(x^1, x^2) \subset L_f(f(x^2))$ i.e. $f(\Gamma(x^1, x^2)) \subset \Omega f(x^2)$. Using the transitivity of Ω we obtain

$$L_f(f(\Gamma(x^1, x^2))) \subset L_f(f(x^2)) \subset L_f(f(\{x^1, x^2\}))$$

and hence condition (3) is fulfilled.

Conversely, if $L_f \circ f$ is $(\Gamma, \supset)$-quasiconvex on X then using Lemma 3.1 we conclude that for any points $y \in E_2$ and $x^1, x^2 \in L_f(y)$ we have

$$L_f(f(x^i)) \subset L_f(y), \ \forall\, i \in \{1, 2\}.$$

On the other hand, the reflexivity of Ω implies that $\Gamma(x^1, x^2) \subset L_f(f(\Gamma(x^1, x^2)))$ and using the assumption (3) we finally infer that $\Gamma(x^1, x^2) \subset L_f(y)$, $\forall\, y \in E_2$, which means that f is (Γ, Ω)-quasiconvex on X. $\square$

Remark 3.3. Even if the implication ii) $\Rightarrow$ i) in Theorem 3.1 is valid without the completeness assumption on Ω, this assumption cannot be dropped for the converse implication, as we can see from the following example:

Example 3.3. Consider $E_1 = \mathbb{R}, X = [0,1]$ and let Γ and Ω be given by

$$\Gamma(x^1,x^2) = [\min\{x^1,x^2\},\max\{x^1,x^2\}],\ \forall x^1,x^2 \in \mathbb{R} \text{ and } \Omega y = y - \mathbb{R}^2_+,\ \forall y \in \mathbb{R}^2.$$

It is easy to see that the function $f : X \to \mathbb{R}^2$ defined by $f(x) = (x, 1-x)$, $\forall\, x \in X$ is (Γ,Ω)-quasiconvex on X because f has Γ-convex level sets:

$$L_f(y) = \begin{cases} \{x\} & \text{if} \quad f(x) \in \Omega y \\ \emptyset & \text{if} \quad f(x) \in \Omega^c y. \end{cases}$$

On the other hand, we can see that the function $L_f \circ f$ is not $(\Gamma,\supset)$-quasiconvex on X because for $x^1 = 0$ and $x^2 = 1$ we have

$$L_f(f(\Gamma(x^1,x^2))) = L_f(f([0,1])) = [0,1] \not\subset L_f(f(\{x^1,x^2\})) = L_f(\{0,1\}) = \{0,1\}.$$

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