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Erratum to the paper Bogdan, M., ”Some comments on a linear programming problem”

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Abstract. The present paper corrects an assertion of the author from [1]. The pivoting algorithms referred to, search for solving the linear programming problem.

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1. Corrected assertion for the case of non-singleton solution

The standard form of a linear programming problem (LP) is $\min_{x\in S} c^T x$, where $S = \{x \in \mathbb{R}^n \mid A \cdot x = b, x \geq_{\mathbb{R}^n} 0_{\mathbb{R}^n}\}$, with data $c \in \mathbb{R}^n$, $A \in M_{m,n}(\mathbb{R})$, and $b \in \mathbb{R}^m$ given ([2]). Denote by $\mathcal{S}$ the set of its solutions. This paper corrects an assertion of the author from [1]. The motivation of the mentioned paper started from the clear difference between the two expressions *finding a solution to the problem* and *solving the problem*, especially when the feasible set S is not bounded. The author was not aware by the paper [5], having the same topic.

In order to correct the assertion from Proposition 3.2 in [1] into Proposition 1.3, Proposition 1.4, and Proposition 1.5, we provide the following two examples.

Example 1.1. Let $a > 0$, $b_1, b_2 > 0$ and the linear programming problem

$$\begin{cases} -x_1 - x_2 - ax_3 \longrightarrow min \\ \quad x_1 + x_2 \qquad\qquad \leq b_1 \\ \qquad\qquad x_2 + x_3 \leq b_1 + b_2 \\ \qquad\qquad\qquad x_3 \leq b_2 \\ x_1, x_2, x_3 \geq 0. \end{cases}$$

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The four iterations are given in Figure 2 and by the classic primal simplex algorithm one may find as solutions $x^1 = (0, b_1, b_2)$ or $x^2 = (b_1, 0, b_2)$. By the extended algorithm

$$\mathcal{S} = \begin{cases} \{x^1\}, & \text{if } a > 1 \\ \{x^2\}, & \text{if } a < 1 \\ co\{x^1, x^2\}, & \text{if } a = 1. \end{cases}$$

1	A^4	A^5	A^6	
A^1	1	0	0	1
A^2	1	1	0	1
A^3	0	1	1	a
	b_1	b_1+b_2	b_2	

2	A^2	A^5	A^6	
A^1	1	-1	0	0
A^4	1	-1	0	-1
A^3	0	1	1	a
	b_1	b_2	b_2	

3	A^2	A^5	A^3	
A^1	1	-1	0	0
A^4	1	-1	0	-1
A^6	0	-1	1	-a
	b_1	0	b_2	

4	A^1	A^5	A^3	
A^2	1	1	0	0
A^4	1	0	0	-1
A^6	0	-1	1	-a
	b_1	b_1	b_2	

FIGURE 1. Optimal 0−max dual basis; bounded $\mathcal{S}$

Note that in the third tableau, for $a = 1$ we have $\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0 = \alpha_{10}$ and $\min_{j \in \mathcal{B}} \alpha_{0j} = 0 = \alpha_{05}$. More, it exists $j = 2 \in \mathcal{B}$ such that $\alpha_{12} = 1 > 0$.

Example 1.2. Let $a \in \mathbb{R}$ and the linear programming problem

$$\begin{cases} \qquad x_2 \longrightarrow min \\ -x_1 \qquad\quad + x_3 \qquad = 0 \\ ax_1 + x_2 \qquad\quad + x_4 = 1 \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

From tables in Figure 3 we have

$$\mathcal{S} = \begin{cases} \{(0,0,0,1)\}, & \text{if } a < 0 \\ \{(\alpha/a, 0, \alpha/a, 1-\alpha) \mid \alpha \in [0,1]\}, & \text{if } a > 0 \\ \{(\alpha, 0, \alpha, 1) \mid \alpha \geq 0\}, & \text{if } a = 0. \end{cases}$$

1	A^3	A^4	
A^1	-1	a	0
A^2	0	1	-1
	0	1	

2	A^3	A^1	
A^4	1/a	1/a	0
A^2	1/a	1/a	-1
	1/a	1/a	

FIGURE 2. Optimal 0−max dual basis; unbounded $\mathcal{S}$

Note that in the first tableau, we have $\alpha_{10} = 0$ and $\alpha_{03} = 0$. For $a < 0$, one has $\alpha_{1j} < 0$, $j \in \mathcal{B} = \{3, 4\}$. For $a > 0$, it exists $j = 4 \in \mathcal{B}$ such that $\alpha_{14} = 1 > 0$.

An important step in the implementation of an algorithm should be a criteria that establishes the boundedness of the feasible set, weather or not it is a polytope (bounded, thus compact) or not. In its absence, the property of the solutions set regarding the boundedness is to be stated and the set itself to be obtained while the algorithm works. Related to the set of solutions for (LP), we give the following results.

