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REMODELLING GIVEN BEZIER SPLINE CURVES AND SURFACES

IOAN GÂNSCĂ AND GHEORGHE COMAN, LEON ŢÂMBULEA

Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. Rational Bézier splines offer many possibilities to control the shapes of curves and surfaces, but their relative complex equations lead at rather complicated formulas for derivatives and, consequently, smoothness conditions. In this paper we present a manner of partial or total remodelling given polynomial Bézier spline curve (part 2) and surface (part 4), preserving their class of continuity. The method consists in performing degree elevations that depend by real parameters. The curvatures of Bézier spline curve in the initial, final and joint points are also studied in part 3. The theory is illustred by some figures with initial (as witnesses) and remodeled spline curves and surfaces, respectively.

1. Introduction

1.1. Let g be a given $C^r[u_0, u_L]$ polynomial Bézier spline curve of degree m, corresponding to the control points $b_i \in \Re^3, i = \overline{0, mL}$, with the breakpoints $b_{mI}, I = \overline{1, L-1}$ and the breakvalues of the parameter $u, u_k, k = \overline{1, L}, u_0 < u_1 < \ldots < u_L$. This spline curve is represented over an interval $[u_I, u_{I+1}], I = \overline{0, L-1}$ by the following equation

$$g(u) = \sum_{k=0}^{m} B_{m,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) b_{mI+k}, \qquad u \in [u_I, u_{I+1}], \tag{1}$$

where $B_{m,k}(t) = \binom{m}{k}(1-t)^{m-k}t^k, k = \overline{0, m}$, are the Bernstein polynomials.

The C^r conditions of g, on the jonction points $u_I, I = \overline{1, L-1}$, are

$$(\Delta_I^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_{mI-i+k} = (\Delta_{I-1}^u)^i \sum_{k=o}^{i} \binom{i}{k} (-1)^{i-k} b_{mI+k} \tag{2}$$

$i = \overline{0, r}$, where $\Delta_I^u = u_{I+1} - u_I$, [2], p.92.

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1.2. Consider a polynomial Bézier spline surface G of degrees m and n relative to the parameters u and v respectively, over the two-dimensional interval $D = [u_0, u_L]\mathrm{x}[v_0, v_M]$, having the control points $b_{ij}, i = \overline{0, mL}, j = \overline{0, nM}$, with breakpoints $b_{mI,nJ}, I = \overline{1, L-1}, J = \overline{1, M-1}$ and u_k and v_l, $k = \overline{1, L-1}$, $l = \overline{1, M-1}$ the breakvalues of the parameters.

On $D_{IJ} = [u_I, u_{I+1}]\mathrm{x}[v_J, v_{J+1}]$ the surface G has the equation

$$G(u,v) = \sum_{k=0}^{m}\sum_{l=0}^{n} B_{m,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n,l}\left(\frac{v-v_J}{v_{J+1}-v_J}\right) b_{mI+k,nJ+l} \tag{3}$$

Two patches of G corresponding to the domains $D_{I-1,J}$ and $D_{I,J}$ are r times continuously differentiable across their common curve $G(u_I, v), v \in [v_J, v_{J+1}]$ if the following conditions

$$(\Delta_I^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_{mI-i+k,nJ+l} = (\Delta_{I-1}^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_{mI+k,nJ+l}, \tag{4}$$

are fulfiled [2], p.272, for any $i = \overline{0, r}$, and $l = \overline{0, n}$.

Analogous, two patches of G, corresponding to the domains $D_{I,J-1}$ and $D_{I,J}$ are s times continuously differentiable, across their common curve

$$G(u, v_J), u \in [u_I, v_{I+1}]$$

if are fulfiled conditions

$$(\Delta_J^v)^j \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} b_{mI+k,nJ-j+l} = (\Delta_{J-1}^v)^j \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} b_{mI+k,nJ+l}, \tag{5}$$

for any $j = \overline{0, s}$ and $k = \overline{0, m}$.

