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Self adjoint operator harmonic polynomials induced Chebyshev-Grüss inequalities

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Abstract. We present here very general self adjoint operator harmonic Chebyshev-Grüss inequalities with applications.

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1. Motivation

Here we mention the following inspiring and motivating result.

Theorem 1.1. (Čebyšev, 1882, [3]) *Let $f, g : [a, b] \to \mathbb{R}$ absolutely continuous functions. If $f', g' \in L_\infty([a, b])$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty . \tag{1.1}$$

Also we mention

Theorem 1.2. (Grüss, 1935, [9]) *Let f, g integrable functions from $[a, b]$ into $\mathbb{R}$, such that $m \le f(x) \le M$, $\rho \le g(x) \le \sigma$, for all $x \in [a, b]$, where $m, M, \rho, \sigma \in \mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{4}(M-m)(\sigma-\rho) . \tag{1.2}$$

Next we follow [1], pp. 132-152.

We make

Brief Assumption 1.3. Let $f:\prod_{i=1}^{m}[a_i,b_i]\to\mathbb{R}$ with $\frac{\partial^l f}{\partial x_i^l}$ for $l=0,1,...,n$; $i=1,...,m$, are continuous on $\prod_{i=1}^{m}[a_i,b_i]$.

Definition 1.4. We put

$$q(x_i,s_i)=\begin{cases} s_i-a_i, \text{ if } s_i\in[a_i,x_i], \\ s_i-b_i, \text{ if } s_i\in(x_i,b_i], \end{cases} \tag{1.3}$$

$x_i\in[a_i,b_i]$, $i=1,...,m$.

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n'=P_{n-1}$, $n\in\mathbb{N}$, $P_0=1$.

Let functions f_λ, $\lambda=1,...,r\in\mathbb{N}-\{1\}$, as in Brief Assumption 1.3, and $n_\lambda\in\mathbb{N}$ associated with f_λ.

We set

$$A_{i\lambda}(x_i,...,x_m):=\frac{n_\lambda^{i-1}}{\prod_{j=1}^{i-1}(b_j-a_j)}$$

$$\times\left[\sum_{k=1}^{n_\lambda-1}(-1)^{k+1}P_k(x_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^k f_\lambda(s_1,...,s_{i-1},x_i,...,x_m)}{\partial x_i^k}ds_1...ds_{i-1}\right. \tag{1.4}$$

$$+\sum_{k=1}^{n_\lambda-1}\frac{(-1)^k(n_\lambda-k)}{b_i-a_i}$$

$$\times\left[P_k(b_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^{k-1}f_\lambda(s_1,...,s_{i-1},b_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}}ds_1...ds_{i-1}\right.$$

$$\left.\left.-P_k(a_i)\int_{a_1}^{b_1}...\int_{a_{i-1}}^{b_{i-1}}\frac{\partial^{k-1}f_\lambda(s_1,...,s_{i-1},a_i,x_{i+1},...,x_m)}{\partial x_i^{k-1}}ds_1...ds_{i-1}\right]\right],$$

and

$$B_{i\lambda}(x_i,...,x_m):=\frac{n_\lambda^{i-1}(-1)^{n_\lambda+1}}{\prod_{j=1}^{i}(b_j-a_j)} \tag{1.5}$$

$$\times\left[\int_{a_1}^{b_1}...\int_{a_i}^{b_i}P_{n_\lambda-1}(s_i)q(x_i,s_i)\frac{\partial^{n_\lambda}f_\lambda(s_1,...,s_i,x_{i+1},...,x_m)}{\partial x_i^{n_\lambda}}ds_1...ds_i\right],$$

for all $i=1,...,m$; $\lambda=1,...,r$.

We also set

$$A_1:=\left(\frac{\left(\prod_{j=1}^{m}(b_j-a_j)\right)}{3}\right)\cdot\left[\sum_{\lambda=1}^{r}\left\{\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}\|f_\rho\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right)\right.\right. \tag{1.6}$$

$$\times\left(\sum_{i=1}^{m}\left[\left(b_i-a_i\right)n_\lambda^{i-1}\left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right)\Bigg\}\Bigg],$$

(let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$)

$$A_2:=\sum_{\lambda=1}^{r}\sum_{i=1}^{m}\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_p\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left\|B_{i\lambda}\right\|_{L_q\left(\prod_{j=i}^{m}[a_j,b_j]\right)}\left(\prod_{j=1}^{i-1}\left(b_j-a_j\right)\right)^{\frac{1}{q}},\qquad(1.7)$$

and

$$A_3:=\frac{1}{2}\left\{\sum_{\lambda=1}^{r}\left\{\left\|\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\right\|_{L_1\left(\prod_{j=1}^{m}[a_j,b_j]\right)}\left[\sum_{i=1}^{m}\left[\left(b_i-a_i\right)n_\lambda^{i-1}\right.\right.\right.\right.\qquad(1.8)$$

$$\left.\left.\left.\left.\times\ \left\|P_{n_\lambda-1}\right\|_{\infty,[a_i,b_i]}\left\|\frac{\partial^{n_\lambda}f_\lambda}{\partial x_i^{n_\lambda}}\right\|_{\infty,\prod_{j=1}^{m}[a_j,b_j]}\right]\right]\right\}\right\}.$$

