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Positive definite kernels on the set of integers, stability, some properties and applications

Arnaldo De La Barrera, Osmin Ferrer and José Sanabria

Abstract. We define and investigate a class of positive definite kernel so called equivalent-kernel. We formulate and prove an analogous of Paley-Wiener theorem in the context of positive definite kernel. The main ingredient in the proof is Kolmogorov decomposition. Finally, some applications to stochastic processes are given.

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Introduction

Positive definite kernels play a prominent role in some applications such as numerical solution of partial differential equations, machine learning, computer graphics, problem moment and probability theory. In the present work we explore some properties of positive definite kernels. For this kernels one obtains some similar results to equivalents bases in Banach spaces and Riesz bases in Hilbert spaces. An important tool to be used is a version of a classic result due to Kolmogorov, which will be called a Kolmogorov decomposition of the positive definite kernel K (see [3]). We will use Kolmogorov decomposition of a positive definite kernel to obtain a characterization results of equivalents kernels (see Theorem 3.3). This result is similar to a known result for equivalents bases, Riesz bases and stochastic processes. Using the above, one obtains an analogue Paley-Wiener Theorem (see [8]) in the context of positive definite kernels (see Theorem 3.4). Finally, some applications to stochastic processes are given.

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1. Paley-Wiener theorem

Orthonormal bases are very important in Hilbert space theory. There is another less known but also very useful type of bases: the Riesz bases. This section will be devoted to them. More about these bases can be found in Young's book [8].

Definition 1.1. A basis in a Hilbert space is a *Riesz basis* if it is equivalent to an orthonormal basis.

The fundamental criterium of stability, and historically the first one, is due to Paley and Wiener [7]. It is based on the known fact that a linear bounded operator T on a Banach space is invertible if

$$\|I - T\| < 1.$$

Theorem 1.2. *(Paley -Wiener) Let $\{x_n\}_{n\in\mathbb{N}}$ be a basis in the Banach space X, and suppose that $\{y_n\}_{n\in\mathbb{N}}$ is a sequence of elements of X such that*

$$\left\|\sum_{n=1}^{N} c_n(x_n - y_n)\right\| \leq \lambda \left\|\sum_{n=1}^{N} c_n x_n\right\|,$$

for all $N \in \mathbb{N}$, some constant λ, with $0 \leq \lambda < 1$ and for any sequence of scalars $\{c_n\}_{n\in\mathbb{N}}$. Then $\{y_n\}_{n\in\mathbb{N}}$ is a basis for X equivalent to $\{x_n\}_{n\in\mathbb{N}}$.

See [8, Theorem 10] for a proof.

2. Kolmogorov decomposition theorem

2.1. The Hilbert space associated to a positive definite operator valued kernel

Let $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ be a family of Hilbert spaces. An *operator valued kernel* on $\mathbb{Z}$ to $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is an application $K : \mathbb{Z}\times\mathbb{Z} \to \bigcup_{m,n\in\mathbb{Z}} \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n)$ such that

$$K(n,m) \in \mathcal{L}(\mathcal{H}_m, \mathcal{H}_n) \quad \text{for } n, m \in \mathbb{Z}.$$

In this section and the following one, unless it is otherwise stated, all the kernels will be operator valued ones.

A sequence $\{h_n\}$ in $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ is said to have *finite support* if $h_n = 0$ except for a finite number of integers n.

A kernel K on $\mathbb{Z}$ to $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a *positive definite kernel* if

$$\sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}_n} \geq 0,$$

for every sequence $\{h_n\}$ in $\oplus_{n\in\mathbb{Z}}\mathcal{H}_n$ with finite support.

Let K be a positive definite kernel. Let $\mathcal{F}$ be the linear space of elements $\bigoplus_{n\in\mathbb{Z}}\mathcal{H}_n$ and $\mathcal{F}_o$ be the space of elements in $\mathcal{F}$ with finite support.

Define $B_K : \mathcal{F}_o \times \mathcal{F}_o \to \mathbb{C}$ with

$$B_K(f,g) = \sum_{m,n\in\mathbb{Z}} \langle K(n,m)f_m, g_n\rangle_{\mathcal{H}_n}, \tag{2.1}$$

for $f, g \in \mathcal{F}_o$, $f = \{f_n\}$, $g = \{g_n\}$, $f_n, g_n \in \mathcal{H}_n$.

