STUDIA UNIV. "BABEŞ-BOLYAI", MATHEMATICA, Volume XLIII, Number 1, March 1998

INDUCTION OF GRADED INTERIOR ALGEBRAS

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. We introduce interior $\mathcal{O}^\alpha H$-algebras graded by a finite group Γ and generalized induction for these algebras. This situation occurs in the study of source algebras of blocks of normal subgroups and our construction unifies various constructions introduced by Lluis Puig.

1. Introduction

Induction for interior G-algebras was introduced by L. Puig [P1], this being the fundamental construction linking an interior G-algebra with its source algebra. Given a subgroup H of G and an interior H-algebra H over a complete discrete valuation ring $\mathcal{O}$, the induced interior G-algebra is $\mathcal{O}G \otimes_{\mathcal{O}H} B \otimes_{\mathcal{O}H} \mathcal{O}G$, with multiplication inspired by that of the endomorphism algebra $\mathrm{End}_{\mathcal{O}G}(\mathcal{O}G \otimes_{\mathcal{O}H} M)$, where M is an $\mathcal{O}H$-module.

Later some generalizations were needed in order to deal with more involved problems. Algebras interior for a twisted group algebra were considered in [P2]; dealing with blocks of normal subgroups in [KP] imposed the construction of of G-algebra extensions; finally, noninjective induction was introduced in [P3] and [P4] in order to study bimodules inducing equivalences between interior algebras.

The aim of this note is to unify these constructions. We shall consider $\mathcal{O}$-algebras A graded by a group Γ, endowed with a grade preserving $\mathcal{O}$-algebra map $\mathcal{O}^\alpha H \to A$, where $\mathcal{O}^\alpha H$ is the twisted group algebra defined by the cocycle $\alpha \in Z^2(H, \mathcal{O}^*)$, and H has a a normal subgroup N such that $G = H/N$ is a subgroup of Γ. This degree of generality is needed; this situation occurs for instance when one considers the source algebra of a G-invariant block of $\mathcal{O}^\alpha N$. We have in mind later applications to Clifford theory, and recall that similar contexts have been considered in recent work of E. Dade.

1991 *Mathematics Subject Classification.* 16W50.
Key words and phrases. interior algebras,blocks.

Most of our conventions and notations will follow those of [P2], [T] and [NV], except that we use the notation of [K] for twisted group algebras. The needed definitions will be given in each section, but some standard facts from these sources will be used without comments. In Section 2 we discuss graded algebras and their exomorphisms; Γ-graded interior $\mathcal{O}^\alpha H$-algebras are introduced in Section 3. Injective induction for these algebras is defined and studied in Section 4, while in the last section we introduce the generalized induction.

2. Twisted group algebras, interior algebras and group extensions

2.1. We fix a p-modular system $(\mathcal{K}, \mathcal{O}, k)$, where $\mathcal{O}$ is a complete discrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$ and $k = \mathcal{O}/J(\mathcal{O})$ is the residue field of $\mathcal{O}$. The case $k = \mathcal{O} = \mathcal{K}$ is not excluded.

2.2. Let $A = \bigoplus_{g\in G} A_g$ be a G-graded $\mathcal{O}$ algebra, where G is a finite group and the additive subgroups A_g, $g \in G$ are $\mathcal{O}$-free of finite rank.

We shall be interested in some particular cases. Recall that A is *strongly graded* if $A_g A_h = A_{gh}$ for all $g, h \in G$, and A is a *crossed product* if $A_g \cap U(A) \neq \emptyset$. In this case, denoting

$$hU(A) = \bigcup_{g\in G} (A_g \cap U(A)),$$

we have the group extension

$$\epsilon(A): \qquad 1 \to U(A_1) \to hU(A) \overset{deg}{\to} G \to 1.$$

If $\epsilon(A)$ splits, then A is a *skew group algebra.* We shall also discuss twisted group algebras later.

2.3. If $B = \bigoplus_{g\in G} B_g$ is another G-graded $\mathcal{O}$-algebra, then a homomorphism $f: A \to B$ of $\mathcal{O}$-algebras (not necessarily unital) is called *G-graded* (*grade preserving*) if $f(A_g \subseteq B_g$ for all $g \in G$.

More generally, let $\phi: G \to H$ be a group homomorphism, $B = \bigoplus_{h\in H} B_h$ a H-graded $\mathcal{O}$-algebra, and denote by $\mathrm{Res}_\phi(B)$ the G-graded algebra $\mathrm{Res}_\phi(B) = \bigoplus_{g\in G} B_{\phi(g)}$. Then a homomorphism $f: A \to B$ of $\mathcal{O}$-algebras is called *graded* if $f(A_g) \subseteq B_{\phi(g)}$ for all $g \in G$, that is, f induces a grade preserving map, still denoted $f: A \to \mathrm{Res}_\phi(B)$.