We finally set

$$W:=r\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\rho=1}^{r}f_\rho\left(x\right)\right)dx\qquad(1.9)$$

$$-\frac{1}{\prod_{j=1}^{n}\left(b_j-a_j\right)}\sum_{\lambda=1}^{r}n_\lambda^{m}\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\left(x\right)\right)dx\right)\left(\int_{\prod_{j=1}^{m}[a_j,b_j]}f_\lambda\left(s\right)ds\right)$$

$$-\sum_{\lambda=1}^{r}\int_{\prod_{j=1}^{m}[a_j,b_j]}\left(\left(\prod_{\substack{\rho=1\\ \rho\neq\lambda}}^{r}f_\rho\left(x\right)\right)\left(\sum_{i=1}^{m}A_{i\lambda}\left(x_i,...,x_m\right)\right)\right)dx.$$

We mention

Theorem 1.5. ([1], p. 151-152) *It holds*

$$\left|W\right|\leq\min\left\{A_1,A_2,A_3\right\}.\qquad(1.10)$$

2. Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C\left(Sp\left(A\right)\right)$

of all continuous functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [8, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$.

Equivalently, we use (see [6], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle), \tag{2.1}$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

An important formula used a lot here is

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H. \tag{2.2}$$

As a symbol we can write

$$f(U) = \int_{m-0}^{M} f(\lambda)\, dE_\lambda. \tag{2.3}$$

Above,

$$m = \min\{\lambda | \lambda \in Sp(U)\} := \min Sp(U),\ M = \max\{\lambda | \lambda \in Sp(U)\} := \max Sp(U).$$

The projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_\lambda \le E_{\lambda'}$ for $\lambda \le \lambda'$;

(b) $E_{m-0} = 0_H$ (zero operator), $E_M = 1_H$ (identity operator) and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

Furthermore

$$E_\lambda := \varphi_\lambda(U), \ \forall \lambda \in \mathbb{R}, \tag{2.4}$$

is a projection which reduces U, with

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

The spectral family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [10], pp. 256-266, and for more detalis see there pp. 157-266. See also [5].

Some more basics are given (we follow [6], pp. 1-5):

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A = A^*$, iff $\langle Ax, x\rangle \in \mathbb{R}$, $\forall\, x \in H$, and if A is selfadjoint, then

$$\|A\| = \sup_{x\in H:\|x\|=1} |\langle Ax, x\rangle|. \tag{2.5}$$

Let A, B be selfadjoint operators on H. Then $A \le B$ iff $\langle Ax, x\rangle \le \langle Bx, x\rangle$, $\forall\, x \in H$.

In particular, A is called positive if $A \ge 0$.

Denote by

$$\mathcal{P} := \left\{\varphi(s) := \sum_{k=0}^{n} \alpha_k s^k | n \ge 0,\ \alpha_k \in \mathbb{C},\ 0 \le k \le n\right\}. \tag{2.6}$$

If $A \in \mathcal{B}(H)$ (the Banach algebra of all bounded linear operators defined on H, i.e. from H into itself) is selfadjoint, and $\varphi(s) \in \mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\| = \max\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{2.7}$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A := \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\}. \tag{2.8}$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [6], p. 4, Theorem 7).

Hence it holds

$$\begin{aligned} \||\varphi(A)|\| &= \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\} \\ &= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|, \end{aligned}$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|. \tag{2.9}$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda$$

$$= \int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}. \quad (2.10)$$

Let $A, B \in \mathcal{B}(H)$, then

$$\|AB\| \le \|A\| \|B\|, \quad (2.11)$$

by Banach algebra property.

3. Main results

Let $(P_n)_{n \in \mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $n \in \mathbb{N}$, $P_0 = 1$. Furthermore, let $[a, b] \subset \mathbb{R}$, $a \ne b$, and $h : [a, b] \to \mathbb{R}$ be such that $h^{(n-1)}$ is absolutely continuous function for some $n \in \mathbb{N}$.