Note that B_K satisfies all the properties of an inner product, except for the fact that the set

$$\mathcal{N}_K = \{h \in \mathcal{F}_o : B_K(h,h) = 0\},$$

could be non-trivial.

According to the Cauchy-Schwarz inequality

$$\mathcal{N}_K = \{h \in \mathcal{F}_o : B_K(h,g) = 0, \text{ for all } g \in \mathcal{F}_o\},$$

hence $\mathcal{N}_K$ is a linear subspace of $\mathcal{F}_o$.

The quotient space $\mathcal{F}_o/\mathcal{N}_K$ is also a linear subspace. If $[h]$ stands for the class of the element h in $\mathcal{F}_o/\mathcal{N}_K$, then the application

$$\langle [h],[g]\rangle = B_K(h,g), \quad h,g \in \mathcal{F}_o,$$

is well defined. To prove that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{F}_o/\mathcal{N}_K$ is straightforward.

The completion of $\mathcal{F}_o/\mathcal{N}_K$ with respect to the norm induced by this inner product is a Hilbert space. It is known as *the Hilbert space associated to the positive definite kernel K* and it is denoted by $\mathcal{H}_K$. The inner product and the norm of $\mathcal{H}_K$ will be represented as $\langle\cdot,\cdot\rangle_{\mathcal{H}_K}$ and $\|\cdot\|_{\mathcal{H}_K}$ respectively. This norm will be named as *the norm induced by K*.

2.2. Kolmogorov Decomposition Theorem

The following theorem is a version of the classic result of Kolmogorov (see [5] for a historical review).

Theorem 2.1 (Kolmogorov). *Let K be a positive definite kernel. Then there exists a Hilbert space $\mathcal{H}_K$ and a map V defined on $\mathbb{Z}$ such that $V(n)$ belongs to $\mathcal{L}(\mathcal{H}_n,\mathcal{H}_K)$ for each $n\in\mathbb{Z}$ and*

(a) $K(n,m) = V^*(n)V(m)$ *if* $n,m\in\mathbb{Z}$.

(b) $\mathcal{H}_K = \bigvee_{n\in\mathbb{Z}} V(n)\mathcal{H}_n$.

(c) *The decomposition is unique in the following sense: if $\mathcal{H}'$ is another Hilbert space and V' defined on $\mathbb{Z}$ is an application such that $V'(n)\in\mathcal{L}(\mathcal{H}_n,\mathcal{H}_K)$ for each $n\in\mathbb{Z}$ that satisfies* (a) *and* (b), *then there exists a unitary operator $\Phi:\mathcal{H}_K\to\mathcal{H}'$ such that $\Phi V(n) = V'(n)$ for all $n\in\mathbb{Z}$.*

A proof of this theorem can be found in [3, Theorem 3.1].

An application V that satisfies the property (a) in Theorem 2.1 will be called *The Kolmogorov Decomposition of the Kernel K* or simply, a *Decomposition of the kernel K* (see [3]). The property (b) is referred to as the *minimality property* of Kolmogorov Decomposition. The meaning of property (c) is that, under the minimality condition (b), the Kolmogorov decomposition is essentially unique.

3. Some results for positive definite kernels

3.1. Equivalent definite positive kernels

Suppose the family of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ reduces to a single space, i.e. $\mathcal{H}_n = \mathcal{H}$ for all $n\in\mathbb{Z}$.

In this section some results given in [1] are extended to the case of kernel to operator valued.

Definition 3.1. Let $K_1, K_2 : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be two positive definite kernels.

It is said that K_1 and K_2 are *equivalent* if there exist two constants A, B with $0 < A \leq B$ such that

$$A\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}} \leq \|[h]_{K_2}\|^2_{\mathcal{H}_{K_2}} \leq B\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}},$$

for $h \in \mathcal{F}_o$.

Remark 3.2. Let $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be a positive definite kernel. Let $h \in \mathcal{F}_o$ and $\{h_n\}_{n\in\mathbb{Z}}$ a sequence in $\mathcal{H}$ with finite support.

By virtue of the definition of norm induced by the kernel K and Kolmogorov decomposition theorem it is obtained

$$\begin{aligned}\|[h]\|^2_{\mathcal{H}_K} &= \langle [h], [h] \rangle_{\mathcal{H}_K} = \sum_{n,m\in\mathbb{Z}} \langle K(n,m)h_m, h_n \rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_K(n)^* V_K(m) h_m, h_n \rangle_{\mathcal{H}} = \left\| \sum_{n\in\mathbb{Z}} V_K(n) h_n \right\|^2_{\mathcal{H}}.\end{aligned}$$

The following is one of our results.

