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ON THE CONVEXITY OF SUPPORTED SETS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. It is proved that if Y is a closed subset with nonvoid interior of a real topological vector space, such that the support points are dense in its boundary, then the set Y is convex.

1. Introduction

Given a subset Y of a real topological vector space (t.v.s. for short) X, an element x_0 of $\overline{Y}$ is said to be a *support point* for Y, if there is $x^* \in X^* x^* \neq 0$, such that $x^*(x_0) = \sup x^*(Y)$ or $x^*(x_0) = \inf x^*(Y)$. As usual we denote by X^* the dual space to X. Taking $-x^*$ instead of x^* it is obvious that we can always suppose that x^* attains its supremum at x_0. In Euclidean spaces this notion was first considered by H. Minkowski [12]. Denoting by spt$Y$ the set of all support points of the set Y and by bd$Y$ its boundary then spt$Y \subset$bd$Y$ (see Lemma 1.1 below). By a famous result of E. Bishop and R.R. Phelps [2], if X is a Banach space and $Y \subset X$ is closed and convex, then spt$Y$ is dense in the boundary of Y. A kind of converse of this result will be proved in Section 3 of this paper: if Y is closed with nonvoid interior and spt$Y$ is dense in bd$Y$ then Y is convex. The study of necessary conditions for some optimization problems (such as best approximation, optimal control, mathematical economics) motivates the investigation of the opposite inclusion bd$Y \subset$ spt$Y$. We say that the set Y is *totally supported* provided spt$Y$ =bd$Y$.

In the following theorem we present a list of some known supported sets.

Theorem 1.1. *A subset Y of a real t.v.s. X is totally supported provided one of the following (not necessarily independent) conditions is fulfilled:*

(a) X is the Euclidean space $\mathbb{R}^n$ and Y is convex (see [13]*)*

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(b) X is a normed space and Y is closed convex with nonvoid interior.

(c) X is a t.v.s. and Y is a p-convex subset of X with nonvoid interior (see [15]*).*

Recall that a subset Y of a vector space X is called p-*convex,* for $0 < p < 1$, if $px + (1-p)y \in Y$ for all $x, y \in Y$. Remark that there exist convex sets consisting only of support points. Namely, S. Rolewicz [18] constructed a closed convex set in a nonseparable Hilbert space X, not contained in any closed hyperplane in X, such that $\text{int}Y = \emptyset$ and $\text{spt}Y = Y$. A similar example is performed in [11] for Banach spaces containing uncountable minimal systems.

The present note is concerned with the investigation of the converse problem: In what conditions on the space X and on the set Y the total supportability of the set Y guarantees its convexity ? In the following theorem we list some known results in this direction.

Theorem 1.2. *Let X be a real t.v.s. and Y a closed subset of X with nonvoid interior. Then the implication "Y is totally supported $\Rightarrow$ Y is convex " is valid whenever one of the following (again not independent) conditions holds:*

(a) X is the Euclidean space $\mathbb{R}^n$ (see [4] *for Y bounded and* [13] *in general).*

(b) X is a prehilbertian space (see [13]*).*

(c) X is a general topological vector space (see [20]*, Th.3)*

The proofs given in [5] and [20] are only sketched, so that we shall present in Section 2 detailed proofs, preparing the proofs of some more general results given in Section 3, namely the cases when the support points are only dense in the boundary of Y or when the topology of X is not a vector toplogy, but merely a vectorial group topology (see[14]).

Infinite dimensional spaces may contain convex closed sets which are not totally supported (*a fortiori* they must have empty interior). Such a set is $Y = \{x \in L^2[0,1] : |x(t)| \leq 1, \text{a.e.} t \in [0,1]\}$ in the Hilbert space $L^2[0,1]$ (see [13, pp. 531–532]). Moreover in [10, pp. 97–98], it is constructed a precompact closed convex set Y in an incomplete inner product space with $\text{spt}Y = \emptyset$, and in [3, Corollary 2] it is proved that every normed space of countable algebraic dimension contains such a set. In the last quoted work it is stated also the conjecture: A real normed space is incomplete if and only if it contains a bounded closed convex set Y with spt$Y = \emptyset$. This conjecture is false for non-normable t.v.s. : there exist metrizable complete locally convex spaces containing bounded closed convex sets having no support points (see [16]). Other supportless convex sets in separable normed spaces are constructed in [8, 9].

In Banach spaces the situation radically improves, namely, in [13, Theorem 14] and, subsequently in [2, Theorem 1], it is proved that any closed convex set Y of a real Hilbert space, or of a real Banach space, respectively, satisfies $\overline{\text{spt}Y} = \text{bd}Y$. A subset Y of a t.v.s. is called ***densely supported*** provided $\overline{\text{spt}Y} = \text{bd}Y$. In Section 3 we shall prove that every densely supported closed set with nonvoid interior is convex.