We set

$$q(x,t) = \begin{cases} t-a, & \text{if } t \in [a,x], \\ t-b, & \text{if } t \in (x,b], \end{cases} \quad x \in [a,b]. \quad (3.1)$$

By [4], and [1], p. 133, we get the generalized Fink type representation formula

$$h(x) = \sum_{k=1}^{n-1} (-1)^{k+1} P_k(x) h^{(k)}(x)$$

$$+ \sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{b-a} \left[P_k(b) h^{(k-1)}(b) - P_k(a) h^{(k-1)}(a) \right] \quad (3.2)$$

$$+ \frac{n}{b-a} \int_a^b h(t) \, dt + \frac{(-1)^{n+1}}{b-a} \int_a^b P_{n-1}(t) q(x,t) h^{(n)}(t) \, dt,$$

$\forall \, x \in [a, b]$, $n \in \mathbb{N}$, when $n = 1$ the above sums are zero.

For the harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \in \mathbb{Z}_+$, (3.2) reduces to Fink formula, see [7].

Next we present very general harmonic Chebyshev-Grüss operator inequalities based on (3.2). Then we specialize them for $n = 1$.

We give

Theorem 3.1. *Let $n \in \mathbb{N}$ and $f, g \in C^n([a,b])$ with $[m, M] \subset (a,b)$, $m < M$. Here A is a selfadjoint linear bounded operator on the Hilbert space H with spectrum $Sp(A) \subseteq [m, M]$. We consider any $x \in H : \|x\| = 1$.*

Then

$$\langle (\Delta(f,g))(A)x, x\rangle := |\langle f(A)g(A)x, x\rangle - \langle f(A)x,x\rangle\langle g(A)x,x\rangle$$
$$-\frac{1}{2}\left[\sum_{k=1}^{n-1}(-1)^{k+1}\left\{\left[\left\langle P_k(A)\left(g(A)f^{(k)}(A)+f(A)g^{(k)}(A)\right)x,x\right\rangle\right]\right.\right. \tag{3.3}$$
$$-\left.\left.\left[\left\langle P_k(A)f^{(k)}(A)x,x\right\rangle\langle g(A)x,x\rangle+\left\langle P_k(A)g^{(k)}(A)x,x\right\rangle\langle f(A)x,x\rangle\right]\right\}\right]\Bigg|$$
$$\leq \frac{\left[\|g(A)\|\left\|f^{(n)}\right\|_{\infty,[m,M]}+\|f(A)\|\left\|g^{(n)}\right\|_{\infty,[m,M]}\right]}{2(M-m)}$$
$$\|P_{n-1}\|_{\infty,[m,M]}\left[\left\|(M1_H-A)^2\right\|+\left\|(A-m1_H)^2\right\|\right].$$

Proof. Here $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is the spectral family of A. Set

$$k(\lambda,t) := \begin{cases} t-m, & m \leq t \leq \lambda, \\ t-M, & \lambda < t \leq M. \end{cases} \tag{3.4}$$

where $\lambda \in [m, M]$.

Hence by (3.2) we obtain

$$f(\lambda) = \sum_{k=1}^{n-1}(-1)^{k+1}P_k(\lambda)f^{(k)}(\lambda) \tag{3.5}$$
$$+\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]$$
$$+\frac{n}{M-m}\int_m^M f(t)\,dt+\frac{(-1)^{n+1}}{M-m}\int_m^M P_{n-1}(t)k(\lambda,t)f^{(n)}(t)\,dt,$$

and

$$g(\lambda) = \sum_{k=1}^{n-1}(-1)^{k+1}P_k(\lambda)g^{(k)}(\lambda) \tag{3.6}$$
$$+\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)g^{(k-1)}(M)-P_k(m)g^{(k-1)}(m)\right]$$
$$+\frac{n}{M-m}\int_m^M g(t)\,dt+\frac{(-1)^{n+1}}{M-m}\int_m^M P_{n-1}(t)k(\lambda,t)g^{(n)}(t)\,dt,$$

$\forall\, \lambda \in [m, M]$.

By applying the spectral representation theorem on (3.5), (3.6), i.e. integrating against E_λ over $[m, M]$, see (2.3), (ii), we obtain:

$$f(A) = \sum_{k=1}^{n-1}(-1)^{k+1}P_k(A)f^{(k)}(A) \tag{3.7}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]\right)1_H$$

$$+\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)1_H+\frac{(-1)^{n+1}}{M-m}\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_\lambda,$$

and

$$g(A)=\sum_{k=1}^{n-1}(-1)^{k+1}P_k(A)\,g^{(k)}(A) \tag{3.8}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)g^{(k-1)}(M)-P_k(m)g^{(k-1)}(m)\right]\right)1_H$$

$$+\left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)1_H+\frac{(-1)^{n+1}}{M-m}\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_\lambda.$$

We notice that

$$g(A)\,f(A)=f(A)\,g(A) \tag{3.9}$$

to be used next.

