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DISTRIBUTIVE NONCOMMUTATIVE LATTICES OF TYPE (S)

G. LASLO AND GH.FĂRCAŞ

Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. The noncommutative lattices of type (S) were first defined in [1]. In this paper on introduces the notion of distributive noncomutative lattice of type (S) and one studies their properties as compared to those known from the classical lattices theory.

1. The triplet $(L, \wedge, \vee)$, where L is a nonvoid set, $\wedge$ and $\vee$ are two binary operations defined in L, is named noncommutative lattice of type (S), if, for all $a, b, c \in L$:

$$(A).\begin{cases} (a \wedge b) \wedge c = a \wedge (b \wedge c) \\ (a \vee b) \vee c = a \vee (b \vee c) \end{cases}$$

$$(B).\begin{cases} a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a \end{cases}$$

$$(S).\begin{cases} a \wedge (b \vee c) = a \wedge (c \vee b) \\ a \vee (b \wedge c) = a \vee (c \wedge b). \end{cases}$$

We observe that this system of laws is selfdual, so in $(L, \wedge, \vee)$ holds the duality principle.

This special class of noncommutative lattices was first defined in [1]. Then, in [1] and [2] are presented a few properties of this class of noncommutative lattices. From these properties, we mention the following:

(1.1). If $(L, \wedge, \vee)$ is a noncommutative lattice of type (S), then for all $a, b, c \in L$:

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$$(1).\begin{cases} a \wedge a = a \\ a \vee a = a \end{cases} \qquad (2).\begin{cases} a \wedge b = (a \wedge b) \vee (b \wedge a) \\ a \vee b = (a \vee b) \wedge (b \vee a) \end{cases}$$

$$(3).\begin{cases} a \wedge (b \vee a) = a \\ a \vee (b \wedge a) = a \end{cases} \qquad (4).\begin{cases} a \wedge (b \wedge c) = a \wedge (c \wedge b) \\ a \vee (b \vee c) = a \vee (c \vee b) \end{cases}$$

$$(5).\begin{cases} a \wedge b \wedge a = a \wedge b \\ a \vee b \vee a = a \vee b. \end{cases}$$

2. In this paper we introduce the notion of distributive noncommutative lattice of type (S) and we study its properties in analogy with the well known ones from the classical theory of lattices.

Let $(L, \wedge, \vee)$ a noncommutative lattice of type (S). If the identities below hold for all $a, b, c \in L$, let us accept the following notations:

$(D_r^{\wedge\vee})$. $\quad (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

$(D_r^{\vee\wedge})$. $\quad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$

$(D_l^{\wedge\vee})$. $\quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$(D_l^{\vee\wedge})$. $\quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(S_r). $\quad a \wedge c = b \wedge c$ and $a \vee c = b \vee c \Rightarrow a = b$

(S_{rl}). $\quad a \wedge c = b \wedge c$ and $c \vee a = c \vee b \Rightarrow a = b$

(S_{lr}). $\quad c \wedge a = c \wedge b$ and $a \vee c = b \vee c \Rightarrow a = b$

(S_l). $\quad c \wedge a = c \wedge b$ and $c \vee a = c \vee b \Rightarrow a = b.$

The noncommutative lattice of type (S), is named distributive if, for all $a, b, c \in L$, it verifies $(D) = \{(D_r^{\wedge\vee}), (D_r^{\vee\wedge}), (D_l^{\wedge\vee}), (D_l^{\vee\wedge})\}$.

We obtain an example of distributive noncommutative lattice of type (S), if we define in the cartesian product $P(M) \times P(M) = \{(A, B) \mid A \subseteq M,\ B \subseteq M\}$, the operations "$\wedge$" and "$\vee$" thus:

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1, B_1 \bigcap B_2)$$

$$(A_1, B_1) \vee (A_2, B_2) = (A_1, B_1 \bigcup B_2)$$

We observe that these operations are not commutative.

The noncommutative lattice of type (S) is named with simplifications if for any $a, b, c \in L$, it verifies the system of laws: $(S) = \{(S_r), (S_{rl}), (S_{lr}), (S_l)\}$.