The Bézier spline surface G is $C^{r,s}(D)$ if the conditions (4) and (5) are fulfiled for every $I = \overline{0, L-1}$ and $J = \overline{0, M-1}$.

2. **Remodelling a Given C^r Bézier Spline Curve**

Consider the polynomial Bézier spline curve g defined in part 1. We will remodel this curve by making a degree elevation corresponding to the following variable points, determined by a set of real parameters $\alpha_k^{(I)} \in (0,1), k = \overline{1, m},\ I = \overline{0, L-1}$,

$$b^*_{(m+1)I+k} = \begin{cases} b_{mI}, & \text{if } k = 0, \\ (1-\alpha_k^{(I)}) b_{mI+k-1} + \alpha_k^{(I)} b_{mI+k}, & \text{if } k = \overline{1, m}, \\ b_{m(I+1)}, & \text{if } k = m+1. \end{cases} \tag{6}$$

One observes that if

$$\alpha_k^{(I)} = \frac{m+1-k}{m+1},$$

then $b^*_{(m+1)I+k}$, $k = \overline{0, m+1}$, are the known Bézier points, which are used in the classical degree elevation, [2], p.52.

The Bézier spline curve g^* corresponding to the points (6), over the interval $[u_I, u_{I+1}]$, has the equation

$$g^*(u) = \sum_{k=0}^{m+1} B_{m+1,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) b^*_{(m+1)I+k}, \; u \in [u_I, u_{I+1}], \; I = \overline{0, L-1}. \quad (7)$$

The spline curve g^* will be $C^r[u_0, u_L]$ if, similar to (2), are fulfiled the conditions

$$(\Delta_I^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b^*_{(m+1)I-i+k} = (\Delta_{I-1}^u)^i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b^*_{(m+1)I+k} \quad (8)$$

for any $i = \overline{0, r}$ and $I = \overline{1, L-1}$.

From here, taking into account by (6) and that the conditions (2) hold, results that g^* is $C^r[u_0, u_L]$, $r \le [m/2]$ if and only if,

$$\begin{cases} \alpha_p^{(I)} = 1 - p\alpha_m^{(I-1)}, p = \overline{1, r}, I = \overline{1, L-1} \\ \\ \alpha_{m-p+1}^{(I)} = p\alpha_m^{(I)}, p = \overline{2, r}, I = \overline{0, L-2} \end{cases} \quad (9)$$

The other parameters $\alpha_p^{(0)}$, $p = \overline{1, m-r}$, $\alpha_p^{(L-1)}$, $p = \overline{r+1, m}$, $\alpha_p^{(I)}$, $p = \overline{r+1, m-r}$, $I = \overline{1, L-2}$ and $\alpha_m^{(I)}$, $I = \overline{0, L-2}$, from the open interval $(0,1)$, are arbitrary. Therefore, the Bézier spline curve $g^* \in C^r[u_0, u_L]$ has the following control points

$$b_k^* = \begin{cases} b_0, & \text{if } k = 0 \\ (1-\alpha_k^{(0)})b_{k-1} + \alpha_k^{(0)} b_k, & \text{if } k = \overline{1, m-r} \\ \left[1-(m+1-k)\alpha_m^{(0)}\right] b_{k-1} + & \\ \quad (m+1-k)\alpha_m^{(0)} b_k, & \text{if } k = \overline{m-r+1, m} \\ b_m, & \text{if } k = m+1 \end{cases} \quad (10)$$

$$b^*_{(m+1)I+k} = \begin{cases} b_{mI}, & \text{if } k = 0 \\ k\alpha_m^{(I-1)} b_{mI+k-1} + \\ \quad +(1 - k\alpha_m^{(I-1)}) b_{mI+k}, & \text{if } k = \overline{1,r} \\ (1 - \alpha_k^{(I)}) b_{mI+k-1} + \alpha_k^{(I)} b_{mI+k}, & \text{if } k = \overline{r+1, m-r} \\ \left[1 - (m+1-k)\alpha_m^{(I)}\right] b_{mI+k-1} + \\ \quad +(m+1-k)\alpha_m^{(I)} b_{mI+k}, & \text{if } k = \overline{m-r+1, m} \\ b_{m(I+1)}, & \text{if } k = m+1 \end{cases} \tag{11}$$