Then it holds

$$g(A)\,f(A)=\sum_{k=1}^{n-1}(-1)^{k+1}g(A)\,P_k(A)\,f^{(k)}(A) \tag{3.10}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)f^{(k-1)}(M)-P_k(m)f^{(k-1)}(m)\right]\right)g(A)$$

$$+\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)g(A)$$

$$+\frac{(-1)^{n+1}}{M-m}g(A)\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_\lambda,$$

and

$$f(A)\,g(A)=\sum_{k=1}^{n-1}(-1)^{k+1}f(A)\,P_k(A)\,g^{(k)}(A) \tag{3.11}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)g^{(k-1)}(M)-P_k(m)g^{(k-1)}(m)\right]\right)f(A)$$

$$+\left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)f(A)$$

$$+\frac{(-1)^{n+1}}{M-m}f(A)\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_\lambda.$$

Here from now on we consider $x \in H : \|x\| = 1$; immediately we get

$$\int_{m-0}^{M} d\langle E_\lambda x, x\rangle = 1.$$

Then it holds (see (2.2))

$$\langle f(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) f^{(k)}(A)x, x\right\rangle \tag{3.12}$$
$$+\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) f^{(k-1)}(M) - P_k(m) f^{(k-1)}(m)\right]$$
$$+\frac{n}{M-m}\int_m^M f(t)\,dt + \frac{(-1)^{n+1}}{M-m}\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right) d\langle E_\lambda x, x\rangle,$$

and

$$\langle g(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) g^{(k)}(A)x, x\right\rangle \tag{3.13}$$
$$+\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) g^{(k-1)}(M) - P_k(m) g^{(k-1)}(m)\right]$$
$$+\frac{n}{M-m}\int_m^M g(t)\,dt + \frac{(-1)^{n+1}}{M-m}\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right) d\langle E_\lambda x, x\rangle.$$

Then we get

$$\langle f(A)x, x\rangle \langle g(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) f^{(k)}(A)x, x\right\rangle \langle g(A)x, x\rangle \tag{3.14}$$
$$+\left(\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) f^{(k-1)}(M) - P_k(m) f^{(k-1)}(m)\right]\right)\langle g(A)x, x\rangle$$
$$+\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)\langle g(A)x, x\rangle$$
$$+\frac{(-1)^{n+1}\langle g(A)x, x\rangle}{M-m}\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right) d\langle E_\lambda x, x\rangle,$$

and

$$\langle g(A)x, x\rangle \langle f(A)x, x\rangle = \sum_{k=1}^{n-1} (-1)^{k+1} \left\langle P_k(A) g^{(k)}(A)x, x\right\rangle \langle f(A)x, x\rangle \tag{3.15}$$
$$+\left(\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) g^{(k-1)}(M) - P_k(m) g^{(k-1)}(m)\right]\right)\langle f(A)x, x\rangle$$
$$+\left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)\langle f(A)x, x\rangle$$

$$+\frac{(-1)^{n+1}\langle f(A)x,x\rangle}{M-m}\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle .$$

Furthermore we obtain

$$\langle f(A)\,g(A)\,x,x\rangle \overset{(3.10)}{=} \sum_{k=1}^{n-1}(-1)^{k+1}\left\langle g(A)\,P_k(A)\,f^{(k)}(A)\,x,x\right\rangle \tag{3.16}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)\,f^{(k-1)}(M)-P_k(m)\,f^{(k-1)}(m)\right]\right)\langle g(A)\,x,x\rangle$$

$$+\left(\frac{n}{M-m}\int_m^M f(t)\,dt\right)\langle g(A)\,x,x\rangle$$

$$+\frac{(-1)^{n+1}}{M-m}\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle,$$

and

$$\langle f(A)\,g(A)\,x,x\rangle \overset{(3.11)}{=} \sum_{k=1}^{n-1}(-1)^{k+1}\left\langle f(A)\,P_k(A)\,g^{(k)}(A)\,x,x\right\rangle \tag{3.17}$$

$$+\left(\sum_{k=1}^{n-1}\frac{(-1)^k(n-k)}{M-m}\left[P_k(M)\,g^{(k-1)}(M)-P_k(m)\,g^{(k-1)}(m)\right]\right)\langle f(A)\,x,x\rangle$$

$$+\left(\frac{n}{M-m}\int_m^M g(t)\,dt\right)\langle f(A)\,x,x\rangle$$

$$+\frac{(-1)^{n+1}}{M-m}\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle.$$