If ϕ is just the inclusion $G \subseteq H$, then we shall simply denote $B_G = \bigoplus_{g\in G} B_g$. Observe that construction of B_H is functorial, that is, if $f: B \to B'$ is a homomorphism of H-graded algebras, then f induces in an obvious way a homomorphism of G-graded algebras $f_G : B_G \to B'_G$. In this situation, the G-graded algebra A can be trivially regarded as an H-graded algebra by defining $A_h = 0$ for $h \in H \setminus G$.

Another important situation which will occur in Section 5 is when $\phi: G \to H$ is surjective. Then the G-graded algebra A can be made into a H-graded algebra by defining $A_h = \bigoplus_{g\in\phi^{-1}(h)} A_g$.

Returning to the case when both A and B are G-graded, remark further that $f: A \to B$ induces a group homomorphism $f^*: U(A) \to U(B)$ by $f^*(a) = f(a-1)+1$. We also have that

$$f(a^{a^*}) = f(a)^{f^*(a^*)},$$

where $a^{a^*} = (a^*)^{-1}aa^*$. Moreover, if f is unital, and A and B are crossed products, then f induces a homomorphism of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(A_1) & \longrightarrow & hU(A) & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \phi} & & \\ 1 & \longrightarrow & U(B_1) & \longrightarrow & hU(B) & \longrightarrow & H & \longrightarrow & 1 \end{array}$$

2.4. The group $hU(A)$ acts on A_1 as $\mathcal{O}$-algebra automorphisms, and on $U(A_1)$ as group automorphisms. Moreover, $hU(A_{Z(G)}) = deg^{-1}(Z(G))$ acts on A as grade-preserving automorphisms, and on $hU(A)$ as automorphisms of group extensions.

2.5. Let A and B be two G-graded $\mathcal{O}$-algebras. Then $A\otimes_{\mathcal{O}} B$ is naturally $G\times G$-graded, and if $\delta(G)$ denotes the diagonal subgroup of $G\times G$, then $(A\otimes_{\mathcal{O}} B)_{\delta(G)} = \bigoplus_{g\in G}(A_g \otimes_{\mathcal{O}} B_g)$ is again a G-graded algebra. We shall denote $\Delta(A, B^{op}) = (A\otimes_{\mathcal{O}} B)_{dG}$, this being coherent with the notation of [M]. The G-grading of B^{op} is given by $B^{op}_g = B_{g^{-1}}$, and by this convention, $\Delta(A,B) = (A\otimes_{\mathcal{O}} B^{op})_{\delta(G)}$. Moreover, if A and B are strongly graded (crossed products), then $A\otimes_{\mathcal{O}} B$ and $\Delta(A,B)$ are strongly graded (crossed products).

2.6. Definition. Let A and B be G-graded algebras. A *graded exomorphism* $\tilde{f}: A \to B$ is the set obtained by composing the grade-preserving homomorpfism $f: A \to B$ with the inner automorphisms of A and B given by conjugation with elements of of A_1 and B_1 respectively. Denote by $\widetilde{\mathrm{Hom}}_{gr}(A,B)$ the set of graded exomorphisms $\tilde{f}: A \to B$.

To obtain $\tilde{f}$ it suffices to compose f only with the above inner automorphisms of B. This implies that graded exomorphisms can be composed.

The exomorphism $\tilde{f}$ is called an *embedding* if $\mathrm{Ker} f = 0$ and $\mathrm{Im} f = f(1)Bf(1)$. Clearly, $\tilde{f}$ is an embedding if and only if $\tilde{f}_1 : A_1 \to B_1$ is an embedding of $\mathcal{O}$-algebras, where $f_1 : A_1 \to B_1$, $f_1(a) = f(a)$.

Let $\tilde{f} \in \widetilde{\mathrm{Hom}}_{gr}(A,B)$, $\tilde{g} \in \widetilde{\mathrm{Hom}}_{gr}(B,C)$ and $\tilde{h} = \tilde{g} \circ \tilde{f}$. It follows by this remark and [P2, Lemma 3.4] that: if $\tilde{g}$ is an embedding then $\tilde{f}$ is uniquely determined by $\tilde{h}$, and $\tilde{f}$ is an embedding if and only if $\tilde{h}$ is an embedding.

2.7. We end this section with by discussing an important example. Let $\alpha: H \times H \to \mathcal{O}^*$ a 2-cocycle (where $\mathcal{O}^* = U(\mathcal{O})$), and consider the twisted group algebra $\mathcal{O}^\alpha H = \{a\bar{x} \mid x \in H,\ a \in \mathcal{O}\}$ with multiplication $\bar{x}\bar{y} = \alpha(x,y)\overline{xy}$ for all $x, y \in H$. Clearly, $\mathcal{O}^\alpha H$ is a particular case of an H-graded crossed product, and if β is another 2-cocycle, then $\mathcal{O}^\alpha H \simeq \mathcal{O}^\beta H$ as H-graded algebras if and only if $\alpha\beta^{-1} \in B^2(H, \mathcal{O}^*)$. If N is a subgroup of H, we shall still denote $\mathcal{O}^\alpha N = \mathcal{O}^{res^H_N \alpha} N$, where $res^H_N \alpha \in Z^2(N\mathcal{O}^*)$.