for any $I = \overline{1, L-2}$, and

$$b^*_{(L-1)(m+1)+k} = \begin{cases} b_{(L-1)m}, & \text{if } k = 0 \\ k\alpha_m^{(L-2)} b_{(L-1)m+k-1} + \\ \quad +(1 - k\alpha_m^{(L-2)}) b_{(L-1)m+k}, & \text{if } k = \overline{1,r} \\ (1 - \alpha_k^{(L-1)}) b_{(L-1)m+k-1} + \\ \quad \alpha_k^{(L-1)} b_{(L-1)m+k}, & \text{if } k = \overline{r+1, m} \\ b_{Lm}, & \text{if } k = m+1 \end{cases} \tag{12}$$

Example. Consider the quadratic spline Bézier curve of $C^1[0,4]$ corresponding to the following points $(m = 2, L = 5)$:

$$b_0(10,16),\quad b_1(7,21),\quad b_2(3,9),\quad b_3(0,0),\quad b_4(-4,0),\quad b_5(-8,0),$$
$$b_6(-6,3),\quad b_7(-4,6),\quad b_8(2,2),\quad b_9(8,-2),\quad b_{10}(10,3),$$

with the breakpoints $b_{2I}, I = \overline{1,4}$ and breakvalues of the parameter u, deduced with the chord length parametrization method: $u_0 = 0, u_1 = 1, u_2 = 7/4, u_3 = 5/2, u_4 = 13/4, u_5 = 4$. The dotted curves are Bézier spline g, as witness curves.

The parameter values corresponding to these figures are:

Fig.1: $\alpha_1^0 = 2/3$, $\alpha_2^0 = 1/20$, $\alpha_2^1 = 1/3$, $\alpha_2^2 = 1/3$, $\alpha_2^3 = 1/3$, $\alpha_2^4 = 1/10$;

Fig.2: $\alpha_1^0 = 1/10$, $\alpha_2^0 = 1/10$; $\alpha_2^1 = 1/3$, $\alpha_2^2 = 1/3$, $\alpha_2^3 = 1/100$, $\alpha_2^4 = 1/5$.

We remark that if $\alpha_2^{(I-1)}$, $I = \overline{1, L-1}$, decreases, then the Bézier spline curve one extends in the vicinity of joint point $b_{(m+1)I}$.

3. Curvature of Bézier Spline Curve g^*

Let $K(u)$ and $K^*(u)$ be the curvatures of the curves g and g^* respectively, with $r \geq 2$. Next we will deduce the dependence of $K^*(u)$ by $K(u)$ and the introduced

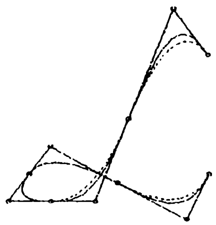

FIGURE 1

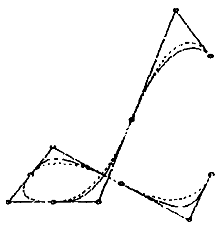

FIGURE 2

parameters, on $u = u_I, I = \overline{0, L}$. One knows that the formulas of $K(u_I)$ and $K^*(u_I)$ are

$$K(u_I) = \begin{cases} \dfrac{m-1}{m} \dfrac{\|\Delta b_{mI+1} \wedge \Delta b_{mI}\|}{\|\Delta b_{mI}\|^3}, & \text{for } I = \overline{0, L-1}, \\ \\ \dfrac{m-1}{m} \dfrac{\|\Delta b_{mL-2} \wedge \Delta b_{mL-1}\|}{\|\Delta b_{mL-1}\|^3}, & \text{for } I = L, \end{cases} \tag{13}$$