By (3.14) and (3.16) we obtain

$$E := \langle f(A)\,g(A)\,x,x\rangle-\langle f(A)\,x,x\rangle\langle g(A)\,x,x\rangle \tag{3.18}$$

$$=\sum_{k=1}^{n-1}(-1)^{k+1}\left[\left\langle g(A)\,P_k(A)\,f^{(k)}(A)\,x,x\right\rangle-\left\langle P_k(A)\,f^{(k)}(A)\,x,x\right\rangle\langle g(A)\,x,x\rangle\right]$$

$$+\frac{(-1)^{n+1}}{M-m}\left[\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.$$

$$\left.-\langle g(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right],$$

and by (3.15) and (3.17) we derive

$$E := \langle f(A)\,g(A)\,x,x\rangle-\langle f(A)\,x,x\rangle\langle g(A)\,x,x\rangle \tag{3.19}$$

$$=\sum_{k=1}^{n-1}(-1)^{k+1}\left[\left\langle f(A)\,P_k(A)\,g^{(k)}(A)\,x,x\right\rangle-\left\langle P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\langle f(A)\,x,x\rangle\right]$$

$$+\frac{(-1)^{n+1}}{M-m}\left[\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.$$
$$\left.=\langle f(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right].$$

Consequently, we get that

$$2E=\sum_{k=1}^{n-1}(-1)^{k+1}\left\{\left[\left\langle g(A)\,P_k(A)\,f^{(k)}(A)\,x,x\right\rangle+\left\langle f(A)\,P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\right]\right.$$
$$\left.-\left[\left\langle P_k(A)\,f^{(k)}(A)\,x,x\right\rangle\langle g(A)\,x,x\rangle+\left\langle P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\langle f(A)\,x,x\rangle\right]\right\}$$
$$+\frac{(-1)^{n+1}}{M-m}\left\{\left[\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.\right.$$
$$\left.+\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right]$$
$$-\left[\langle g(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right.$$
$$\left.\left.+\langle f(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right]\right\}. \tag{3.20}$$

We find that

$$\langle f(A)\,g(A)\,x,x\rangle-\langle f(A)\,x,x\rangle\langle g(A)\,x,x\rangle$$
$$-\frac{1}{2}\left[\sum_{k=1}^{n-1}(-1)^{k+1}\left\{\left[\left\langle P_k(A)\left(g(A)\,f^{(k)}(A)+f(A)\,g^{(k)}(A)\right)x,x\right\rangle\right]\right.\right.$$
$$\left.\left.-\left[\left\langle P_k(A)\,f^{(k)}(A)\,x,x\right\rangle\langle g(A)\,x,x\rangle+\left\langle P_k(A)\,g^{(k)}(A)\,x,x\right\rangle\langle f(A)\,x,x\rangle\right]\right\}\right]$$
$$=\frac{(-1)^{n+1}}{2(M-m)}\left\{\left[\left\langle\left(g(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right.\right.$$
$$\left.+\left\langle\left(f(A)\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_{\lambda}\right)x,x\right\rangle\right]$$
$$-\left[\langle g(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right.$$
$$\left.\left.+\langle f(A)\,x,x\rangle\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)d\langle E_{\lambda}x,x\rangle\right]\right\}=:R. \tag{3.21}$$

Therefore it holds

$$|R|\le\frac{1}{2(M-m)}\left\{\left[\|g(A)\|\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_{\lambda}\right\|\right.\right.$$

$$+ \|f(A)\| \left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) g^{(n)}(t) dt \right) dE_\lambda \right\| \Bigg] \tag{3.22}$$

$$+ \Bigg[\|g(A)\| \left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) dE_\lambda \right\|$$

$$+ \|f(A)\| \left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) g^{(n)}(t) dt \right) dE_\lambda \right\| \Bigg] \Bigg\}$$

$$= \frac{1}{(M-m)} \Bigg\{ \|g(A)\| \left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) dE_\lambda \right\|$$

$$+ \|f(A)\| \left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) g^{(n)}(t) dt \right) dE_\lambda \right\| \Bigg\} =: (\xi_1). \tag{3.23}$$

We notice the following:

$$\left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) dE_\lambda \right\|$$

$$= \sup_{x \in H: \|x\|=1} \left| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) d\langle E_\lambda x, x\rangle \right|$$

$$\leq \sup_{x \in H: \|x\|=1} \left(\int_{m-0}^{M} \left(\int_{m}^{M} |P_{n-1}(t)| |k(\lambda,t)| \left| f^{(n)}(t) \right| dt \right) d\langle E_\lambda x, x\rangle \right) \tag{3.24}$$

$$\leq \left(\|P_{n-1}\|_{\infty,[m,M]} \left\| f^{(n)} \right\|_{\infty,[m,M]} \right)$$

$$\sup_{x \in H: \|x\|=1} \left(\int_{m-0}^{M} \left(\int_{m}^{M} |k(\lambda,t)| dt \right) d\langle E_\lambda x, x\rangle \right) =: (\xi_2).$$