2. Convexity of totally supported sets

We start by a lemma.

Lemma 2.1. *Let X be a t.v.s. over the field $\mathbb{K}$ of real or complex numbers. If $x^* : X \to \mathbb{K}$ is a non-null continuous linear functional and Y a nonvoid open convex subset of X then the set $x^*(Y)$ is open in $\mathbb{K}$.*

*Proof.*Let $\lambda_0 \in x^*(Y)$ and $y_0 \in Y$ be such that $\lambda_0 = x^*(y_0)$. Since $x^* \neq 0$, there exists $x_0 \in X$ with $x^*(x_0) = 1$. As $0x_0 = 0$ and $Y - y_0$ is a neighborhood of $0 \in X$, there exists a number $r > 0$ such that

$$D_r x_0 \subset Y - y_0 \tag{2.1}$$

where $D_r = \{\lambda \in \mathbb{K} : |\lambda| < r\}$. If $\lambda \in D_r$ then, by (2.1), $\lambda x_0 \in Y - y_0$ and

$$\lambda_0 + \lambda = x^*(y_0) + \lambda x^*(x_0) = x^*(y_0) + \lambda x_0) \in x^*(Y)$$

showing that $\lambda_0 + D_r \subset x^*(Y)$.□

Lemma 2.2. *If Y is a subset of a t.v.s. X then spt$Y \subset$ bd$Y$.*

Proof. Suppose there exists a point $x_0 \in$spt$Y\setminus$bd$Y$. It follows the existence of a functional $x^* \in X^*$, $x^* \neq 0$, such that $c = x^*(x_0) = \sup x^*(Y)$. Since $x_0 \in$int$Y$, by Lemma 2.1, there exists $\epsilon > 0$ such that the interval $[c-\epsilon, c+\epsilon]$ is contained in $x^*(Y)$, yielding the contradiction

$$c = \sup x^*(Y) \geq \sup x^*(\text{int}Y) \geq c + \epsilon.$$

Lemma is proved.□

Lemma 2.3. *Let Y be a closed subset of a real t.v.s. X and let $x,y,z \in Y$. If there exist $\alpha,\beta,\gamma > 0$, $\alpha+\beta+\gamma = 1$, such that $\alpha x+\beta y+\gamma z \in$ sptY then $x,y,z \in$ sptY.*

Proof. Supposing $x_0 = \alpha x+\beta y+\gamma z \in \mathrm{spt}Y$ it follows that there exists $x^* \in X^*$, $x^* \neq 0$, such that

$$c = x^*(x_0) = \sup x^*(Y).$$

implying

$$x^*(x-x_0) \leq 0, \qquad x^*(y-x_0) \leq 0, \qquad x^*(z-x_0) \leq 0. \tag{2.2}$$

¿From the identity

$$(\alpha+\beta+\gamma)x^*(x_0) = \alpha x^*(x)+\beta x^*(y)+\gamma x^*(z)$$

one obtains

$$\alpha x^*(x-x_0)+\beta x^*(y-x_0)+\gamma x^*(z-x_0) = 0$$

which, by (2.2), gives

$$x^*(x-x_0) = x^*(y-x_0) = x^*(z-x_0) = 0$$

showing that $x,y,z \in \mathrm{spt}Y$.□

Theorem 2.4. *Let X be a real t.v.s. and Y a closed subset of X with nonvoid interior. If Y is totally supported then Y is convex.*

Proof. Suppose the contrary, i.e. there exist $x,y \in Y$ and $\lambda \in]0,1[$ such that

$$u = \lambda x+(1-\lambda)y \notin Y \tag{2.3}$$

Let $z \in \mathrm{int}Y$ and let $f : [0,1] \to X$ be defined by $f(t) = tu+(1-t)z$, $t \in [0,1]$. Put

$$t_0 = \sup\{t \in [0,1] : f(t) \in Y\} \tag{2.4}$$

and let $(t_n) \subset]0,1[$ be such that $f(t_n) \in Y$ and $t_n \to t_0$. It follows

$$x_0 = f(t_0) = \lim_{n \leftarrow \infty} f(t_n) \in \overline{Y} = Y$$

and $t_0 < 1$ (by the definition (2.3) of u).

Show now that $x_0 \notin \mathrm{int}Y$. For if contrary, then as Y is a neighborhood of x_0 it would exists $\epsilon > 0$ such that $t_0+\epsilon < 1$ and $f(t_0+\epsilon) \in Y$, in contradiction to (2.4).