and

$$K^*(u_I) = \begin{cases} \dfrac{m}{m+1} \dfrac{\|\Delta b^*_{(m+1)I+1} \wedge \Delta b^*_{(m+1)I}\|}{\|\Delta b^*_{(m+1)I}\|^3}, & \text{for } I = \overline{0, L-1}, \\ \\ \dfrac{m}{m+1} \dfrac{\|\Delta b^*_{Im-1} \wedge \Delta b^*_{Im}\|}{\|\Delta b^*_{Im}\|^3}, & \text{for } I = L, \end{cases} \tag{14}$$

where "$\wedge$" denotes the vector product.

By direct calculus, taking into account by (10), (11), (12), and (13), the formula (14) becomes

$$\begin{aligned} K^*(u_0) = & \ \frac{m^2}{m^2-1} \frac{\alpha_2^{(0)}}{\left(\alpha_1^{(0)}\right)^2} K(u_0), \\ K^*(u_I) = & \ \frac{m^2}{m^2-1} \frac{1-2\alpha_m^{(I-1)}}{\left(1-\alpha_m^{(I-1)}\right)^2} K(u_I), I = \overline{1, L-1}, \\ K^*(u_L) = & \ \frac{m^2}{m^2-1} \frac{1-\alpha_{m-1}^{(L-1)}}{\left(1-\alpha_m^{(L-1)}\right)^2} K(u_L). \end{aligned} \tag{15}$$

In figure 3, the variation of $K^*(u_I)$, with respect to $\alpha_m^{(I-1)}, 0 < \alpha_m^{(I-1)} < 1/2$, $I = \overline{1, L-1}$, is shown.

Remarks: a) From (15) results that we have the possibility to control the curvature on any breakpoint $u_I, I = \overline{0, L}$.

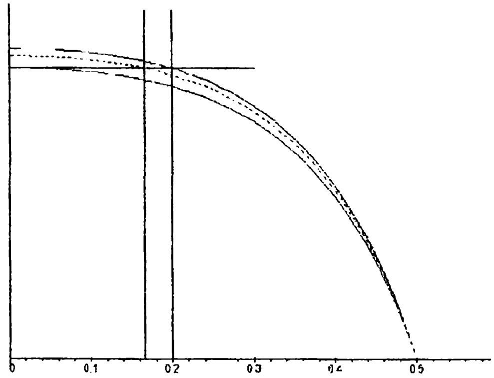

FIGURE 3

b) For any positive integer $m, 4 \le m < \infty$, and $0 \le \alpha_m^{(I-1)} \le 1/2, I = \overline{1, L-1}$, we have

$$\frac{1-2\alpha_m^{(I-1)}}{\left(1-\alpha_m^{(I-1)}\right)^2} K(u_I) \le K^*(u_I) \le \frac{16}{15} \frac{1-2\alpha_m^{(I-1)}}{\left(1-\alpha_m^{(I-1)}\right)^2} K(u_I), I = \overline{1, L-1}.$$

c) If $0 < \alpha_m^{(I-1)} \le \frac{1}{m+1}$, then $K^*(u_I) \ge K(u_I)$,
if $\frac{1}{m+1} \le \alpha_m^{(I-1)} \le \frac{1}{2}$, then $K^*(u_I) \le K(u_I)$,
if $\alpha_m^{(I-1)} = \frac{1}{m+1}$ results $K^*(u_I) = K(u_I)$,
for any $I = \overline{1, L-1}$, and any positive integer $m, 4 \le m < \infty$.
d) If $\alpha_1^{(0)} = \frac{m}{m+1}$, $\alpha_2^{(0)} = \frac{m-1}{m+1}$, then $K^*(u_0) = K(u_0)$,
and for $\alpha_{m-1}^{(L-1)} = \frac{2}{m+1}$, $\alpha_m^{(L-1)} = \frac{1}{m+1}$ results $K^*(u_L) = K(u_L)$.