(Notice that

$$\int_{m}^{M} |k(\lambda,t)| dt = \int_{m}^{\lambda} (t-m) dt + \int_{\lambda}^{M} (M-t) dt = \frac{(\lambda-m)^2 + (M-\lambda)^2}{2}.) \tag{3.25}$$

Hence it holds

$$(\xi_2) \overset{(3.25)}{=} \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\| f^{(n)} \right\|_{\infty,[m,M]}}{2} \right)$$

$$\times \sup_{x \in H: \|x\|=1} \left[\left\langle (M1_H - A)^2 x, x \right\rangle + \left\langle (A - m1_H)^2 x, x \right\rangle \right]$$

$$\leq \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\| f^{(n)} \right\|_{\infty,[m,M]}}{2} \right) \left[\left\| (M1_H - A)^2 \right\| + \left\| (A - m1_H)^2 \right\| \right]. \tag{3.26}$$

We have proved that

$$\left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) k(\lambda,t) f^{(n)}(t) dt \right) dE_\lambda \right\| \tag{3.27}$$

$$\leq \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|f^{(n)}\right\|_{\infty,[m,M]}}{2} \right) \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right].$$

Similarly, it holds

$$\left\| \int_{m-0}^{M} \left(\int_{m}^{M} P_{n-1}(t) \, k(\lambda, t) \, g^{(n)}(t) \, dt \right) dE_\lambda \right\|$$

$$\leq \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|g^{(n)}\right\|_{\infty,[m,M]}}{2} \right) \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right]. \quad (3.28)$$

Next we apply (3.27), (3.28) into (3.23), we get

$$(\xi_1) \leq \frac{1}{(M-m)} \left\{ \|g(A)\| \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|f^{(n)}\right\|_{\infty,[m,M]}}{2} \right. \right. \quad (3.29)$$

$$\left. \times \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right] \right) + \|f(A)\|$$

$$\left. \times \left(\frac{\|P_{n-1}\|_{\infty,[m,M]} \left\|g^{(n)}\right\|_{\infty,[m,M]}}{2} \right) \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right] \right\}$$

$$= \frac{1}{2(M-m)} \left\{ \left[\|g(A)\| \left\|f^{(n)}\right\|_{\infty,[m,M]} + \|f(A)\| \left\|g^{(n)}\right\|_{\infty,[m,M]} \right] \right.$$

$$\left. \|P_{n-1}\|_{\infty,[m,M]} \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right] \right\}. \quad (3.30)$$

We have proved that

$$|R| \leq \frac{\left(\|g(A)\| \left\|f^{(n)}\right\|_{\infty,[m,M]} + \|f(A)\| \left\|g^{(n)}\right\|_{\infty,[m,M]} \right)}{2(M-m)}$$

$$\|P_{n-1}\|_{\infty,[m,M]} \left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right]. \quad (3.31)$$

The theorem is proved. □

It follows the case $n = 1$.

Corollary 3.2. (to Theorem 3.1) *Let $f, g \in C^1([a,b])$ with $[m,M] \subset (a,b)$, $m < M$. Here A is a selfadjoint bounded linear operator on the Hilbert space H with spectrum $Sp(A) \subseteq [m,M]$. We consider any $x \in H : \|x\| = 1$.*

Then

$$|\langle f(A) \, g(A) \, x, x \rangle - \langle f(A) \, x, x \rangle \langle g(A) \, x, x \rangle| \quad (3.32)$$

$$\leq \frac{\left[\|g(A)\| \|f'\|_{\infty,[m,M]} + \|f(A)\| \|g'\|_{\infty,[m,M]} \right]}{2(M-m)}$$

$$\left[\left\|(M1_H - A)^2\right\| + \left\|(A - m1_H)^2\right\| \right].$$

We continue with

Theorem 3.3. *All as in Theorem* 3.1. *Let* $\alpha,\beta,\gamma>1:\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=1$. *Then*

$$\langle(\Delta(f,g))(A)x,x\rangle\leq\frac{\|P_{n-1}\|_{\alpha,[m,M]}}{(M-m)(\beta+1)^{\frac{1}{\beta}}}$$
$$\left[\|g(A)\|\left\|f^{(n)}\right\|_{\gamma,[m,M]}+\|f(A)\|\left\|g^{(n)}\right\|_{\gamma,[m,M]}\right] \tag{3.33}$$
$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].$$

Proof. As in (3.24) we have

$$\left\|\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_\lambda\right\|$$
$$=\sup_{x\in H:\|x\|=1}\left|\int_{m-0}^{M}\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d(\langle E_\lambda x,x\rangle)\right|=:\psi_1. \tag{3.34}$$