We shall be interested in other gradings, too. If N is a normal subgroup of H, and $G = H/N$, then $\mathcal{O}^\alpha H$ is naturally graded by G.

We recall from [K] some properties of twisted group algebras.

(1.7.1) $\mathcal{O}^\alpha H \otimes_{\mathcal{O}} \mathcal{O}^{\alpha'} H' \simeq \mathcal{O}^{\alpha \times \alpha'}(H \times H')$ via $(\bar{h} \otimes \bar{h}') \leftrightarrow \overline{(h, h')}$.

(1.7.2) $(\mathcal{O}^\alpha H)^{op} \simeq \mathcal{O}^{\alpha^{-1}} H$ via $\bar{h} \leftrightarrow \overline{h^{-1}}$.

(1.7.3) If $\alpha, \beta \in Z^2(H, \mathcal{O}^*)$ then $\mathcal{O}^{\alpha\beta} H \simeq (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} \mathcal{O}^\beta H)_{\delta(H)}$ via $\tilde{h} \leftrightarrow \bar{h} \otimes_{\mathcal{O}} \hat{h}$. (Notice that we have taken here the diagonal with respect to the H-grading.)

3. Graded interior algebras

We shall now describe our main object of study.

3.1. Definition. Let H and Γ be finite groups, $\mu: H \to \Gamma$ a group homomorphism, $N = \mathrm{Ker}\mu$, and $G = \mathrm{Im}\mu$. We also denote by μ the induced injective homomorphism $G \to \Gamma$. Let $\alpha \in Z^2(H, \mathcal{O}^*)$ and A a Γ-graded $\mathcal{O}$-algebra endowed with a graded homomorphism $\psi: \mathcal{O}^\alpha H \to A$ (that is, $\psi(\bar{h}) \in A_{\mu(h)}$ for all $h \in H$). Then (A, μ, ψ) (or simply A is called a Γ-*graded interior* $\mathcal{O}^\alpha H$-*algebra*, and μ, ψ are the *structural maps* of A.

3.2. Examples. a) Clearly, if N is a normal subgroup of H and $G = H/N$, then $\mathcal{O}^\alpha H$ is a G-graded interior $\mathcal{O}^\alpha H$-algebra.

b) If $e \in Z(\mathcal{O}^\alpha N)$ is a G-invariant idempotent, then $e\mathcal{O}^\alpha H$ is a G-graded interior $\mathcal{O}^\alpha H$-algebra with structural map $a\bar{h} \mapsto ea\bar{h}$, for all $a \in \mathcal{O}$, $h \in H$.

c) Let U be an $\mathcal{O}^\alpha N$-module and $M = \mathcal{O}^\alpha H \otimes_{\mathcal{O}^\alpha N} U = \operatorname{Ind}_{\mathcal{O}^\alpha N}^{\mathcal{O}^\alpha H} U$ with the usual G-grading. The $\mathcal{O}$-algebra $A = \operatorname{End}_{\mathcal{O}}(M)^{op}$ has a G-grading given by $A_g = \{f \in A \mid f(M_x) \subseteq M_{gx}$ for all $x \in G\}$. Now define $\psi\colon \mathcal{O}^\alpha H \to A$ by $\psi(\bar{h})(\bar{h}' \otimes u) = \bar{h}'\bar{h} \otimes u$. One can easily verify that A becomes a G-graded interior $\mathcal{O}^\alpha H$-algebra.

3.3. Let (A, μ, ψ) be a Γ-graded interior $\mathcal{O}^\alpha H$-algebra and $A, \mu', \psi')$ a γ-graded interior $\mathcal{O}^\beta H$-algebra. We have the group extension $N \times N \to H \times H \to G \times G$, and denote by $\delta_G(H) = \{(x, y) \in H \times H \mid xN = yN\}$ the "diagonal" of $H \times H$ w.r.t. G. Then $A \otimes_{\mathcal{O}} B$ is a $\Gamma \times \Gamma$-graded interior $\mathcal{O}^{\alpha\times\beta}(H \times H)$-algebra, and $\Delta(A, B^{op})$ is a $\delta(\Gamma)$-graded interior $\delta_G(H)$-algebra (and also a $\delta(H)$-algebra by restriction).