4. Remodelling a Given $C^{r,s}$ Bézier Spline Surface

4.1. First we consider the particular Bézier spline surface which results from §1, part 1.2, for $M = 1$. Over any domain $D_{I,1} = [u_I, u_{I+1}]\text{x}[\text{v}_0, \text{v}_1]$, $\text{I} = \overline{0, \text{L}-1}$, the Bézier spline surface denoted G_1, has the equation

$$G_1(u,v) = \sum_{k=0}^{m} \sum_{l=0}^{n} B_{m,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n,l}\left(\frac{v-v_0}{v_1-v_0}\right) b_{mI+k,l}. \tag{16}$$

Assuming that $G_1(u,v)$ is $C^r[u_0, u_L]$ with respect to variable u (G_1 is evidently indefinite differentiable with respect to variable v) then the conditions (4), in the particular case $I = 0$, are fulfiled, for any $l = \overline{0, n}$.

We will remodel the above surface, preserving its class of smoothness, performing a degree elevation relative to the variable u, with the aid of following control points

$$b^*_{(m+1)I+k,l} = \begin{cases} b_{mI,l}, & \text{if } k = 0 \\ \left(1-\alpha^{(I)}_{k,l}\right) b_{mI+k-1,l} + \alpha^{(I)}_{k,l} b_{mI+k,l}, & \text{if } k = \overline{1,m} \\ b_{m(I+1),l}, & \text{if } k = m+1, \end{cases} \tag{17}$$

where the parameters $\alpha^{(I)}_{k,l} \in (0,1)$, $I = \overline{0,L-1}$, $l = \overline{0,n}$. We again remark that if $\alpha^{(I)}_{k,l} = \dfrac{m+1-k}{m+1}$, $l = \overline{0,n}$, $I = \overline{0,L-1}$, then (17) are the prints of the degree elevation with respect to variable u.

Bézier spline surface corresponding to these points over domain $D_{I,1}, I = \overline{0,L-1}$, has the equation

$$G^*_1(u,v) = \sum_{k=0}^{m+1}\sum_{l=0}^{n} B_{m+1,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n,l}\left(\frac{v-v_0}{v_1-v_0}\right) b^*_{(m+1)I+k,l}. \tag{18}$$

This surface will be C^r across the curve $G^*_1(u_I, v)$, $v \in [v_0, v_1]$ if, similar to (4) for $I = 0$, the following conditions are fulfiled

$$(\Delta^u_I)^i \sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k} b^*_{(m+1)I-i+k,l} = (\Delta^u_{I-1})^i \sum_{k=0}^{i}\binom{i}{k}(-1)^{i-k} b_{(m+1)I+k,l},$$

for any $i = \overline{0,r}$ and $l = \overline{0,n}$.

From here, taking into account by (17) and (4), with $I = 0$, one obtains, similar to (9) that

$$\begin{aligned} \alpha^{(I)}_{p,l} &= 1 - p\alpha^{(I-1)}_{m,l}; \quad p = \overline{1,r}, I = \overline{1,L-1}, \\ \alpha^{(I)}_{m-p+1,l} &= p\alpha^{(I)}_{m,l}; \quad p = \overline{2,r}, I = \overline{0,L-2}. \end{aligned} \tag{19}$$

The parameters $\alpha^{(0)}_{p,l}$, $p = \overline{1,m-r}$; $\alpha^{(L-1)}_{p,q}$, $p = \overline{r+1,m}$; $\alpha^{(I)}_{p,q}$, $p = \overline{r+1,m-r}$, $I = \overline{1,L-2}$, and $\alpha^{(I)}_{m,l}$, $I = \overline{0,L-2}$ take arbitrary values from the interval $(0,1)$, for any $l = \overline{0,n}$.