Theorem 3.3. *Let $K_1, K_2 : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be two positive definite kernels. Then the following conditions are equivalent:*

(i) *The kernels K_1 y K_2 are equivalents.*

(ii) *There exists a linear bounded bijective application, with bounded inverse*

$$\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2},$$

such that

$$\Phi V_{K_1}(n) = V_{K_2}(n) \quad \textit{for all } n \in \mathbb{Z}.$$

(iii) *There exist two constants A, B with $0 < A \leq B$ such that*

$$\begin{aligned} A \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n \rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n \rangle_{\mathcal{H}} \\ &\leq B \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n \rangle_{\mathcal{H}}, \end{aligned}$$

for all sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$.

Proof. Let V_{K_1} and V_{K_2} be the Kolmogorov decomposition of the kernels K_1, K_2 and Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ the associated Hilbert spaces.
Remark 3.2 allows us to write condition (iii) in the following way: there exist two constants A and B with $0 < A \leq B$ such that

$$A\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}} \leq \|[h]_{K_2}\|^2_{\mathcal{H}_{K_2}} \leq B\|[h]_{K_1}\|^2_{\mathcal{H}_{K_1}},$$

for $h \in \mathcal{F}_o$.
Consequently the conditions (i) and (iii) are equivalents.

Next, suppose that condition (ii) is true. Since Φ is a linear bounded and invertible operator, then there exist two constants a_o, b_o with $0 < a_o \leq b_o$ such that

$$a_o\|f\|_{\mathcal{H}_{K_1}} \leq \|\Phi(f)\|_{\mathcal{H}_{K_2}} \leq b_o\|f\|_{\mathcal{H}_{K_1}},$$

for all $f \in \mathcal{H}_{K_1}$.
Let $f \in \mathcal{H}_{K_1}$ given by

$$f = \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
Then

$$a_o^2\left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}} \leq \left\|\sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n\right\|^2_{\mathcal{H}_{K_2}} \leq b_o^2\left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}}.$$

On the other hand, since K_1 and K_2 are positive definite kernels, by the Kolmogorov decomposition theorem we have

$$K_1(n,m) = V^*_{K_1}(n)V_{K_1}(m), \quad m,n \in \mathbb{Z}$$

and

$$K_2(n,m) = V^*_{K_2}(n)V_{K_2}(m), \quad m,n \in \mathbb{Z}.$$

Taking in to account the above expression we have that

$$\begin{aligned}\left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|^2_{\mathcal{H}_{K_1}} &= \left\langle \sum_{m\in\mathbb{Z}} V_{K_1}(m)h_m, \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\rangle_{\mathcal{H}_{K_1}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_{K_1}(n)^*V_{K_1}(m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}\,,\end{aligned}$$

similarly,

$$\left\|\sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n\right\|^2_{\mathcal{H}_{K_2}} = \sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}}\,.$$

Thus, choosing $A = a_o^2$ and $B = b_o^2$ we have

$$\begin{aligned}A\sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\leq B\sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}\,,\end{aligned}$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
Now, let us suppose that condition (iii) is valid.
The application $\Phi_o : \mathcal{F}_{o,K_1} \to \mathcal{F}_{o,K_2}$ is defined as follows

$$\Phi_o\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right) = \sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support. It is not hard to prove that Φ_o is a linear operator.
In what follows we will proof that Φ_o is a bounded above and bounded below operator. By the Kolmogorov decomposition theorem we obtain

$$\sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} = \sum_{m,n\in\mathbb{Z}} \langle V_{K_2}(n)^* V_{K_2}(m)h_m, h_n\rangle_{\mathcal{H}}\,.$$

Taking into account the above result and the way that the operator Φ_o was defined we arrive to the next result

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} &= \left\langle \sum_{m\in\mathbb{Z}} V_{K_2}(m)h_m, \sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n \right\rangle_{\mathcal{H}_{K_2}} \\ &= \left\| \sum_{n\in\mathbb{Z}} V_{K_2}(n)h_n \right\|^2_{\mathcal{H}_{K_2}} = \left\| \Phi_o\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right)\right\|^2_{\mathcal{H}_{K_2}}.\end{aligned}$$

