ON A CLASS OF MODULES WHOSE NON-ZERO ENDOMORPHISMS ARE MONOMORPHISMS

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Dedicated to Professor Ioan Purdea at his 60^{th} anniversary

Abstract. In this paper are established some results concerning a class of modules, denoted by $\mathcal{M}$, consisting of all non-zero R-modules with the property that every non-zero endomorphism of A is a monomorphism. If $A \in \mathcal{M}$, then A is indecomposable, $End_R(A)$ is a domain and $Ann_Ra = Ann_RA$ for every $0 \neq a \in A$. If R is commutative and $A \in \mathcal{M}$, it is shown that Ann_RA is a prime ideal of R, A is a torsion-free R/Ann_RA-module and if A is uniform then A is isomorphic to a submodule of $Ann_{E(R/Ann_RA)}(Ann_RA)$.

1. Introduction

In this paper we denote by R an associative ring with non-zero identity and all R-modules are left unital R-modules. The ring R will be considered as a left module over itself. By an homomorphism we understand an R-homomorphism.

Let A be an R-module. Then we denote by $E(A)$ an injective envelope of A and by $End_R(A)$ the ring of endomorphisms of A. If $0 \neq B \subseteq A$ and $0 \neq I \subseteq R$, we denote $Ann_RB = \{r \in R \mid rb = 0, \forall b \in B\}$ and $Ann_AI = \{a \in A \mid ra = 0, \forall r \in I\}$. If $0 \neq a \in A$, $Ann_R\{a\}$ is denoted by Ann_Ra. An R-module A is said to be faithful if $Ann_RA = 0$.

A submodule B of an R-module A is said to be essential in A if $B \cap Ra \neq 0$ for every $0 \neq a \in A$ ([2], Chapter 1, Definition 2.12.1). By $B \leq A$ we shall denote that B is a submodule of the R-module A and if A is an essential extension of B, this will be denoted by $B \trianglelefteq A$. A non-zero R-module A is said to be uniform in case each of its non-zero submodules is essential in A ([1], p.294).

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Let R be a domain and let A be an R-module. Then A is called divisible if $rA = A$ for every $0 \neq r \in R$ and A is called torsion-free if $ra \neq 0$ for every $0 \neq r \in R$ and $0 \neq a \in A$ ([4], p.32 and p.34).

An R-module A is said to be quasi-injective if for every $B \leq A$ each homomorphism $f : B \to A$ extends to an endomorphism of A ([3], p.333).

Throughout this paper we denote by $\mathcal{M}$ a class of non-zero R-modules which has the following property: a non-zero R-module A belongs to $\mathcal{M}$ if and only if every non-zero endomorphism $f \in End_R(A)$ is a monomorphism.

Remarks. *a*) For example, every simple R-module is contained in the class $\mathcal{M}$.

b) If A and B are two R-modules such that $A \in \mathcal{M}$ and $B \cong A$, then $B \in \mathcal{M}$.

2. Main results

Theorem 1. *Let $A \in \mathcal{M}$. Then:*

(i) A is indecomposable ;

(ii) $End_R(A)$ is a domain ;

(iii) A is a left torsion-free $End_R(A)$-module.

Proof. *(i)* Suppose that A is not indecomposable. Then there exist non-zero R-modules B and C such that $A = B \oplus C$. Define the homomorphisms $f : A \to B$ by $f(b, c) = b$ and $g : B \to A$ by $g(b) = (b, 0)$ for every $b \in B$ and $c \in C$. It follows that $0 \neq gf \in End_R(A)$ and gf is not a monomorphism, hence $A \notin \mathcal{M}$, which represents a contradiction.

(ii). Let $f, g \in End_R(A)$ non-zero endomorphisms. Then f and g are monomorphisms. Suppose that $fg = 0$. Then $f(g(a)) = 0$ for every $a \in A$. Since f is a monomorphism, we have $g(a) = 0$ for every $a \in A$, i.e. $g = 0$. This provides a contradiction.

(iii). It is well-known that A is an $End_R(A)$-module if we define $fa = f(a)$ for every $a \in A$ and $f \in End_R(A)$. Let $0 \neq f \in End_R(A)$ and $0 \neq a \in A$. Then $fa = f(a) \neq 0$, because f is a monomorphism. Hence A is a left torsion-free $End_R(A)$-module.