4.2. Now we consider the remodelling, in this manner, of a Bézier spline surface $G(u,v)$, given in part 1.2, preserving its class of $C^{r,s}(D)$. As in previous case, we make a degree elevation using the following control points, depending by two sets of parameters

$$b^*_{(m+1)I+k,(n+1)I+l} = \begin{cases} b_{m(I+\frac{k}{m+1}),n(I+\frac{l}{n+1})}, \\ \qquad \text{if } (k,l) \in \{0, m+1\} \text{x} \{0, n+1\}, \\ \left[1-\alpha_{k,l}^{(I,J)}, \alpha_{k,l}^{(I,J)}\right] \; A \; \left[1-\beta_{k,l}^{(I,J)}, \beta_{k,l}^{(I,J)}\right]^T, \\ \qquad \text{if } \begin{array}{l} k=\overline{1,m}, l=\overline{1,n}, \\ I=\overline{0,L-1}, J=\overline{0,M-1}, \end{array} \end{cases} \tag{20}$$

where A is the matrix

$$A = \begin{bmatrix} b_{mI+k-1,nJ+l-1} & b_{mI+k-1,nJ+l} \\ b_{mI+k,nJ+l-1} & b_{mI+k,nJ+l} \end{bmatrix}$$

The Bézier spline surface G^* corresponding to these control points has, on the domain D_{IJ}, the equation

$$G^*(u,v) = \sum_{k=0}^{m+1} \sum_{l=0}^{n+1} B_{m+1,k}\left(\frac{u-u_I}{u_{I+1}-u_I}\right) B_{n+1,l}\left(\frac{v-v_J}{v_{J+1}-v_J}\right) b^*_{(m+1)I+k,(n+1)J+l} \tag{21}$$

By direct calculus one deduces that G^* is $C^{r,s}(D)$, in hypothesis that G is $C^{r,s}(D)$, if and only if

$$\alpha_{k,n-j}^{(I,J-1)} = \alpha_{k,j}^{(I,J)}, I=\overline{0,L-1}, J=\overline{1,M-2}, k=\overline{0,m}, j=\overline{1,s},$$
$$\beta_{m-i,l}^{(I-1,J)} = \beta_{i,l}^{(I,J)}, I=\overline{1,L-2}, J=\overline{0,M-1}, l=\overline{0,n}, i=\overline{1,r}$$

and, similar to (19),

$$\begin{array}{l} \alpha_{m-p,l}^{(I,J)} = p\alpha_{m,l}^{(I,J)} = \alpha_m^{(I,J)}, p=\overline{2,r},\ I=\overline{0,L-2},\ J=\overline{0,M-1} \\ \alpha_{p,l}^{(I,J)} = 1-p\alpha_{m,l}^{(I-1,J)}, p=\overline{1,r},\ I=\overline{1,L-1},\ J=\overline{0,M-1} \\ \beta_{k,n-q}^{(I,J)} = q\beta_{k,n}^{(I,J)} = \beta_n^{(I,J)}, q=\overline{2,s},\ I=\overline{0,L-1},\ J=\overline{0,M-2} \\ \beta_{k,q}^{(I,J)} = 1-q\beta_{k,n}^{(I,J-1)} = \beta_n^{(I,J)}, q=\overline{1,s},\ I=\overline{0,L-1},\ J=\overline{1,M-1}. \end{array} \tag{22}$$

The other parameters, from the interval $(0,1)$,

$$\alpha_{p,l}^{(0,J)}, p=\overline{1,m-r}; \alpha_{p,l}^{(I,J)}, p=\overline{r+1,m-r};$$
$$\alpha_m^{(I,J)}, I=\overline{0,L-2}, J=\overline{0,M-1}; \alpha_{p,l}^{(L-1,J)}, p=\overline{m-r,m}$$

and

$$\beta_{k,q}^{(I,0)}, q = \overline{1, n-s};\ \beta_{k,q}^{(I,J)}, q = \overline{s+1, n-s};$$
$$\beta_{n}^{(I,J)}, I = \overline{0, L-1}, J = \overline{0, M-2};\ \beta_{k,q}^{(I,M-1)}, q = \overline{n-s, n},$$

are arbitrary.