In a similar way we have

$$\sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} = \left\| \sum_{n\in\mathbb{Z}} V_{K_1} h_n \right\|^2_{\mathcal{H}_{K_1}}.$$

By (iii),

$$A\left\| \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\|^2_{\mathcal{H}_{K_1}} \leq \left\| \Phi_o\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right)\right\|^2_{\mathcal{H}_{K_2}} \leq B\left\| \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\|^2_{\mathcal{H}_{K_1}}.$$

The last chain of inequalities shows us that Φ_o is a bounded above and bounded below operator. Even more the domain and the range of Φ_o are dense in the spaces $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ respectively. Then this operator can be extended to a bounded operator with bounded inverse say $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$. By construction

$$\Phi V_{K_1}(n) = V_{K_2}(n) \quad \text{for all} \quad n \in \mathbb{Z}. \qquad \square$$

Theorem 3.3 has similarities with results referring to equivalent basic sequences in Banach spaces, for more details on the topic (see [6, 2]).

Our next stability result for positive definite kernels is similar to a stability theorem for equivalent bases due to Paley-Wiener (see [8, Theorem 10]).

In first place we will fix the notation. Given two positive definite kernels $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ and $K_1 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$, let V_K and V_{K_1} the Kolmogorov decompositions of K and K_1 respectively and let $\mathcal{H}_K$ and $\mathcal{H}_{K_1}$ the induced Hilbert spaces.

Theorem 3.4. *Let $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ and $K_1 : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be two positive definite kernels. If $V_{K_1}(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$ and satisfies*

$$\left\| \sum_{n\in\mathbb{Z}} (V_K(n) - V_{K_1}(n))h_n \right\|_{\mathcal{H}_K} \leq \lambda \left\| \sum_{n\in\mathbb{Z}} V_K(n)h_n \right\|_{\mathcal{H}_K},$$

for any sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$, where $\lambda \in (0,1)$, then K_1 is equivalent to K.

Proof. Let us define the operator $T : \mathcal{H}_K \to \mathcal{H}_K$ as follows

$$T\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right) = \sum_{n\in\mathbb{Z}} (V_K(n) - V_{K_1}(n))h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
By hypothesis T is well defined and it is a linear operator. From the definition of T and by hypothesis we have.

$$\left\|T\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|^2_{\mathcal{H}_K} \leq \lambda^2 \left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|^2_{\mathcal{H}_K}.$$

Hence, T is a bounded operator and moreover

$$\|T\| \leq |\lambda| < 1.$$

Next, let us consider the operator $I - T : \mathcal{H}_K \to \mathcal{H}_K$, as usual $I : \mathcal{H}_K \to \mathcal{H}_K$ is the identity operator.
Since $\|T\| < 1$, by a well known functional analysis Theorem, $I - T$ is an invertible bounded linear operator. Moreover,

$$\begin{aligned}
(I-T)\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right) &= \sum_{n\in\mathbb{Z}} V_K(n)h_n - T\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\\
&= \sum_{n\in\mathbb{Z}} V_K(n)h_n - \left(\sum_{n\in\mathbb{Z}} (V_K(n) - V_{K_1}(n))h_n\right)\\
&= \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n.
\end{aligned}$$

From the above, it follows that there are positive constants m and M with $m \leq M$ such that

$$\begin{aligned}
m\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|_{\mathcal{H}_K} &\leq \left\|(I-T)\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|_{\mathcal{H}_K}\\
&= \left\|\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right\|_{\mathcal{H}_K}\\
&\leq M\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|_{\mathcal{H}_K}.
\end{aligned}$$

By Remark 3.2

$$\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|^2_{\mathcal{H}_K} = \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}}.$$

By hypothesis $V_{K_1}(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$, thus $V_{K_1}(n)h_n \in \mathcal{H}_K$. Then

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &= \sum_{m,n\in\mathbb{Z}} \langle V_{K_1}(n)^* V_{K_1}(m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_{K_1}(m)h_m, V_{K_1}(n)h_n\rangle_{\mathcal{H}_K} \\ &= \left\| \sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n \right\|^2_{\mathcal{H}_K} .\end{aligned}$$

Replacing these expressions in the above inequalities, we derive the existence of positive constants A and B with $A \leq B$ such that

$$\begin{aligned}A \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{m,n\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\leq B \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} ,\end{aligned}$$

for all sequences $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ with finite support.