Lemma 2. *Let $A \in \mathcal{M}$ be a quasi-injective R-module and let $0 \neq B \leq A$. Then $B \in \mathcal{M}$.*

Proof. Denote by $i : B \to A$ the inclusion monomorphism and let $0 \neq f \in End_R(B)$. Since A is quasi-injective, there exists $h \in End_R(A)$ such that $hi = if$. It follows that $h \neq 0$ and thus h is a monomorphism. Therefore f is a monomorphism. Hence $B \in \mathcal{M}$.

In the sequel, we shall suppose that the ring R is commutative.

Theorem 3. *Let $A \in \mathcal{M}$. Then:*

(i) $Ann_R a = Ann_R A$ for every non-zero element $a \in A$;

(ii) $Ann_R A$ is a prime ideal of R ;

(iii) A is a torsion-free $R/Ann_R A$-module ;

(iv) If A is uniform, then A is isomorphic to a submodule of the module $Ann_{E(R/Ann_R A)}(Ann_R A)$.

Proof. *(i)* Let $r \in R$ such that $r \notin Ann_R A$ and let $0 \neq a \in A$. Then exists $b \in A$ such that $rb \neq 0$. We define the endomorphism $g : A \to A$ by $g(x) = rx$ for every $x \in A$. Since $g(b) = rb \neq 0$, it follows that g is a monomorphism. Therefore $g(a) = ra \neq 0$, i.e. $r \notin Ann_R A$. Hence $Ann_R a \subseteq Ann_R A$. Obviously we have $Ann_R A \subseteq Ann_R a$. Therefore $Ann_R A = Ann_r a$.

(ii). Let $r, s \in R$ such that $rs \in Ann_R A$ and let $0 \neq a \in A$. Then we have $Ann_R A = Ann_R a$. Suppose that $s \notin Ann_R A$. It follows that $sa \neq 0$. But $rs \in Ann_R A$, hence $rsa = 0$. Therefore $r \in Ann_R(sa) = Ann_R A$. Hence $Ann_R A$ is a prime ideal of R.

(iii). Since $Ann_R A$ is a prime ideal of R, $R/Ann_R A$ is an integral domain. Denote $\bar{r} = r + Ann_R A$ for every $r \in R$. Then A becomes an $R/Ann_R A$-module if we define $\bar{r}a = ra$ for every $r \in R$ and $a \in A$. Let $0 \neq a \in A$ and $0 \neq \bar{r} \in R/Ann_R A$. Suppose that $\bar{r}a = 0$. Then $ra = 0$, hence $r \in Ann_R a = Ann_R A$, i.e. $\bar{r} = 0$. This provides a contradiction. Therefore $\bar{r}a \neq 0$. Thus A is a torsion-free $R/Ann_R A$-module.

(iv). We have $A \trianglelefteq E(A)$. Denote $p = Ann_R A = Ann_R a$ for every $a \in A$. Then $Ra \cong R/p$ for every $a \in A$. Since A is uniform, it follows that $E(A) \cong E(Ra) \cong$ $\cong E(R/p)$ ([4], Chapter 2, Proposition 2.28). Hence A is isomorphic to a submodule of $Ann_{E(R/Ann_R A)}(Ann_R A)$.

Corollary 4. *Let $A \in \mathcal{M}$ be a faithful R-module. Then:*

(i) R is an integral domain ;

(ii) A is a torsion-free R-module ;

(iii) If A is uniform, then A is isomorphic to a submodule of $E(R)$.

Proof. By Theorem 3, $Ann_R A = 0$ is a prime ideal of R, hence R is an integral domain.

Theorem 5. *Let A be a non-zero R-module. Then the following statements are equivalent:*

(i) A is uniform and $A \in \mathcal{M}$;

(ii) $A \cong B$, where $0 \neq B \trianglelefteq Ann_{E(R/p)}p$ for a prime ideal p of R.

Proof. *(i)$\Longrightarrow$(ii).* Assume (i). Let $p = Ann_R A$, which is a prime ideal of R. Now the result follows by Theorem 3.

$(ii) \Longrightarrow (i)$. Assume (ii). For every $0 \neq a \in E(R/p)$ we have $Ann_R a \subseteq p$ ([4], Lemma 2.31). Hence $Ann_R a = p$ for every $0 \neq a \in Ann_{E(R/p)}p$. Therefore we have $Ann_R B = Ann_R a = p$ for every $0 \neq a \in B$. Since $E(R/p)$ is an indecomposable injective R-module, it follows that B is uniform ([4], Chapter 2, Proposition 2.28). Let $0 \neq f \in End_R(B)$. Then there exists $0 \neq a \in B$ such that $f(a) \neq 0$. Suppose that f is not a monomorphism. Then there exists $0 \neq b \in B$ such that $f(b) = 0$. Since B is uniform, there exist $r, s \in R$ such that $0 \neq ra = sb \in Ra \cap Rb$. Hence $rf(a) = f(ra) = f(sb) = sf(b) = 0$, i.e. $r \in Ann_R f(a) = p$. Therefore $ra = 0$, which is a contradiction. Thus f is a monomorphism. It follows that $B \in \mathcal{M}$, which means that A is uniform and $A \in \mathcal{M}$.