Example. We consider the following Bézier control points:

		l=0	l=1	l=2	l=3
	k=0	(0,0,3)	(0,1,2)	(0,3,1)	(0,4,3)
$b_{2\cdot 0+k,l}$	k=1	(1,0,1)	(1,1,4)	(1,3,2)	(1,4,4)
	k=2	(2,0,2.5)	(2,1,3)	(2,3,1)	(2,4,3)
	k=0	(2,0,2.5)	(2,1,3)	(2,3,1)	(2,4,3)
$b_{2\cdot 1+k,l}$	k=1	(3,0,4)	(3,1,2)	(3,3,0)	(3,4,2)
	k=2	(4,0,2.5)	(4,1,3)	(4,3,1)	(4,4,2.5)
	k=0	(4,0,2.5)	(4,1,3)	(4,3,1)	(4,4,2.5)
$b_{2\cdot 2+k,l}$	k=1	(5,0,1)	(5,1,4)	(5,3,2)	(5,4,3)
	k=2	(6,0,3)	(6,1,0)	(6,3,1)	(6,4,2)

For $\alpha_{k,l}^{(I)} = \frac{1}{3}$, $I = \overline{0,2}$, $k = \overline{0,2}$, $l = \overline{0,3}$ one obtains the Bézier spline surface from Figure 4, corresponding to the usual degree elevation. The parameter values are: $u \in [0,3]$ and $v \in [0,1]$; in Figure 5 the parameter values are: $\alpha_{1,l}^{0} = 1/2$, $\alpha_{2,l}^{0} = 1/500$, $\alpha_{2,l}^{1} = 1/1000$, $\alpha_{2,l}^{2} = 4/5$; $l = \overline{0,3}$.

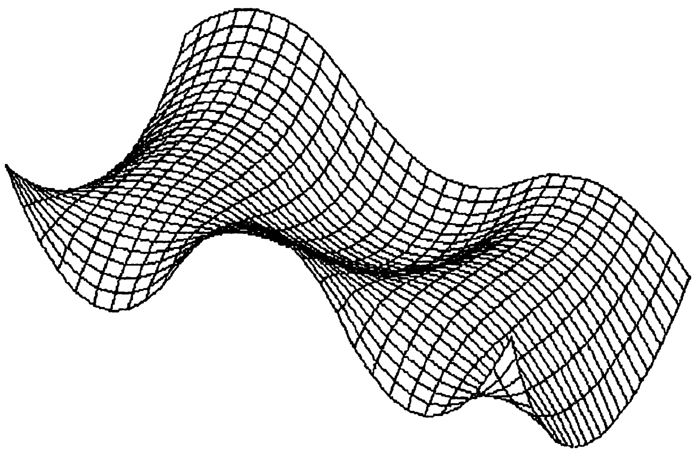

FIGURE 4. Surface 1

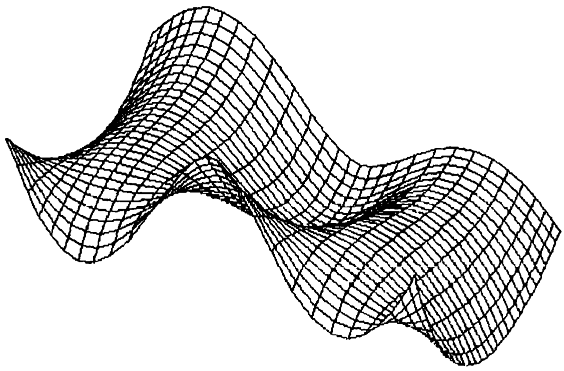

FIGURE 5. Surface 2

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DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY CLUJ-NAPOCA, ROMANIA

MATHEMATICS AND COMPUTER SCIENCE FACULTY, "BABES-BOLYAI" UNIVERSITY CLUJ-NAPOCA, ROMANIA