3.4. Observe that $A_{\mu(G)}$ is a $\mu(G)$-graded crossed product and μ, ψ induce the homomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(\mathcal{O}^\alpha N) & \longrightarrow & hU(\mathcal{O}^\alpha H) & \longrightarrow & G & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mu} & & \\ 1 & \longrightarrow & U(A_1) & \longrightarrow & hU(A_{\mu(G)}) & \longrightarrow & \mu(G) & \longrightarrow & 1 \end{array}$$

of group extensions. Although H may not be a subgroup of $hU(\mathcal{O}^\alpha H)$, it still acts on A_1 by conjugation. Actually, $A_{\mu(G)}$ is determined by A_1, the group extension $N \to H \to G$, and the action of H on A_1. Indeed, the homomorphism $\mathcal{O}^\alpha H \to A_{\mu(G)}$ of G-graded algebras (identifying G with $\mu(G)$) determines a structure of a G-graded $\mathcal{O}^\alpha H$-bimodule on $A_{\mu G}$ and also a map

$$(\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op})_{\Delta_\Gamma(\mathcal{O}^\alpha H)} A_1 \to (A \otimes_{\mathcal{O}} A^{op}) \otimes_{\Delta(A)} A_1$$

of G-graded $(\mathcal{O}^\alpha H, \mathcal{O}^\alpha H)$-bimodules (where $\Delta(A) = (A \otimes_{\mathcal{O}} A^{op})_{\delta(G)}$, see [M, Section 2]). Since the 1-component of this map is just the identity map of A_1, it follows that

$$(\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op})_{\Delta_\Gamma(\mathcal{O}^\alpha H)} A_1 \to A \quad (\bar{x} \otimes \bar{y}) \otimes a \mapsto xay^{-1}$$

is an isomorphism of G-graded $\mathcal{O}^\alpha H$-bimodules.

3.5. Definition. A *homomorphism* $f\colon A \to A'$ of Γ-graded interior $\mathcal{O}^\alpha H$-algebras is a graded $\mathcal{O}$-algebra map satisfying $f\bar{x} \cdot a \cdot \bar{y}) = \bar{x} \cdot f(a) \cdot \bar{d}$ for all $x, y \in H$ and $a \in A$. We still denote by $\operatorname{Hom}_{gr}(A, A')$ the set of these homomorphisms.

The *exomorphism* $\tilde{f}\colon A \to A'$ is the orbit of f under the action of $U(A_1^H) \times U({A'_1}^H)$ on $\operatorname{Hom}_{gr}(A, A')$.

Since this orbit coincides with the orbit under the action of $U(A_1'^H)$, it follows that the exomorphisms of Γ-graded interior $\mathcal{O}^\alpha H$-algebras can be composed, and we denote by $\widetilde{\mathrm{Hom}}_{[}gr(A, A')$ the set of exomorphisms $\tilde{f}\colon A \to A'$.

3.6. Let ρ be a homomorphism

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

of group extensions such that $G_1 \to G$ is injective (otherwise we replace N_1 with the kernel of the composition $H_1 \to H \to G$). Then $\mathrm{Res}_\rho A$ is, by definition, the Γ-graded interior $\mathcal{O}^{\alpha_1} H_1$-algebra $(A, \mu \circ \rho, \psi \circ \rho)$, where $\alpha_1 = res_\rho \alpha \in Z^2(H_1, \mathcal{O}^*)$.

Moreover, a homomorphism $f : A \to B$ of Γ-graded interior $\mathcal{O}^\alpha H$ algebras induces obviously the homomorphism $\mathrm{Res}_\rho(f) : \mathrm{Res}_\rho A \to \mathrm{Res}_\rho B$ of Γ-graded interior $\mathcal{O}^{\alpha_1} H_1$-algebras.

4. Injective induction for graded interior algebras

4.1. Consider the group extension $N \to H \to G$ and the subgroups K of H, $N \cap K$ of N and $K/K \cap N \simeq KN/N$ of G, and let $[H/K]$ be a complete set of representatives for the left cosets of K in H. Denote by ρ all these inclusion maps and, for $\alpha \in Z^2(H, \mathcal{O}^*)$, we also denote by α the element $\mathrm{Res}^H_K \alpha \in Z^2(K, \mathcal{O}^*)$. Let finally $\mu\colon G \to \Gamma$ an injective group homomorphism.

4.2. Let B be a Γ-graded interior $\mathcal{O}^\alpha K$-algebra with structural maps $\mu' = \mu \circ \rho : KN/N \to \Gamma$ and $\psi'\colon \mathcal{O}^\alpha K \to B$. Consider the $(\mathcal{O}^\alpha H, \mathcal{O}^\alpha H)$-bimodule $A = \mathcal{O}^\alpha H \otimes_{\mathcal{O}^\alpha K} B \otimes_{\mathcal{O}^\alpha K} \mathcal{O}^\alpha H$, and define the $\mathcal{O}$-bilinear multiplication

$$(\bar{x} \otimes b \otimes \bar{y})(\bar{x}' \otimes b' \otimes \bar{y}') = \begin{cases} 0, & \text{if } yx' \in K \\ \bar{x} \otimes b \cdot \bar{y}\bar{x}' \cdot b' \otimes \bar{y}', & \text{if} yx' \in K \end{cases}$$

and the map

$$\psi\colon \mathcal{O}^\alpha H \to A, \quad \bar{x} \mapsto \sum_{y \in [H/K]} \bar{x}\bar{y} \otimes 1_B \otimes \bar{y}^{-1}.$$

4.3. Proposition. *A is a Γ-graded interior $\mathcal{O}^\alpha H$-algebra with structural maps μ and ψ.*

Proof. It can be easily verified that the multiplication is well defined and associative, and that A is an $\mathcal{O}$-algebra with unit element $1_A = \sum_{y\in[H/K]} \bar{y} \otimes 1_B \otimes \bar{y}^1$. Also, ψ is a well defined $\mathcal{O}$-algebra map.