Applying Theorem 3.3, it follows that K_1 is equivalent to K. □

4. Applications to stochastic processes

4.1. Multivariate stochastic processes

In this section it will be used the decomposition of the covariance Kernels of the stochastic processes (see [3], Section 1, Chapter 6).

Let (Ω, F, P) be a probability space, where F is a σ-algebra of subsets of Ω and P is a probability measure on F. A *stochastic variable* is a function $x : \Omega \to \mathbb{C}$, which is measurable with respect to the σ-algebra F. *A stochastic process* is a family $\{x_n\}_{n\in\mathbb{Z}}$ of stochastic variables. Let $L^2(P)$ be the Hilbert space of the measurable functions from F to Ω with integrable square, this is,

$$L^2(P) = \left\{ x : \Omega \to \mathbb{C} : x \text{ is a measurable function and } \int_\Omega |x(\omega)|^2 dP(\omega) < +\infty \right\}$$

equipped with the inner product

$$\langle x, y\rangle_{L^2(P)} = \int_\Omega x(\omega)\overline{y(\omega)} dP(\omega).$$

From here on, only stochastic processes with variables in $L^2(P)$ will be considered.

The mean-value variable is defined by

$$m_n = E(x_n) = \int_\Omega x_n(\omega) dP(\omega)$$

and it is convenient to assume that $m_n = 0$ for all $n \in \mathbb{Z}$. The correlation of the stochastic process $\{x_n\}_{n\in\mathbb{Z}}$ is given by

$$K(m,n) = K_{mn} = \int_\Omega x_n(\omega)\overline{x_m(\omega)} dP(\omega) = \langle x_n, x_m\rangle_{L^2(P)}.$$

for all $m, n \in \mathbb{Z}$.

It is straightforward that the correlation kernel of this process is a positive definite kernel. In fact

$$\begin{aligned}\sum_{i,j=m}^{n} K_{ij}\lambda_j\overline{\lambda}_i &= \sum_{i,j=m}^{n} \langle x_j, x_i\rangle_{L^2(P)}\lambda_j\overline{\lambda}_i \\ &= \sum_{i,j=m}^{n} \langle \lambda_j x_j, \lambda_i x_i\rangle_{L^2(P)} \\ &= \left\| \sum_{j=m}^{n} \lambda_j x_j \right\|^2_{L^2(P)} \geq 0,\end{aligned}$$

for all $m, n \in \mathbb{Z}, m \leq n$, and $\lambda_k \in \mathbb{C}$, where $k = m, m+1, ..., n$.

A stochastic process $\{x_n\}_{n\in\mathbb{Z}}$ is said to be *stationary (in a wide sense)* if its correlation kernel is a Toeplitz kernel, that is

$$K(m,n) = K_{n-m} \quad \text{for all} \quad m, n \in \mathbb{Z}.$$

In this case it can be used the Naimark Decomposition Theorem in order to associate the stationary stochastic process $\{x_n\}_{n\in\mathbb{Z}}$ with the Hilbert space $\mathcal{H}_K$, the unitary operator $S \in L(\mathcal{H}_K)$ and the operator $Q \in L(\mathbb{C}, \mathcal{H}_K)$ such that

$$K_n = Q^* S^n Q, \quad n \in \mathbb{Z}.$$

The geometric settings for the prediction problem can be extended in order to deal with the multivariate case too. Let notice that a random variable $x_n : \Omega \to \mathbb{C}$, of a stochastic process $\{x_n\}_{n\in\mathbb{Z}} \subset L^2(P)$, can be interpreted as an operator from $\mathbb{C}$ to $L^2(P)$ defining $\widetilde{x}_n : \mathbb{C} \to L^2(P)$ as

$$\widetilde{x}_n(\lambda) = \lambda x_n,$$

and the elements of the correlation kernel of the process can be calculated according to the rule

$$K(m,n) = (\widetilde{x}_m)^* \widetilde{x}_n.$$

Also, it must be noticed that many stochastic processes have the same correlation kernel. Having this in mind it is convenient to adopt the following terminology. The main object used to describe a *multivariate process* will be its correlation kernel K which is supposed to be positive definite and $K(m,n) \in \mathcal{L}(\mathcal{H}_n, \mathcal{H}_m)$ for all $m, n \in \mathbb{Z}$, where $\mathbf{H} = \{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ is a family of Hilbert spaces.