Corollary 6. *For every prime ideal p of R, $R/p \in \mathcal{M}$.*

Corollary 7. *Let $A \in \mathcal{M}$. Then $Ra \in \mathcal{M}$ for every $0 \neq a \in A$.*

Theorem 8. *Let $A \in \mathcal{M}$ be a faithful R-module which is not injective. Then there exists $0 \neq f \in End_R(A)$ which is not an isomorphism.*

Proof. Suppose that every non-zero $f \in End_R(A)$ is an isomorphism. By Corollary 4, R is an integral domain and A is a torsion-free R-module. It follows that R is isomorphic to a subring of the ring $End_R(A)$, hence $rA = A$ for every non-zero element $r \in R$, i.e. A is divisible. Then A is injective ([4], Chapter 2, Proposition 2.7). This provides a contradiction. Thus there exists $0 \neq f \in End_R(A)$ which is not an isomorphism.

Example 9. *Let R be an integral domain. Then R is uniform and the ideal $Ann_R(E(R)) = 0$ is a prime ideal of R. If A is a non-zero submodule of $E(R)$, then $A \in \mathcal{M}$. Hence $E(R) \in \mathcal{M}$. Since $E(R)$ is an indecomposable injective R-module, every non-zero endomorphism $f \in End_R(E(R))$ is an isomorphism ([4], Chapter 3, Lemma 3.10). If A is a non-zero proper submodule of $E(R)$, then A is not injective because $E(R)$ is indecomposable. By Theorem 8, there exists $0 \neq f \in End_R(A)$ which is not an isomorphism.*

Theorem 10. *Let $A \in \mathcal{M}$ be an injective R-module and denote $p = Ann_R A$. Then $A \cong E(R/p)$ and $End_R(A)$ is a division ring.*

Proof. Let $0 \neq a \in A$. By Theorem 3, $p = Ann_R A = Ann_R a$ is a prime ideal of R. But $aR \cong R/Ann_R a = R/p$, hence $E(R/p) \cong E(aR) \leq A$. By Theorem 1, A is indecomposable, hence $A \cong E(R/p)$. Let $0 \neq f \in End_R(A)$. Then f is a monomorphism. Since A is an indecomposable injective R-module, it follows that f is an isomorphism ([4], Chapter 3, Lemma 3.10). Therefore $End_R(A)$ is a division ring.

Example 11. *Let $R = \mathbb{Z}$ be the ring of integers and $\mathbb{Q}$ the additive group of rational numbers. Then $\mathbb{Q} \in \mathcal{M}$, $\mathbb{Q} = E(\mathbb{Z})$ and $End_{\mathbb{Z}}(\mathbb{Q}) \cong \mathbb{Q}$ is a field.*

Theorem 12. *Let $A \in \mathcal{M}$ be a quasi-injective R-module and denote $p = Ann_R A$. If $A \leq B \leq Ann_{E(A)} p$, then $B \in \mathcal{M}$.*

Proof. We have $Ann_R a = p$ for every $0 \neq a \in Ann_{E(A)} p$. Let $0 \neq f \in End_R(B)$. Then there exists $0 \neq b \in B$ such that $f(b) \neq 0$. Since $A \trianglelefteq B$, there exists $r \in R$ such that $0 \neq rb \in A \cap Rb$. Therefore $r \notin p$ and $f(rb) = rf(b) \neq 0$, i.e. $f|_A \neq 0$. But f extends to a $g \in End_R(E(A))$. Since A is quasi-injective, we have $g(A) \subseteq A$, hence $f(A) \subseteq A$ ([2], p.252). Let $h \in End_R(A)$ be defined by $h(a) = f(a)$ for every $a \in A$. Since $h(b) = f(b) \neq 0$, it follows that h is a monomorphism. Suppose now that f is not a monomorphism. Then there exists $0 \neq c \in B$ such that $f(c) = 0$. Also there exists $s \in R$ such that $0 \neq sc \in A \cap Rc$. We have $h(sc) = f(sc) = sf(c) = 0$, which is a contradiction. Therefore f is a monomorphism.

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