All the theorems below refer to the noncommutative lattices of type (S). We will prove a few properties of distributive noncommutative lattices of type (S).

(2.1). $(D_r^{\wedge\vee}) \bigcap (D_l^{\wedge\vee}) \Rightarrow (D_l^{\vee\wedge})$ and $(D_r^{\vee\wedge}) \bigcap (D_l^{\vee\wedge}) \Rightarrow (D_l^{\wedge\vee})$

Proof. If for all $a, b, c \in L$ hold $(D_r^{\wedge\vee})$ and $(D_l^{\wedge\vee})$, then, using the properties from theorem (1.1) we obtain:

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [a \wedge (a \vee c)] \vee [b \wedge (a \vee c)] = a \vee [b \wedge (a \vee c)] = \\ &= a \vee [(b \wedge a) \vee (b \wedge c)] = [a \vee (b \wedge a)] \vee (b \wedge c)] = a \vee (b \wedge c),\end{aligned}$$

So $(D_l^{\vee\wedge})$ is true, namely $(D_r^{\wedge\vee}) \bigcap (D_l^{\wedge\vee}) \Rightarrow (D_l^{\vee\wedge})$.

The other implication from (2.1) is the dual of this first.

The following sentence is a result of the theorem (2.1).

(2.2). The noncommutative lattice of type (S) is distributive if and only if, it verifies the system of laws $\{(D_r^{\wedge\vee}), (D_r^{\vee\wedge}), (D_l^{\wedge\vee})\}$ or the system $\{(D_r^{\wedge\vee}), (D_r^{\vee\wedge}), (D_l^{\vee\wedge})\}$.

(2.3). $(D_r^{\wedge\vee}) \bigcap (D_l^{\wedge\vee}) \Leftrightarrow (D_r^{\vee\wedge}) \bigcap (D_l^{\vee\wedge})$

Proof. If for all $a, b, c \in L$, the rules $(D_r^{\wedge\vee})$ and $(D_l^{\wedge\vee})$ are true, then, using the definiton of noncommutative lattices of type (S) and the properties from (1.1) we obtain:

$$\begin{aligned}(a \vee c) \wedge (b \vee c) &= [a \wedge (b \vee c)] \vee [c \wedge (b \vee c)] = [a \wedge (b \vee c)] \vee c = \\ &= [(a \wedge b) \vee (a \wedge c)] \vee c = (a \wedge b) \vee [(a \wedge c) \vee c] = \\ &= (a \wedge b) \vee [c \vee (a \wedge c)] = (a \wedge b) \vee c,\end{aligned}$$

and respective

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [a \wedge (a \vee c)] \vee [b \wedge (a \vee c)] = a \vee [b \wedge (a \vee c)] = \\ &= a \vee [(b \wedge a) \vee (b \wedge c)] = [a \vee (b \wedge a)] \vee (b \wedge c) = \\ &= a \vee (b \wedge c),\end{aligned}$$

So, the equalities $(D_r^{\vee\wedge})$ and $(D_l^{\vee\wedge})$ are true, namely $(D_r^{\wedge\vee}) \bigcap (D_l^{\wedge\vee}) \Rightarrow (D_r^{\vee\wedge}) \bigcap (D_l^{\vee\wedge})$. The inverse one is the dual of the first.

An immediate result of this theorem is the following sentence:

(2.4). The noncommutative lattice of type (S) is distributive if and only if, it verifies the system of laws $\{(D_r^{\wedge\vee}), (D_l^{\wedge\vee})\}$ or the system $\{(D_r^{\vee\wedge}), (D_l^{\vee\wedge})\}$.

(2.5). $(D_r^{\wedge\vee}) \Rightarrow (D_l^{\vee\wedge})$ and $(D_r^{\vee\wedge}) \Rightarrow (D_l^{\wedge\vee})$

Proof. Obviously, it will be enough to prove the first implication, because the second is the dual of the first.