The grading of A is defined as follows. If $g \in \Gamma$, $x, y \in H$ and $b \in B_g$, then $\bar{x} \otimes b \otimes \bar{y}$ is a homogeneous element of degree $\mu(xN)g\mu(yN) \in \Gamma$. It follows that for $g = xN \in G$ we have

$$A_g = \sum_{y\in[H/K]} \bar{x}\bar{y} \otimes_{\mathcal{O}^\alpha K} B_1 \otimes_{\mathcal{O}^\alpha K} \bar{y}^{-1}.$$

In particular, $A_1 = \sum_{y\in[H/K]} \bar{y} \otimes B_1 \otimes \bar{y}^{-1}$ is a subalgebra of A. It also follows that the structural map ψ is grade-preserving.

4.4. Definition. We shall say that the Γ-graded interior $\mathcal{O}^\alpha H$-algebra A is ***induced*** from B, and we denote $A = \mathrm{Ind}_\rho(B) = \mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(B)$.

The construction is functorial, since if $f : B \to B'$ is a homomorphism of Γ-graded interior $\mathcal{O}^\alpha K$-algebras, then $\mathrm{Ind}_\rho(f) = id \otimes f \otimes id : \mathrm{Ind}_\rho(B) \to \mathrm{Ind}_\rho(B')$ is a homomorphism of Γ-graded interior $\mathcal{O}^\alpha H$-algebras.

4.5. Since the subalgebra A_G is a G-graded crossed product, it can be constructed from B_1 in an alternative way. Indeed, we have that

$$\begin{aligned} A_G &= (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op}) \otimes_{\mathcal{O}^\alpha K \otimes_{\mathcal{O}} (\mathcal{O}^\alpha K)^{op}} B_G \\ &\simeq (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op}) \otimes_{\mathcal{O}^\alpha K \otimes_{\mathcal{O}} (\mathcal{O}^\alpha K)^{op}} ((\mathcal{O}^\alpha K \otimes_{\mathcal{O}} (\mathcal{O}^\alpha K)^{op}) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1) \\ &\simeq (\mathcal{O}^\alpha H \otimes_{\mathcal{O}} (\mathcal{O}^\alpha H)^{op}) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha H)} (\Delta_\Gamma(\mathcal{O}^\alpha H) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1) \end{aligned}$$

where we have also denoted $\Delta_\Gamma(\mathcal{O}^\alpha K) = \Delta_{K/K\cap N}(\mathcal{O}^\alpha K)$. Then, for $g = xH \in G$,

$$A_g \simeq (\bar{x} \otimes_{\mathcal{O}} 1) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha H)} (\Delta_\Gamma(\mathcal{O}^\alpha H) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1),$$

and A_1 is the $\Delta_\Gamma(\mathcal{O}^\alpha H)$-module $\Delta_\Gamma(\mathcal{O}^\alpha H) \otimes_{\Delta_\Gamma(\mathcal{O}^\alpha K)} B_1$.

4.6. Proposition. *Let M be an $\mathcal{O}^\alpha(K \cap N)$-module and $B = \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\mathcal{O}(K\cap N)}^{\mathcal{O}^\alpha K} M)$. Then*

$$\mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(B) \simeq \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha H}(M))$$

as Γ-graded interior $\mathcal{O}^\alpha H$-algebras.

Proof. By construction, B is a KN/N-graded crossed product, which can be trivially regarded as a Γ-graded interior $\mathcal{O}^\alpha K$-algebra, since $KN/N \leq G \leq \Gamma$. Denote $A =$

$\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha H}(M))$. Since $\mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha H}(M)) \simeq \mathrm{Ind}_{\mathcal{O}^\alpha N}^{\mathcal{O}^\alpha H}(\mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha n}(M))$, A is a G-graded crossed product by Example 2.2.c).

First, we define an $\mathcal{O}$-linear action of $\mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(B)$ on $\mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha H}(M)$. Let $f \in B_g$, $v \in \mathrm{Ind}_{\mathcal{O}^\alpha(K\cap N)}^{\mathcal{O}^\alpha K}(M))$ and $x, y, z \in H$, and define

$$(\bar{x} \otimes f \otimes \bar{y})(\bar{z} \otimes v) = \begin{cases} \bar{x} \otimes f(\bar{y}^{-1}\bar{z}v), & \text{if } y^{-1}z \in K \\ 0, & \text{otherwise.} \end{cases}$$

If f is homogeneous of degree $g \in G$ and v is homogeneous of degree $h \in KN/N$, then $z \otimes v$ is homogeneous of degree $xNgy^{-1}zNh$. By [T, (6.4)], this action induces an isomorphism of interior $\mathcal{O}^\alpha H$-algebras, and by the above remarks, it is also grade-preserving.