Therefore

$$x_0 = t_0 u + (1 - t_0)z \in \mathrm{bd}Y = \mathrm{spt}Y. \tag{2.5}$$

By (2.5) and (2.3), x_0 can be written in the form

$$x_0 = t_0(\lambda x + (1-\lambda)y) + (1-t_0)z = \alpha x + \beta y + \gamma z \tag{2.6}$$

where

$$\alpha = \lambda t_0 > 0, \quad \beta = (1-\lambda)t_0 > 0, \text{ and } \gamma = 1 - t_0 > 0.$$

Since $\alpha + \beta + \gamma = 1$, by Lemma 2.3 one obtains $z \in \mathrm{spt}Y = \mathrm{bd}Y$, in contradiction to $z \in \mathrm{int}Y$. □

Remark. The following examples show that the conditions " Y is closed" and "$\mathrm{int}Y \neq \emptyset$" are essential for the validity of Theorem 2.4.

The set

$$Y = [0,1] \times [0,1] \setminus \{(x_1, 0) : 0 < x_1 < 1\}$$

contained in $X = \mathbb{R}^2$, is totally supported without being convex.

Also, the set $Y = \{0,1\} \subset \mathbb{R}$ verifies $\mathrm{spt}Y = \mathrm{bd}Y$ and is not convex.

3. The convexity of densely supported sets

Recall that a subset Y of a t.v.s. X is called densely supported if $\overline{\mathrm{spt}Y} = \mathrm{bd}Y$.

Theorem 3.1. *Let X be areal t.v.s. and Y a closed subset of X having nonvoid interior. If Y is densely supported then Y is convex.*

Proof. Supposing the contrary, there exist $x, y \in Y$ and $\lambda \in]0,1[$ such that

$$u = \lambda x + (1-\lambda)y \notin Y. \tag{3.1}$$

Let $z' \in \mathrm{int}Y$ and let $V = \mathrm{int}Y$. Then V is an open neghborhood of z' and, reasoning like in the proof of Theorem 2.4, one can find a number $t_0 \in]0,1[$ such that

$$x_0' = t_0 + (1-t_0)z' \in \mathrm{bd}Y. \tag{3.2}$$

It follows $z' = \mu(x_0' - t_0 u)$, where $\mu = (1-t_0)^{-1} > 1$, so that

$$z' = u + \mu(x_0' - u). \tag{3.3}$$

Since spt$Y$ is dense in bd$Y$, $x_0' \in$ bd$Y$ and $W = x_0' + \mu^{-1}(V - z')$ is a neighborhood of x_0', there exists $x_0 \in W \cap$ spt$Y$, implying

$$x_0 - x_0' \in \mu^{-1}(V - z'). \tag{3.4}$$

Let

$$z = u + \mu(x_0' - u) \tag{3.5}$$

By (3.3),(3.4) and (3.2) we get

$$z - z' = \mu(x_0 - x_0') \in V - z',$$

showing that $z \in V = \text{int}Y$. By (3.5) and (3.2)

$$\alpha x + \beta y + \gamma z = x_0 \in \text{spt}Y$$

where

$$\alpha = \lambda(\mu - 1)/\mu, \qquad \beta = (1 - \lambda)(\mu - 1)/\mu \text{ and } \gamma = 1/\mu.$$

Since $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$, by Lemma 2.3 one obtains $z \in \text{spt}Y \subset \text{bd}Y$, in contradiction with $z \in \text{int}Y$, □

Corollary 3.2. *Let X be a real t.v.s. and Y a closed convex subset of X with nonvoid interior. Then the following assertions are equivalent:*

(a) Y is convex;

(b) Y is totally supported;

(c) Y is densely supported.

Proof. The implication $(b) \Rightarrow (c)$ is obvious. The implication $(c) \Rightarrow (a)$ is contained in Theorem 3.1 and the implication $(a) \Rightarrow (b)$ is contained in assertion (c) of Theorem 1.1. □

Remarks

1.Archimedes (see [22]) defined the convexity of a set in $\mathbb{R}^3$ as a totally supported set, a definition which is in concordance with Corollary 3.2.

2. A careful examination of the proofs shows that Lemma 2.3 and Theorems 2.4 and 3.1 remain valid in the case when the topology of the vector space X is not a vectorial topology. Namely, it is sufficient to suppose that

(i) the addition $(x, y) \to x + y$ is continuous from $X \times X$ to X, and

(ii) for every fixed $x \in X$ the multiplication by scalars $(\lambda, x) \to \lambda x$ is a continuous function from $\mathbb{R}$ to X

(see [14] for properties of such spaces called topological vector groups).

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