Definition 4.1. A pair $[\mathcal{K}, X]$, where $\mathcal{K}$ is a Hilbert space and $X = \{X_n\}_{n\in\mathbb{Z}}$ is a family of operators X_n in $\mathcal{L}(\mathcal{H}_n, \mathcal{K})$, is called a *geometric model of the multivariate process* with correlation kernel K, if

$$K(m,n) = X_m^* X_n.$$

The Kolmogorov Decomposition Theorem shows that given a positive definite kernel K, there exists a geometric model of the multivariate process with correlation kernel K. If $[\mathcal{K}, X]$ is the geometric model of the multivariate process with covariance kernel K then $\mathcal{H}_X$ will be the subspace of $\mathcal{K}$ generated for this model, that is,

$$\mathcal{H}_X = \bigvee_{n\in\mathbb{Z}} X_n \mathcal{H}_n. \tag{4.1}$$

If $[\mathcal{K}', X']$ is another geometric model of the same process, then the Kolmogorov Decomposition Theorem guarantees the existence of an unitary operator $\Phi : \mathcal{H}_X \to \mathcal{H}_{X'}$ such that $\Phi X_n = X'_n$ for all $n \in \mathbb{Z}$. This means that the geometry of the process is essentially determined by the choise of a geometric model such that

$$\mathcal{K} = \bigvee_{n\in\mathbb{Z}} X_n \mathcal{H}_n. \tag{4.2}$$

4.2. Equivalent multivariate stochastic processes

From here on, $\mathcal{H}_n = \mathcal{H}$ for all $n \in \mathbb{Z}$ and the covariance kernels of the processes will be positive definite.

Theorem 4.2 (Isomorphism). *Let $[\mathcal{W}, X]$ be the geometric model of a multivariate process and let $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be the kernel of covariance associated with the process. Then there exists an unit operator $\Phi : \mathcal{H}_K \to \mathcal{H}_X$ such that*

$$\Phi V_K(n) = X_n \quad \textit{for all} \quad n \in \mathbb{Z}.$$

Proof. Let $[\mathcal{W}, X]$, $X = \{X_n\}_{n\in\mathbb{Z}}$ be a geometric model of a multivariate process and $K : \mathbb{Z}\times\mathbb{Z} \to \mathcal{L}(\mathcal{H})$ be the kernel of covariance associated with the process.
It follows that the covariance kernel and the space generated by the process is given by

$$K(n,m) = X_n^* X_m \quad \text{and} \quad \mathcal{H}_X = \bigvee_{n\in\mathbb{Z}} X_n \mathcal{H}.$$

On the other hand, since K is a positive definite kernel one more time by the Kolmogorov decomposition theorem there exists a Hilbert space $\mathcal{H}_K$ and an application $V_K(n) \in \mathcal{L}(\mathcal{H}, \mathcal{H}_K)$ for all $n \in \mathbb{Z}$ such that

$$K(n,m) = V_K^*(n) V_K(m) \quad \text{and} \quad \mathcal{H}_K = \bigvee_{n\in\mathbb{Z}} V_K(n)\mathcal{H}.$$

Let us define the application $\Phi : \mathcal{H}_K \to \mathcal{H}_X$ in the following way

$$\Phi\left(\sum_{n\in\mathbb{Z}} V_K(n) h_n\right) = \sum_{n\in\mathbb{Z}} X_n h_n,$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence with finite support in $\mathcal{H}$.

Then we have

$$\begin{aligned}\left\|\Phi\left(\sum_{n\in\mathbb{Z}} V_K(n)h_n\right)\right\|^2_{\mathcal{H}_X} &= \left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} = \sum_{m,n\in\mathbb{Z}} \langle X_m h_m, X_n h_n\rangle_{\mathcal{H}_X} \\ &= \sum_{m,n\in\mathbb{Z}} \langle K(n,m)h_m, h_n\rangle_{\mathcal{H}} = \sum_{m,n\in\mathbb{Z}} \langle V_K^*(n)V_K(m)h_m, h_n\rangle_{\mathcal{H}} \\ &= \sum_{m,n\in\mathbb{Z}} \langle V_K(m)h_m, V_K(n)h_n\rangle_K = \left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|^2_{\mathcal{H}_K},\end{aligned}$$

all of this show us that the application Φ can be extended by continuity to an unit operator from $\mathcal{H}_K$ over $\mathcal{H}_X$ and moreover $\Phi V_K(n) = X_n$ for all $n \in \mathbb{Z}$. □

Definition 4.3. Two geometric models of multivariate processes $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are said to be *equivalent* , if $\dim(\mathcal{H}_X) = \dim(\mathcal{H}_Y)$ and there are two constants A, B with $0 < A \leq B$ such that

$$A\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} \leq \left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|^2_{\mathcal{H}_Y} \leq B\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X},$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.