If for all $a, b, c \in L$, $(D_r^{\wedge\vee})$ is true, then, using the definition of noncommutative lattices of type (S) and the theorem (1.1), we obtain:

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= [a \wedge (a \vee c)] \vee [b \wedge (a \vee c)] = \\
&= a \vee [b \wedge (a \vee c)] = \\
&= a \vee [(a \vee c) \wedge b] = \\
&= a \vee [(a \wedge b) \vee (c \wedge b] = \\
&= [a \vee (a \wedge b)] \vee (c \wedge b) = \\
&= a \vee (c \wedge b) = \\
&= a \vee (b \wedge c),
\end{aligned}$$

So, the rule $(D_l^{\vee\wedge})$ is true. The second implication is the dual of the first.

An immediate result of this theorem is the following:

(2.6). The noncommutative lattice of type (S), is distributive, if and only if, it verifies the system of laws $\{(D_r^{\wedge\vee}), (D_r^{\vee\wedge})\}$.

We observe that theorem (2.1) can be considered a result of the theorem (2.5).

(2.7). $(D) \Rightarrow (S_r) \bigcap (S_{rl}) \bigcap (S_{lr})$

Proof. We suppose that, for every $a, b, x \in L$, the equalities: $a \wedge x =$ $= b \wedge x$, $a \vee x = b \vee x$ are true. Using the distributivity laws, we obtain:

$$\begin{aligned}
a &= a \wedge (a \vee x) = a \wedge (b \vee x) = (a \wedge b) \vee (a \wedge x) = \\
&= (a \wedge b) \vee (b \wedge x) = (a \wedge b) \vee (x \wedge b) = (a \vee x) \wedge b = (b \vee x) \wedge b = \\
&= (b \wedge b) \vee (x \wedge b) = b \vee (x \wedge b) = b,
\end{aligned}$$

so (S_r) is true, namely $(D) \Rightarrow (S_r)$.

Then, if we suppose that for $x, a, b \in L$ the equalities $a \wedge x = b \wedge x, x \vee a = x \vee b$ are true, then, using the distributivity laws, we obtain

$$\begin{aligned}
a &= a \wedge (x \vee a) = a \wedge (x \vee b) = (a \wedge x) \vee (a \wedge b) = (b \wedge x) \vee (a \wedge b) = \\
&= [b \vee (a \wedge b)] \wedge [x \vee (a \wedge b)] = b \wedge [x \vee (a \wedge b)] = b \wedge [(x \vee a) \wedge (x \vee b)] = \\
&= b \wedge [(x \vee b) \wedge (x \vee b)] = b \wedge (x \vee b) = b,
\end{aligned}$$

so (S_{rl}) is true, namely $(D) \Rightarrow (S_{rl})$.

The imolication $(D) \Rightarrow (S_{lr})$ is the dual of $(D) \Rightarrow (S_{rl})$.

(2.8). $(S_l) \Rightarrow (S_r) \bigcap (S_{rl}) \bigcap (S_{lr})$

Proof. We suppose that, for $x, a, b \in L$, the equalities $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$ are true.

Using the laws wich define the noncommutative lattices of type (S), and the property (5) of theorem (1.1) we obtain:

$$x \wedge a = x \wedge a \wedge x = x \wedge b \wedge x = x \wedge b$$

$$x \vee a = x \vee a \vee x = x \vee b \vee x = x \vee b,$$

Applying (S_l) we obtain that $a = b$, namely $(S_l) \Rightarrow (S_r)$.

If for $a, b, x \in L$, the equalities $a \wedge x = b \wedge x$, $x \vee a = x \vee b$ are true, then, $x \wedge a = x \wedge a \wedge x = x \wedge b \wedge x = x \wedge b$, and, by aplying (S_l) we have that $a = b$, namely $(S_l) \Rightarrow (S_{rl})$.

The implication $(S_l) \Rightarrow (S_{lr})$ is the dual of $(S_l) \Rightarrow (S_{rl})$

(2.9). $(S_{rl}) \bigcup (S_{lr}) \Rightarrow (S_r)$

Proof. If for $a, b, x \in L$, the equalities $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$ are true, then, $x \vee a = x \vee a \vee x = x \vee b \vee x = x \vee b$, so, aplying (S_{rl}) we obtain $a = b$. The implication $(S_{lr}) \Rightarrow (S_r)$ is the dual of the first.

