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SUFFICIENT CONDITIONS FOR STARLIKENESS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. In this paper we will study a differential subordination of the form:

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec h(z)$$

where h(z) is an univalent function in the unit disc U and we will obtain sufficient conditions of starlikeness for a function $f(z) = z + a_2 z^2 + \dots$ analytic in U.

We will obtain our results by using the differential subordination method developed in [1], [2] and [3].

1. Introduction and preliminaries

Let A denote the class of analytic functions in the unit disc $U=\{z,\ |z|<1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Also, let $S^* = \left\{ f \in A,\ Re\frac{zf'(z)}{f(z)} > 0,\ z \in U \right\}$ be the class of starlike functions in U.

In [7] the authors considered the class of functions $f \in A$ which satisfy the condition:

$$Re\left\{\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in U, \tag{1}$$

for $\alpha \geq 0$ where $\frac{f(z)}{z} \neq 0$, $z \in U$.

In [4] and [7] different types of starlike functions were investigated.

In [5] condition (2) was replaced by:

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec 1 + \lambda z, \tag{2}$$

$$\frac{f(z)}{z} \neq 0,\ z \in U, \text{ where } \alpha > 0 \text{ and } \lambda > 0.$$

In this paper we will consider a more general differential subordination of the form (1),where h is an univalent function in U.

We will need the following notions and lemmas to prove our main results.

If f and F are analytic functions in U,then we say that f is subordinate to F, written $f \prec F$, or $f(z) \prec F(z)$, if there is a function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ and if $f(z) = F(w(z))$, $z \in U$. If F is univalent then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Lemma A. *([1], [2], [3]) Let q be univalent in $\overline{U}$ with $q'(\zeta) \neq 0$, $|\zeta| = 1$, $q(0) = a$ and let $p(z) = a + p_1 z + \ldots$ be analytic in U, $p(z) \not\equiv a$. If $p \not\prec q$ then there exist $z_0 \in U$, $\zeta_0 \in \partial U$ and $m \geq 1$ such that:*

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$.

The function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(z,t) = a_1(t)z + a_2(t)z^2 + a_3(t)z^3 + \ldots$ is analytic and univalent in U for any $t \geq 0$ and if $L(z,t_1) \prec L(z,t_2)$ when $0 \leq t_1 \leq t_2$.

Lemma B. *([6]) The function $L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $a_1(t) \neq 0$ for $t \geq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if there are the constants $r \in (0,1]$ and $M > 0$ such that:*

(i) $L(z,t)$ is analytic in $|z| < r$ for any $t \geq 0$, locally absolute continuous in $t \geq 0$ for every $|z| < r$ and satisfies $|L(z,t)| \leq M|a_1(t)|$ for $|z| < r$ and $t \geq 0$.

(ii) there is a function $p(z,t)$ analytic in U for any $t \geq 0$ and measurable in $[0,\infty)$ for any $z \in U$ so that Re $p(z,t) > 0$ for $z \in U$, $t \geq 0$ and

$$\frac{\partial L(z,t)}{\partial t} = z\frac{\partial L(z,t)}{\partial z}p(z,t) \text{ for } |z| < r \text{ and for almost any } t \geq 0.$$

2. Main results

Theorem 1. *Let the function:*

$$h(z) = 1 + (2\alpha + 1)\mu z + \alpha\mu^2 z^2, \tag{3}$$

where

$$\alpha > 0 \qquad \textit{and} \qquad 0 < \mu \le 1 + \frac{1}{2\alpha}. \tag{4}$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ *is analytic in* U *and satisfies the condition:*

$$\alpha z p'(z) + \alpha p^2(z) + (1-\alpha)p(z) \prec h(z), \tag{5}$$

then $p(z) \prec 1 + \mu z$ *and this result is sharp.*

Proof. If we let $q(z) = 1+\mu z$, $\mu > 0$ and $\psi(p(z), zp'(z)) = \alpha zp'(z)+\alpha p^2(z)+(1-\alpha)p(z)$, then $\psi(q(z), zq'(z)) = h(z)$.

We will show that $\psi(p(z), zp'(z)) \prec h(z)$ implies $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

If we let $L(z,t) = \psi(q(z), (1+t)zq'(z)) = 1 + (\alpha t + 2\alpha + 1)\mu z + 2\alpha\mu^2 z^2$, then it is easy to show that:

$$\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} = \frac{1}{\alpha}[(\alpha t + 2\alpha + 1) + 2\alpha\mu z],\ |z| < 1, \qquad \text{and}$$

$$Re\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} = Re\frac{1}{\alpha}(\alpha t + 2\alpha + 1 + 2\alpha\mu z) \ge \frac{1}{\alpha}(\alpha t + 2\alpha + 1 - 2\alpha\mu)$$

Using now the condition (5) we obtain :

$$Re\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} \ge \frac{1}{\alpha}\left[\alpha t + 2\alpha + 1 - 2\alpha\left(1 + \frac{1}{2\alpha}\right)\right] = \frac{1}{\alpha}(\alpha t) = t \ge 0$$

Hence $Re\frac{z\frac{\partial}{\partial z}L(z,t)}{\frac{\partial}{\partial t}L(z,t)} \ge 0$, and by Lemma B we deduce that $L(z,t)$ is a subordination chain.