Proposition 1.3. *Suppose that $x^0 = x^B$ is an optimal solution for (LP) and let B be the optimal basis. If $\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} < 0$, then there is no other solution generated by B.*

Proof. Suppose $x^{B'}$ is another solution generated by B, thus $c^T x^{B'} = c^T x^0$. Let $h \in \overline{\mathcal{B}}$ and suppose that the vector A^k is replaced by A^h in B. Let $\theta = \frac{\alpha_{0k}}{\alpha_{hk}} \geq 0$ be the rate transfer. Therefore, we have

$$c^T x^0 = c^T x^{B'} = c^T x^0 + \theta \cdot (-\alpha_{h0}).$$

If $\theta > 0$ we get the contradiction since $\alpha_{h0} < 0$. If $\theta = 0 = \alpha_{0k}$, then the pivoting element α_{hk} must be strictly negative. For $j \in \mathcal{B}' = (\mathcal{B} \setminus \{k\}) \cup \{h\}$, the coordinates of $x^{B'}$ are $\alpha'_{0j} = \alpha_{0j}$ and $\alpha_{0h} = \frac{\alpha_{0k}}{\alpha_{hk}} = 0$, that is $x^{B'} = x^0$, a contradiction. □

Proposition 1.4. *Let $x^0 = x^B$ be a solution for (LP) obtained in* Step 1 *of the algorithm and B be the optimal basis. Suppose that*

$$\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0 = \min_{j \in \mathcal{B}} \alpha_{0j}.$$

Denote by $\overline{\mathcal{B}}_0 = \{i \in \overline{\mathcal{B}} \,|\, \alpha_{i0} = 0\}$ and $\mathcal{B}_0 = \{i \in \mathcal{B} \,|\, \alpha_{0j} = 0\}$. The following implications apply:

1. *if $\alpha_{\bar{i}j} \leq 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j \in \mathcal{B}$, then $\mathcal{S}$ is unbounded.*
2. *if $\alpha_{\bar{i}j_0} > 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j_0 \in \mathcal{B}_0$, then there is no other solution generated by B.*
3. *if exist $\bar{i} \in \overline{\mathcal{B}}_0$, $k \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{\bar{i}k} > 0$, then the solution is not unique.*

Proof. 1. Let $\bar{i} \in \overline{\mathcal{B}}_0$. Since $\alpha_{\bar{i}j} \leq 0$, $\forall j \in \mathcal{B}$, there is no pivoting element, consequently another solution cannot be obtained by a classic pivoting operation. There exists $\bar{c} > 0_{\mathbb{R}^n}$ such that

$$\{x^0 + \alpha \cdot c \,|\, \alpha \geq 0\} \subseteq \mathcal{S}.$$

The unboundedness direction $\bar{c} = (\bar{c}_1, ..., \bar{c}_n)$ is given by

$$\bar{c}_j = \begin{cases} -\alpha_{\bar{i}j}, & j \in \mathcal{B} \setminus \mathcal{B}_{\bar{i}0} \\ 1, & j = \bar{i} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_{\bar{i}0} = \{k \in \mathcal{B} \,|\, \alpha_{\bar{i}k} = 0\}$.

2. Let $\bar{i} \in \overline{\mathcal{B}}_0$ and $j_0 \in \mathcal{B}_0$. Since $\alpha_{0j_0} = 0$, by replacing vector A^{j_0} with $A^{\bar{i}}$ in B, the value of the objective function does not change

$$c^T x^0 - \alpha_{\bar{i}0} \cdot \frac{\alpha_{0j_0}}{\alpha_{\bar{i}j_0}} = c^T x^0.$$

Let $x^{B'}$ be the new optimal solution. Its coordinates $x_j^{B'}$ are $x_j^{B'} = x_j^B$, for $j \in \mathcal{B} \setminus \{j_0\}$, $\alpha'_{0\bar{i}} = \alpha_{0j_0} = 0$, and 0 in rest, (i.e. $j \notin \mathcal{B}' = (\mathcal{B} \setminus \{j_0\}) \cup \{\bar{i}\}$), therefore $x^{B'} = x^B$.