4.7. Proposition. *Let $L \leq K \leq H$ and C a Γ-graded interior $\mathcal{O}^\alpha L$-algebra. Then there is an isomorphism of Γ-graded interior $\mathcal{O}^\alpha H$-algebras*

$$\mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(\mathrm{Ind}_{\mathcal{O}^\alpha L}^{\mathcal{O}^\alpha K} C) \simeq \mathrm{Ind}_{\mathcal{O}^\alpha L}^{\mathcal{O}^\alpha H}(C).$$

Proof. Using [T, Proposition 16.3], one can check that the map

$$\gamma\colon \mathrm{Ind}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H}(\mathrm{Ind}_{\mathcal{O}^\alpha L}^{\mathcal{O}^\alpha K} C) \to \mathrm{Ind}_{\mathcal{O}^\alpha L}^{\mathcal{O}^\alpha H}(C), \quad \bar{x} \otimes (\bar{y} \otimes c \otimes \bar{y}') \otimes \bar{x}' \mapsto \bar{x}\bar{y} \otimes c \otimes \bar{y}'\bar{x}'$$

is an isomorphism of of interior $\mathcal{O}^\alpha H$-algebras. By Definition 2.1 it also follows that $\gamma$is grade-preserving.

4.8. Proposition. Let $K \leq H$, A a Γ-graded interior $\mathcal{O}^\alpha H$-algebra and B a Γ-graded interior $\mathcal{O}^\beta K$-algebra, where $\alpha, \beta \in Z^2(H, \mathcal{O}^*)$. Then there is an isomorphism

$$\delta\colon \Delta_\Gamma(A \otimes_{\mathcal{O}} \mathrm{Ind}_{\mathcal{O}^\beta K}^{\mathcal{O}^\beta H}(B)) \to \mathrm{Ind}_{\mathcal{O}^{\alpha\beta} K}^{\mathcal{O}^{\alpha\beta} H}(\Delta_\Gamma(\mathrm{Res}_{\mathcal{O}^\alpha K}^{\mathcal{O}^\alpha H} A \otimes_{\mathcal{O}} B))$$

of Γ-graded interior $\mathcal{O}^{\alpha\beta}H$-algebras.

Proof. Define δ by

$$a \otimes (\hat{x} \otimes b \otimes \hat{y}) \mapsto \tilde{x} \otimes (\bar{x}^{-1} \cdot a \cdot \bar{y}^{-1} \otimes b) \otimes \tilde{y},$$

where $a \in A_g$ and $b \in B_g$. Then δ is an isomorphism of Γ-graded interior $\mathcal{O}^{\alpha\beta}H$-algebras, having inverse δ^{-1} defined by

$$\tilde{x} \otimes (a \otimes b) \otimes \tilde{y} \mapsto \bar{x} \cdot a \cdot \bar{y} \otimes \hat{x} \otimes b \otimes \hat{y}.$$

4.9. Let B be a Γ-graded interior $\mathcal{O}^\alpha K$-algebra, and consider the homomorphism of Γ-graded interior $\mathcal{O}^\alpha K$-algebras

$$d^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}\colon B \to \mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}B, \qquad b \mapsto 1\otimes b\otimes 1.$$

This map determines the *canonical embedding*

$$\tilde{d}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B)\colon B \to \mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}\mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}B, \qquad b \mapsto 1\otimes b\otimes 1.$$

4.10. Proposition. *Let $\tilde{g}\colon B \to \mathrm{Res}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}A$ be an embedding of Γ-graded interior $\mathcal{O}^\alpha K$-algebras, and assume that $1 \in \mathrm{Tr}^H_K(g(1))$, $(g(1)\cdot g(1)^x = 0$ for all $x \in H/K$, and that $g(1)$ centralizes A_1.*

Then there is a unique exomorphism $\tilde{f}\colon \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B) \to A$ such that $\tilde{g} = \mathrm{Res}^H_K(\tilde{f})\circ \tilde{d}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B)$.

Proof. If $\tilde{f}$ exists, we may take $f(\bar{x}\otimes b\otimes \bar{y}) = \bar{x}\cdot g(b)\cdot \bar{y}$ for any $x,y \in H$, $b \in B$. Conversely, let $f : \mathrm{Ind}^{\mathcal{O}^\alpha H}_{\mathcal{O}^\alpha K}(B) \to A$ be defined by this formula; as in [T, Proposition 16.6], we obtain that f is an isomorphism of interior $\mathcal{O}^\alpha H$-algebras, and since g is grade-preserving, f is grade-preserving too. Moreover, $\tilde{f}$ does not depend on the choice of g in $\tilde{g}$, since if $b \in (B_1^K)^*$, then $\mathrm{Tr}^H_K(g(b)) \in (A_1^H)^*$ and $(\bar{x}\cdot g(b)\cdot \bar{y})^{\mathrm{Tr}^H_K(g(b))} = \bar{x}\cdot g(b)\cdot y$ for all $x,y \in H$ and $b \in B$.