In particular, for $t = 0$ we have $L(z,0) = h(z) \prec L(z,t)$, for $t \ge 0$.

If we suppose that $p(z)$, is not subordonate to $q(z)$, then by Lemma A there exist $z_0 \in U$, $\zeta_0 \in \partial U$ such that $p(z_0) = q(\zeta_0)$ with $|\zeta_0| = 1$, and $z_0 p'(z_0) = (1+t)\zeta_0 q'(\zeta_0)$, with $t \ge 0$.

Therefore $\psi_0 = \psi(p(z_0), z_0 p'(z_0)) = \psi(q(\zeta_0), (1+t)\zeta_0 q'(\zeta_0)) = L(\zeta_0, t)$, $t \ge 0$.

Since $h(z_0) = L(z_0, 0)$ we deduce that $\psi_0 \notin h(U)$, which contradicts condition (6). Hence $p(z) \prec q(z)$ and since $\psi(q(z), zq'(z)) = h(z)$ we deduce that q is the best dominant.

Corollary 1. *If $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ is analytic in U and satisfies the condition: $\alpha z p'(z) + \alpha p^2(z) + (1-\alpha)p(z) \prec 1 + \lambda z$, where $\lambda = \mu(2\alpha + 1 - \alpha\mu)$*

$$\text{and} \qquad 0 < \mu \leq \left(1 + \frac{1}{2\alpha}\right), \qquad \text{then} \qquad p(z) \prec 1 + \mu z.$$

Proof. For $|z| = 1$ from (5) we deduce $|h(z) - 1| = \mu|2\alpha + 1 + \alpha\mu z| \geq \mu(2\alpha + 1 - \alpha\mu)$.

If we put $\lambda = \mu(2\alpha + 1 - \alpha\mu)$ we obtain $1 + \lambda z \prec h(z)$ and from Theorem 1 we deduce that $p(z) \prec 1 + \mu z$.

If we put $p = \dfrac{zf'}{f}$, where $f \in A$ then Theorem 1 can be written in the following equivalent form.

Theorem 2. *Let $h(z) = 1 + (2\alpha + 1)\mu z + \alpha\mu^2 z^2$, where $\alpha > 0$, $0 < \mu \leq \left(1 + \frac{1}{2\alpha}\right)$. Let $f \in A$, with $\frac{f(z)}{z} \neq 0$, satisfy the condition:*

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec h(z).$$

Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \mu z$$

and $1 + \mu z$ is the best dominant.

Corollary 2. *Let $f \in A$, with $\frac{f(z)}{z} \neq 0$, satisfy the condition:*

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec 1 + \lambda z$$

where $\lambda = \mu(2\alpha + 1 - \alpha\mu)$ and $0 < \mu \leq \left(1 + \frac{1}{2\alpha}\right)$.

Then

$$\frac{zf'(z)}{f(z)} \prec 1 + \mu z.$$

3. Particular cases

I. If $\alpha = 1$, then $0 < \mu \leq \frac{3}{2}$.

a). If we take $\mu = 1$ then $\lambda = 2$ and from Corollary 2 we obtain:

If $f \in A$ satisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)-1\right| < 2 \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < 1.$$

This result was obtained in [5].

b) If we take $\mu = \frac{1}{2}$ then $\lambda = \frac{5}{4}$ and from Corollary 2 we obtain the following condition for starlikeness. If $f \in A$ statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{5}{4} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{1}{2}.$$

c) If we take $\mu = \frac{2}{3}$ then $\lambda = \frac{9}{4}$ and from Corollary 2 we deduce:

If $f \in A$ statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{9}{4} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{3}{2}.$$

II. If $\alpha = 2$, then $0 < \mu \leq \frac{5}{4}$.

a). If we take $\mu = 1$ then $\lambda = \mu(2\lambda + 1 - \lambda\mu) = 3$ and from Corollary 2 we deduce:

If $f \in A$ satisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right| < 3 \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < 1.$$

b) If $\mu = \frac{1}{4}$ then $\lambda = \frac{9}{8}$ and from Corollary 2 we deduce: If $f \in A$, statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{9}{8} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{1}{4}.$$

c) If $\mu = \frac{5}{4}$, then $\lambda = \frac{25}{8}$ and from Corollary 2 we deduce:

If $f \in A$, statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{25}{8} \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right| < \frac{5}{4}.$$

d) If $\mu = \frac{1}{2}$, then $\lambda = 2$ and from Corollary 2 we deduce:

If $f \in A$, statisfies the condition

$$\left|\frac{zf'(z)}{f(z)}\left(2\frac{zf''(z)}{f'(z)}+1\right)-1\right|<2 \text{ then } \left|\frac{zf'(z)}{f(z)}-1\right|<\frac{1}{2}.$$

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