Here $\alpha,\beta,\gamma>1:\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=1$. By Hölder's inequality for three functions we get

$$\left|\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right|\leq\int_m^M|P_{n-1}(t)|\,|k(\lambda,t)|\left|f^{(n)}(t)\right|dt$$
$$\leq\|P_{n-1}\|_\alpha\left\|f^{(n)}\right\|_\gamma\left(\int_m^M|k(\lambda,t)|^\beta\,dt\right)^{\frac{1}{\beta}}$$
$$=\|P_{n-1}\|_\alpha\left\|f^{(n)}\right\|_\gamma\left(\int_m^\lambda(t-m)^\beta\,dt+\int_\lambda^M(M-t)^\beta\,dt\right)^{\frac{1}{\beta}} \tag{3.35}$$
$$=\|P_{n-1}\|_\alpha\left\|f^{(n)}\right\|_\gamma\left[\frac{(\lambda-m)^{\beta+1}+(M-\lambda)^{\beta+1}}{\beta+1}\right]^{\frac{1}{\beta}}$$
$$\leq\frac{\|P_{n-1}\|_\alpha\left\|f^{(n)}\right\|_\gamma}{(\beta+1)^{\frac{1}{\beta}}}\left[(\lambda-m)^{\frac{\beta+1}{\beta}}+(M-\lambda)^{\frac{\beta+1}{\beta}}\right].$$

I.e. it holds

$$\left|\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right|$$
$$\leq\frac{\|P_{n-1}\|_\alpha\left\|f^{(n)}\right\|_\gamma}{(\beta+1)^{\frac{1}{\beta}}}\left[(\lambda-m)^{1+\frac{1}{\beta}}+(M-\lambda)^{1+\frac{1}{\beta}}\right],\quad\forall\,\lambda\in[m,M]. \tag{3.36}$$

Therefore we get

$$\psi_1\leq\sup_{x\in H:\|x\|=1}\int_{m-0}^{M}\left|\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right|d\langle E_\lambda x,x\rangle$$
$$\leq\left(\sup_{x\in H:\|x\|=1}\int_{m-0}^{M}\left[(\lambda-m)^{1+\frac{1}{\beta}}+(M-\lambda)^{1+\frac{1}{\beta}}\right]d\langle E_\lambda x,x\rangle\right)$$

$$\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}$$

$$\leq\left(\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\right) \tag{3.37}$$

$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].$$

We have proved that

$$\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_\lambda\right\| \tag{3.38}$$

$$\leq\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].$$

Similarly, it holds

$$\left\|\int_{m-0}^{M}\left(\int_{m}^{M}P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_\lambda\right\|$$

$$\leq\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|g^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right]. \tag{3.39}$$

Using (3.23) we derive

$$|R|\leq\frac{1}{(M-m)}\left\{\|g(A)\|\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|f^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}\right. \tag{3.40}$$

$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right]$$

$$+\|f(A)\|\frac{\|P_{n-1}\|_{\alpha,[m,M]}\left\|g^{(n)}\right\|_{\gamma,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}}$$

$$\left.\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right]\right\}$$

$$=\frac{1}{(M-m)}\left[\|g(A)\|\left\|f^{(n)}\right\|_{\gamma,[m,M]}+\|f(A)\|\left\|g^{(n)}\right\|_{\gamma,[m,M]}\right]\frac{\|P_{n-1}\|_{\alpha,[m,M]}}{(\beta+1)^{\frac{1}{\beta}}} \tag{3.41}$$

$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right],$$

proving the claim. □

The case $n=1$ follows.

Corollary 3.4. (to Theorem 3.3) *All as in Theorem* 3.3*. It holds*

$$|\langle f(A)\,g(A)\,x,x\rangle-\langle f(A)\,x,x\rangle\langle g(A)\,x,x\rangle|$$
$$\leq\frac{1}{(M-m)(\beta+1)^{\frac{1}{\beta}}}\left[\|g(A)\|\,\|f'\|_{\gamma,[m,M]}+\|f(A)\|\,\|g'\|_{\gamma,[m,M]}\right] \tag{3.42}$$
$$\left[\left\|(A-m1_H)^{1+\frac{1}{\beta}}\right\|+\left\|(M1_H-A)^{1+\frac{1}{\beta}}\right\|\right].$$

We also give

Theorem 3.5. *All as in Theorem* 3.1*. It holds*

$$\langle(\Delta(f,g))(A)\,x,x\rangle\leq\|P_{n-1}\|_{\infty,[m,M]}$$
$$\left[\|g(A)\|\left\|f^{(n)}\right\|_{1,[m,M]}+\|f(A)\|\left\|g^{(n)}\right\|_{1,[m,M]}\right]. \tag{3.43}$$