5. Generalized induction

We are now going to define the induction of graded interior algebras through an arbitrary group homomorphism $\phi\colon H \to H'$.

5.1. Consider the commutative diagram of groups

$$\begin{array}{ccccc}
K\cap N & \longrightarrow & K & \xrightarrow{\mu} & \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
N = \mathrm{Ker}\mu & \longrightarrow & H & \xrightarrow{\mu} & \Gamma \\
\downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\
N' = \mathrm{Ker}\mu' & \longrightarrow & H' & \xrightarrow{\mu'} & \Gamma'
\end{array}$$

where $K = \mathrm{Ker}(\phi\colon H \to H')$ and $\Lambda = \mathrm{Ker}(\phi\colon \Gamma \to \Gamma')$. Let $G = H/N \simeq \mu(H) \leq \Gamma$ and $G' = H'/N' \simeq \mu'(H') \leq \Gamma'$. Then $K/K \cap N \simeq KN/N$ is isomorphic to a subgroup of Λ, and $H/KN \simeq (H/N)/(KN/N)$ is isomorphic to a subgroup of Γ/Λ, so to a subgroup of Γ'.

Let further $\alpha' \in Z^2(H', \mathcal{O}^*)$ and $\alpha = res_\phi(\alpha) \in Z^2(H, \mathcal{O}^*)$. It follows that $res^H_K\alpha = 1$ and $\mathcal{O}^\alpha K = \mathcal{O}K$, and ϕ induces a homomorphism $\bar{\phi}\colon \mathcal{O}^\alpha H \to \mathcal{O}^{\alpha'}H'$ of $\mathcal{O}$-algebras with image $\bar{\phi}(\mathcal{O}^\alpha H) \simeq \mathcal{O}^{\alpha'}(H/K)$.

We can regard $\mathcal{O}^\alpha H$ and $\mathcal{O}^{\alpha'} H$ as H/KN-graded algebras in the usual way, and also as Γ'-graded algebras (where the components of degree g' not belonging to the image of H are trivial). Similarly, $\mathcal{O}^{\alpha'}H'$ can be regarded as a H'/N'-graded algebra, and also as a Γ'-graded algebra. Then $\bar{\phi}\colon \mathcal{O}^\alpha H \to \mathcal{O}^{\alpha'}H'$ is a homomorphism of Γ'-graded algebras.

5.2. Let (A, μ, ψ) be a Γ-graded interior $\mathcal{O}^\alpha H$-algebra, and as above, regard A as a Γ/Λ-graded algebra, and also as a Γ'-graded algebra. Then the structural map $\psi\colon \mathcal{O}^\alpha H \to A$ is a homomorphism of Γ'-graded algebras.

By [P4, 3.2], $(\mathcal{O} \otimes_{\mathcal{O}K} A)^K$ is an $\mathcal{O}$-algebra with multiplication $(1 \otimes a)(1 \otimes b) = 1 \otimes ab$. Also, ψ factorizes through $\mathcal{O}^{\alpha'}(H/K)$, so $(\mathcal{O} \otimes_{\mathcal{O}K} A)^K$ becomes an interior $\mathcal{O}^{\alpha'}(H/K)$-algebra.

If $g \in \Gamma$, define the grade of $a \otimes_{\mathcal{O}K} a \in \mathcal{O} \otimes_{\mathcal{O}K} A)^K$ to be $\phi(g) \in \Gamma'$. This is clearly well-defined, and $(\mathcal{O} \otimes_{\mathcal{O}K} A)^K$ is a Γ'-graded interior $\mathcal{O}^{\alpha'}(H/K)$-algebra.

5.3. Now, by definition, let

$$\mathrm{Ind}_\phi(A) = \mathrm{Ind}^{H'}_{\phi(H)}((\mathcal{O} \otimes_{\mathcal{O}K} A)^K) = \mathcal{O}^{\alpha'}H' \otimes_{\mathcal{O}^{\alpha'}(H/K)} (\mathcal{O} \otimes_{\mathcal{O}K} A)^K \otimes_{\mathcal{O}^{\alpha'}(H/K)} \mathcal{O}^{\alpha'}H'.$$