By Theorem 4.2 and definitions we have the following.

Proposition 4.4. *Let $[\mathcal{W}, X]$ and $[\mathcal{W}_1, Y]$ be two geometric model of multivariate process and let K_1 and K_2 be two kernels of covariance associated with the processes. Then K_1 and K_2 are equivalent kernels if and only if $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent processes.*

As an application we give the proof of the results obtained in [4].

Theorem 4.5. *Let $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ be two geometric models of multivariate processes. The following conditions are equivalent:*

(i) *The models of the multivariate processes $[\mathcal{K}, X]$ and $[\mathcal{L}, Y]$ are equivalent.*

(ii) *There is a bijective bounded linear application with bounded inverse $\psi : \mathcal{H}_X \to \mathcal{H}_Y$ such that*

$$\psi X_n = Y_n \quad \textit{for all } n \in \mathbb{Z}.$$

(iii) *There exist two constants A, B with $0 < A \leq B$ such that*

$$A\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X} \leq \left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|^2_{\mathcal{H}_Y} \leq B\left\|\sum_{n\in\mathbb{Z}} X_n h_n\right\|^2_{\mathcal{H}_X},$$

for each sequence with finite support $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$.

Proof. The equivalence between (i) and (iii) follows by definition. Next, we are going to show that (i) implies (ii) to this end let us assume that $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent processes let K_1 and K_2 be the kernels of covariance associated with the processes $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ respectively. Since $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$ are equivalent, then by proposition 4.4 we concluded that K_1 and

K_2 are equivalent kernels. By Theorem 3.3, there exists a biyective bounded linear application linear with bounded inverse $\Phi : \mathcal{H}_{K_1} \to \mathcal{H}_{K_2}$ such that

$$\Phi V_{K_1}(n) = V_{K_2}(n) \quad \text{for all } n \in \mathbb{Z}.$$

Let us consider the operators $\phi_1 : \mathcal{H}_{K_1} \to \mathcal{H}_X$ such that

$$\phi_1 V_{K_1}(n) = X_n \quad \text{for all} \quad n \in \mathbb{Z}$$

and $\phi_2 : \mathcal{H}_{K_2} \to \mathcal{H}_Y$ such that

$$\phi_2 V_{K_2}(n) = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

From the above it follows that

$$\phi_2^{-1} \Phi \phi_1^{-1} X_n = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

Now suppose that (ii) holds then there is a bijective bounded linear application with bounded inverse $\psi : \mathcal{H}_X \to \mathcal{H}_Y$ such that

$$\psi X_n = Y_n \quad \text{for all } n \in \mathbb{Z}.$$

Let K_1 and K_2 be two kernels of covariance associated with the processes $X = \{X_n\}_{n\in\mathbb{Z}}$ and $Y = \{Y_n\}_{n\in\mathbb{Z}}$, respectively.
Let us consider the operators $\phi_1 : \mathcal{H}_{K_1} \to \mathcal{H}_X$ such that

$$\phi_1 V_{K_1}(n) = X_n \quad \text{for all} \quad n \in \mathbb{Z}$$

and $\phi_2 : \mathcal{H}_{K_2} \to \mathcal{H}_Y$ such that

$$\phi_2 V_{K_2}(n) = Y_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

From the above it follows that

$$\phi_2^{-1} \psi \phi_1 V_{K_1}(n) = V_{K_2}(n) \quad \text{for all} \quad n \in \mathbb{Z}.$$

By Theorem 3.3, we obtain $\dim(\mathcal{H}_{K_1}) = \dim(\mathcal{H}_{K_2})$ and there exist two positive constants A, B, $A \leq B$ such that

$$\begin{aligned} A \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}} &\leq \sum_{n,m\in\mathbb{Z}} \langle K_2(n,m)h_m, h_n\rangle_{\mathcal{H}} \\ &\leq B \sum_{n,m\in\mathbb{Z}} \langle K_1(n,m)h_m, h_n\rangle_{\mathcal{H}}, \end{aligned}$$

where $\{h_n\}_{n\in\mathbb{Z}}$ is a sequence in $\mathcal{H}$ with finite support.
The result comes up from the fact that $K_1(m,n) = X_m^* X_n$ and $K_2(m,n) = Y_m^* Y_n$. $\square$

In the multivariate stochastic processes setting it is possible to obtain a result similar to that of the theorem on stability (see Theorem 1.2).