Proof. We have that

$$\left|\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right|\leq\int_m^M|P_{n-1}(t)|\,|k(\lambda,t)|\left|f^{(n)}(t)\right|dt$$
$$\leq\|P_{n-1}\|_{\infty,[m,M]}(M-m)\int_m^M\left|f^{(n)}(t)\right|dt$$
$$=\|P_{n-1}\|_{\infty,[m,M]}(M-m)\left\|f^{(n)}\right\|_{1,[m,M]}. \tag{3.44}$$

So that

$$\left|\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right|$$
$$\leq(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{1,[m,M]}.$$

Hence

$$\left\|\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)dE_\lambda\right\|$$
$$=\sup_{x\in H:\|x\|=1}\left|\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,f^{(n)}(t)\,dt\right)d\langle E_\lambda x,x\rangle\right| \tag{3.45}$$
$$\leq(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{1,[m,M]},$$

and similarly,

$$\left\|\int_{m-0}^M\left(\int_m^M P_{n-1}(t)\,k(\lambda,t)\,g^{(n)}(t)\,dt\right)dE_\lambda\right\| \tag{3.46}$$
$$\leq(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|g^{(n)}\right\|_{1,[m,M]}.$$

Using (3.23) we obtain

$$|R|\leq\frac{1}{(M-m)}\left\{\|g(A)\|(M-m)\|P_{n-1}\|_{\infty,[m,M]}\left\|f^{(n)}\right\|_{1,[m,M]}\right.$$

$$+ \|f(A)\| (M - m) \|P_{n-1}\|_{\infty,[m,M]} \left\|g^{(n)}\right\|_{1,[m,M]} \bigg\}$$

$$= \|P_{n-1}\|_{\infty,[m,M]} \left[\|g(A)\| \left\|f^{(n)}\right\|_{1,[m,M]} + \|f(A)\| \left\|g^{(n)}\right\|_{1,[m,M]}\right], \quad (3.47)$$

proving the claim. □

The case $n = 1$ follows.

Corollary 3.6. (to Theorem 3.5) *It holds*

$$|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle|$$
$$\leq \left[\|g(A)\| \|f'\|_{1,[m,M]} + \|f(A)\| \|g'\|_{1,[m,M]}\right]. \quad (3.48)$$

Comment 3.7. The case of harmonic sequence of polynomials $P_k(t) = \frac{(t-x)^k}{k!}$, $k \in \mathbb{Z}_+$, was completely studied in [2], and this work generalizes it.

Another harmonic sequence of polynomials related to this work is

$$P_k(t) = \frac{1}{k!}\left(t - \frac{m+M}{2}\right)^k, \quad k \in \mathbb{Z}_+, \quad (3.49)$$

see also [4].

The Bernoulli polynomials $B_n(t)$ can be defined by the formula (see [4])

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n, \quad |x| < 2\pi, \ t \in \mathbb{R}. \quad (3.50)$$

They satisfy the relation

$$B_n'(t) = nB_{n-1}(t), \quad n \in \mathbb{N}.$$

The sequence

$$P_n(t) = \frac{1}{n!} B_n(t), \quad n \in \mathbb{Z}_+, \quad (3.51)$$

is a harmonic sequence of polynomials, $t \in \mathbb{R}$.

The Euler polynomials are defined by the formula (see [4])

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} \frac{E_n(t)}{n!} x^n, \quad |x| < \pi, \ t \in \mathbb{R}. \quad (3.52)$$

They satisfy

$$E_n'(t) = nE_{n-1}(t), \quad n \in \mathbb{N}.$$

The sequence

$$P_n(t) = \frac{1}{n!} E_n(t), \quad n \in \mathbb{Z}_+, \ t \in \mathbb{R}, \quad (3.53)$$

is a harmonic sequence of polynomials.

Finally:

Comment 3.8. One can apply (3.3), (3.33) and (3.43), for the harmonic sequences of polynomials defined by (3.49), (3.51) and (3.53).

In particular, when (see (3.49))

$$P_n(t) = \frac{1}{n!}\left(t - \frac{m+M}{2}\right)^n, \quad n \in \mathbb{Z}_+, \tag{3.54}$$

we get

$$\|P_{n-1}\|_{\infty,[m,M]} = \frac{1}{(n-1)!}\left(\frac{M-m}{2}\right)^{n-1}, \tag{3.55}$$

and

$$\|P_{n-1}\|_{\alpha,[m,M]} = \frac{1}{(n-1)!\,(\alpha(n-1)+1)^{\frac{1}{\alpha}}}\left(\frac{(M-m)^{\alpha(n-1)+1}}{2^{\alpha(n-1)}}\right), \tag{3.56}$$

where $\alpha, \beta, \gamma > 1 : \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$.

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