(2.10). If in the noncommutative lattice of type (S), the rule (S_l) is true, then the two binary operations are commutative, namely $(L, \wedge, \vee)$ becomes lattice.

Proof. We suppose that in $(L, \wedge, \vee)$, the rule (S_l) is true. Then, using (3) and (4) from theorem (1.1), we obtain that for every $x, a, b \in L$:

$$x \wedge (a \wedge b) = x \wedge (b \wedge a) \quad \text{and} \quad x \vee (a \wedge b) = x \vee (b \wedge a).$$

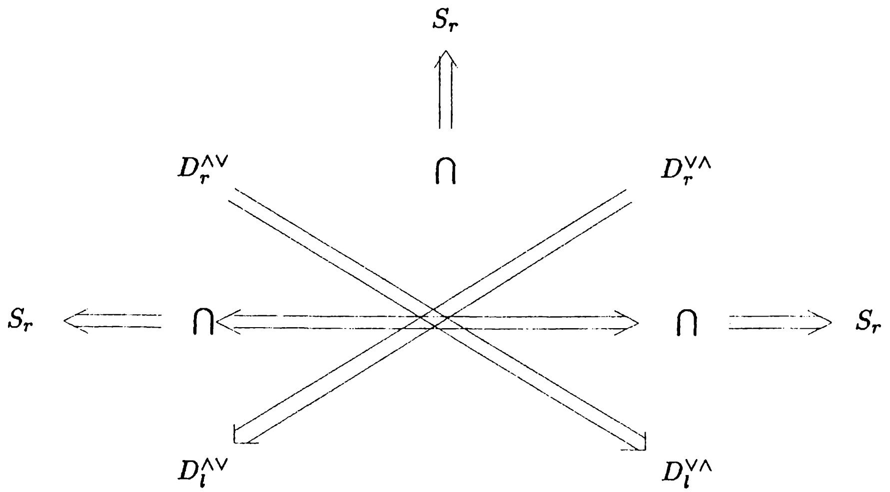
By applying (S_s), we obtain $a \wedge b = b \wedge a$ for all $a, b \in L$, namely the operation "$\wedge$" is commutative. The commutativity of "$\vee$" results analogously.

From this theorem results that in a distributive noncommutative lattice of type (S), the rules (S_s) are not necessarily true.

(2.10) also shows that any noncommutative lattice of type (S) with left simplifying is lattice.

We observe that the theorem (2.8) can be considered a result of theorem (2.10).

The main results from this paper can be represented by the following diagram:



It is known that, if $(L, \wedge\vee)$ is a lattice, then the following equivalences are true:

$(D) \Leftrightarrow (D_r^{\wedge\vee}) \Leftrightarrow (D_r^{\vee\wedge}) \Leftrightarrow (D_l^{\wedge\vee}) \Leftrightarrow (D_l^{\vee\wedge})$

$(S) \Leftrightarrow (S_d) \Leftrightarrow (S_{rl}) \Leftrightarrow (S_{lr}) \Leftrightarrow (S_s)$

$(D) \Leftrightarrow (S)$.

These equivalences are not true if $(L, \wedge\vee)$ is a noncommutative lattice of type (S), without being a lattice.

References

[1] Gh. Fărcaş, *Treillis non-commutatifs de type (S)*, Studia Univ. Babeş-Bolyai, Mathematica, t. XXXII, 2 (1987), 44-48.
[2] Gh. Fărcaş, *Sur les treillis non-commutatifs de type (S)*, Studia Univ. Babeş-Bolyai, Mathematica, t. XXXIII, 2 (1988), 24-27.
[3] G. Laslo and Gh. Fărcaş, *Distributive noncommutative lattices of type (G)*, Mathematica, Vol 38 (61), 1996.

TÂRGU-MUREŞ TECHNICAL UNIVERSITY, 4300 TÂRGU-MUREŞ, ROMANIA