By the preceding section, $\mathrm{Ind}_\phi(A)$ is a Γ'-graded interior $\mathcal{O}^{\alpha'}H'$-algebra with multiplication

$$(\bar{x}' \otimes (1 \otimes a) \otimes \bar{s}')(\bar{y}' \otimes (1 \otimes b) \otimes \bar{t}') = \begin{cases} \bar{x} \otimes (1 \otimes a \cdot \bar{z} \cdot b) \otimes \bar{t}', & \text{if } s'y' = \phi(z) \\ 0 & \text{otherwise,} \end{cases}$$

where z is a suitable element of H. The structural maps are $\mu'\colon H' \to \Gamma'$ and $\psi'\colon \mathcal{O}^{\alpha'}H' \to \mathrm{Ind}\phi(A)$ (preserving Γ'-gradings) defined by

$$\psi'(\bar{x}') = \bar{x}' \cdot \mathrm{Tr}^{H'}_{\phi(H)}(1 \otimes (1 \otimes 1) \otimes 1) = \mathrm{Tr}^{H'}_{\phi(H)}(1 \otimes (1 \otimes 1) \otimes q) \cdot \bar{x}'.$$

5.4. For $\sigma \in \mathrm{Aut}(K)$, denote, as in [P4, 2.3], $N_A^\sigma(K) = \{a \in A \mid a \cdot \bar{x} = \overline{\sigma(x)} \cdot a\}$, and let

$$N_A(K) = \sum_{\sigma \in \mathrm{Aut}(K)} N_A^\sigma(K).$$

Then, recalling that $\mathcal{O}^\alpha K = \mathcal{O}K$ and $\phi(K) = 1$, it follows immediately that $N_A(K)$ inherits from A a structure of a Γ'-graded interior $\mathcal{O}^\alpha H$-algebra. Moreover, the map

$$d_\phi(A)\colon N_A(K) \to \mathrm{Res}_\phi(\mathrm{Ind}_\phi(A)), \quad a \mapsto 1 \otimes (a \otimes a) \otimes 1$$

is a homomorphism of Γ'-graded interior $\mathcal{O}^\alpha H$-algebra, and by [P4, 3.4.4], if $\mathcal{O} \otimes_{\mathcal{O}K} A$ is a projective $\mathcal{O}K$-module, it induces an embedding $N_A(K)/\mathrm{Ker}(d_\phi(A)) \to \mathrm{Res}_\phi(\mathrm{Ind}_\phi(A))$ of Γ'-graded interior $\mathcal{O}^\alpha H$-algebras.

Further, if $f\colon A \to B$ is a homomorphism of Γ-graded interior $\mathcal{O}^\alpha H$-algebras, then f induces a grade-preserving map $f\colon N_A(K) \to N_B(K)$ and $\mathrm{Ind}_\phi(f)\colon \mathrm{Ind}_\phi(A) \to \mathrm{Ind}_\phi(B)$ such that we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Ind}_\phi(A) & \xrightarrow{\mathrm{Ind}_\phi(F)} & \mathrm{Ind}_\phi(B) \\
{\scriptstyle d_\phi(A)}\Big\uparrow & & \Big\uparrow{\scriptstyle d_\phi(B)} \\
N_A(K) & \xrightarrow{\quad f \quad} & N_B(K)
\end{array}$$

It is not hard to see that Propositions 4.6, 4.7 and 4.8 can be generalized to this situation. In essence, one has to check that the maps defined in [P4, 3.7, 3.13 and 3.17] are grade-preserving. We shall only give here a common generalization of Proposition 4.6 and [P4, Proposition 3.7]

5.5. Let M be a G/N-graded $\mathcal{O}^\alpha H$-algebra, that is, there is an $\mathcal{O}^\alpha H$-module M_1 such that $M = \mathcal{O}^\alpha H \otimes_{\mathcal{O}^\alpha N} M_1$.We can regard M as a H/KN-graded $\mathcal{O}^\alpha H$-module, and also as a Γ'-graded $\mathcal{O}^\alpha H$-module. Then $\mathrm{Ind}_\phi(M) = \mathcal{O}^{\alpha'} H' \otimes_{\mathcal{O}^\alpha H} M$ is a Γ'-graded $\mathcal{O}^{\alpha'} H'$-module, where for $h' \in H'$, $g' \in |gamma'$ and $m \in M_{g'}$, the element $\bar{h}' \otimes m$ has, by definition, degree $\mu'(h')g'$.

5.6. Proposition. *If M is $\mathcal{O}$-free, then there is an isomorphism of Γ'-graded interior $\mathcal{O}^{\alpha'} H'$-algebras*

$$ind_{\phi,M}\colon \mathrm{Ind}_\phi(\mathrm{End}_{\mathcal{O}}(M)) \to \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_\phi(M)).$$

Proof. By [P4, 3.7], for $x', s' \in H'$ and $1 \otimes f \in (|CO \otimes_{OK} \mathrm{End}_{\mathcal{O}}(M))^K$, $ind_{\phi,M}(\bar{x} \otimes (1 \otimes f) \otimes \bar{s}')$ is, by definition, the $\mathcal{O}$-linear endomorphism of $\mathrm{Ind}_\phi(M)$ mapping $\bar{y}' \otimes m$ to zero or to $\bar{x}' \otimes f(\bar{z} \cdot m)$, whenever there is $z \in H$ such that $s'y' = \phi(z)$. It is straightforward to check that $ind_{\phi,M}$ is grade-preserving.

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