The following is our result about stability of multivariate stochastic processes.

Theorem 4.6. *Let $[\mathcal{W}, Y]$ be a geometrical model of a multivariate stochastic process, $\mathcal{H}_Y$ the subspace generated by the process, and suppose $X_n \in \mathcal{L}(\mathcal{H}, \mathcal{H}_Y)$ for all $n \in \mathbb{Z}$ such that*

$$\left\| \sum_{n\in\mathbb{Z}} (Y_n - X_n)h_n \right\|_{\mathcal{H}_Y} \leq \delta \left\| \sum_{n\in\mathbb{Z}} Y_n h_n \right\|_{\mathcal{H}_Y}, \tag{4.3}$$

for some constant δ, $0 < \delta < 1$, and any sequence $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ with finite support. Then the geometric model of the multivariate process $[\mathcal{K}, X]$ is equivalent to $[\mathcal{W}, Y]$.

Proof. Let K and K_1 be two kernels of covariance associated with the processes $Y = \{Y_n\}_{n\in\mathbb{Z}}$ and $X = \{X_n\}_{n\in\mathbb{Z}}$, respectively.

Let us consider the operators $\Phi_1 : \mathcal{H}_K \to \mathcal{H}_Y$ such that

$$\Phi_1 V_K(n) = Y_n \quad \text{for all} \quad n \in \mathbb{Z}$$

and $\Phi_2 : \mathcal{H}_{K_1} \to \mathcal{H}_X$ such that

$$\Phi_2 V_{K_1}(n) = X_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

From the above and hypothesis we have

$$\mathcal{H}_{K_1} \subset \mathcal{H}_K \quad \text{and} \quad \Phi_2\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right) = \sum_{n\in\mathbb{Z}} X_n h_n = \Phi_1\left(\sum_{n\in\mathbb{Z}} V_{K_1}(n)h_n\right).$$

Then

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{Z}}(V_K(n) - V_{K_1}(n))h_n\right\|_{\mathcal{H}_K} &= \left\|\Phi_1\sum_{n\in\mathbb{Z}}(V_K(n) - V_{K_1}(n))h_n\right\|_{\mathcal{H}_Y} \\
&= \left\|\sum_{n\in\mathbb{Z}}(\Phi_1 V_K(n) - \Phi_2 V_{K_1}(n))h_n\right\|_{\mathcal{H}_Y} \\
&= \left\|\sum_{n\in\mathbb{Z}}(Y_n - X_n)h_n\right\|_{\mathcal{H}_Y} \\
&\leq \delta\left\|\sum_{n\in\mathbb{Z}} Y_n h_n\right\|_{\mathcal{H}_Y} = \delta\left\|\sum_{n\in\mathbb{Z}}\Phi_1 V_K(n)h_n\right\|_{\mathcal{H}_Y} \\
&= \delta\left\|\sum_{n\in\mathbb{Z}} V_K(n)h_n\right\|_{\mathcal{H}_K},
\end{aligned}$$

for any sequence $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ with finite support.

Finally, by Theorem 3.4 it follows that K_1 and K are equivalent kernels. Therefore $X = \{X_n\}_{n\in\mathbb{Z}}$ is equivalent to $Y = \{Y_n\}_{n\in\mathbb{Z}}$. □

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Arnaldo De La Barrera
Departamento de Matemáticas,
Universidad de Pamplona,
Pamplona, Colombia
e-mail: abarrera1994@gmail.com

Osmin Ferrer
Departamento de Matemáticas,
Universidad de Sucre,
Sincelejo, Colombia
e-mail: osmin.ferrer@unisucre.edu.co
https://orcid.org/0000-0003-1015-6271

José Sanabria
Departamento de Matemáticas,
Universidad de Sucre,
Sincelejo, Colombia
e-mail: jose.sanabria@unisucre.edu.co, jesanabri@gmail.com
https://orcid.org/0000-0002-9749-4099