3. Let $\bar{i} \in \overline{\mathcal{B}}_0$. Consider $\alpha_{\bar{i}k}$ as pivoting element. By replacing vector A^k with $A^{\bar{i}}$, the value of the objective function does not change

$$c^T x^0 - \alpha_{\bar{i}0} \cdot \frac{\alpha_{0k}}{\alpha_{\bar{i}k}} = c^T x^0.$$

The coordinates of the new solution $x^{B'}$ are $x_j^{B'} = x_j^B - \alpha_{\bar{i}j} \cdot \frac{\alpha_{0k}}{\alpha_{\bar{i}k}}$, for $j \in \mathcal{B} \setminus \{k\}$, $\alpha'_{0\bar{i}} = \frac{\alpha_{0k}}{\alpha_{\bar{i}k}} \neq 0$, and 0 in rest, including $\alpha'_{0k} = 0$, thus $x^{B'} \neq x^B$.

□

Corresponding to item 3 above, by Example 1.1, we have $\bar{i} = 1 \in \overline{\mathcal{B}}_0$, $k = 2 \in \mathcal{B} \setminus \mathcal{B}_0$ with $\alpha_{12} > 0$. Similarly, by Example 1.2, case $a > 0$, we have $\overline{\mathcal{B}}_0 = \{1\}$ and it exists $k = 2 \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{14} > 0$.

Proposition 1.5. ([1]) *Suppose that $x^0 = x^B$ is a solution for (LP) obtained in* Step 1 *of the algorithm and that B is the optimal basis. If $\max\limits_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0$ and $\min\limits_{j \in \mathcal{B}} \alpha_{0j} > 0$, then*

$$\{x^0\} \subsetneq \mathit{argmin}_{x \in S} c^T x;$$

if more

α) for some $\bar{i} \in \overline{\mathcal{B}}_0$, it exists $k \in \mathcal{B}$ such that $\alpha_{\bar{i}k} > 0$, then

$$co\{x^0, x^1, ..., x^u\} \subseteq \mathcal{S},$$

with $u \leq card\, \overline{\mathcal{B}}_{0+}$, where

$$\overline{\mathcal{B}}_{0+} = \{i \in \overline{\mathcal{B}}_0 \,|\, \exists k \in \mathcal{B} \text{ such that } \alpha_{ik} > 0\};$$

β) for some $\bar{i} \in \overline{\mathcal{B}}_0$, $\alpha_{\bar{i}k} \leq 0$, $\forall k \in \mathcal{B}$, then the set of solutions is (convex) unbounded.

Proof. α) The proof is the same to 3. from Proposition 1.4 since $\mathcal{B}_0 = \emptyset$.

β) see [1], Proposition 3.2, 3., a3β). □

Most of the works contain background, terminology, usual notations, and basic results. Let us remind some of them. A vector $x \in \mathbb{R}^n$ is seen as a column vector and its transpose, denoted by $x^T = (x_1, ..., x_n) \in \mathbb{R}^n$, as a row vector. In particular, denote by $0^T_{\mathbb{R}^n} = (0, ..., 0) \in \mathbb{R}^n$ and by $e^T_j = (0, ..., 0, 1, 0, ..., 0)$ with 1 for the j^{th} position. The scalar product of $c \in \mathbb{R}^n$ and x is given by

$$c^T x = \sum_{i=1}^{n} c_i \cdot x_i.$$

The relation $x \geq_{\mathbb{R}^n} 0_{\mathbb{R}^n}$ means $x_i \geq 0$, for all $i \in \{1, ..., n\}$; one has $x >_{\mathbb{R}^n} 0_{\mathbb{R}^n}$ iff $x_i \geq 0$, $\forall i$ and $\exists i_0$ with $x_{i_0} > 0$. Also, $x \leq_{\mathbb{R}^n} y$ iff $y - x \geq 0_{\mathbb{R}^n}$, and $x >_{\mathbb{R}^n} y$ iff $x - y >_{\mathbb{R}^n} 0_{\mathbb{R}^n}$. The $j-$th column of a matrix $A \in M_{m,n}(\mathbb{R})$ is denoted by A^j; a matrix B consisting of m independent columns of A, $m < n$, is called *basic*. The remaining columns of A that are not in B are said to be outside the basis or nonbasic. For $A = (A^j)_{1 \leq j \leq n}$,

$$\mathcal{B} = \{j \in \{1, 2, ..., n\} \mid \exists k, A^j = B^k\},$$

$$\overline{\mathcal{B}} = \{1, 2, ..., n\} \setminus \mathcal{B} = \{i \in \{1, 2, ..., n\} \mid \nexists k, A^i = B^k\},$$

so $\{A^j\}_{1 \leq j \leq n} = \{\{A^j\}_{j \in \mathcal{B}}, \{A^i\}_{i \in \overline{\mathcal{B}}}\}$. The linear combination for A^i, $i \in \overline{\mathcal{B}}$, is given by

$$A^i = \sum_{j \in \mathcal{B}} \alpha_{ij} A^j.$$

The coordinates of b are α_{0j}, i.e.

$$b = \sum_{j \in \mathcal{B}} \alpha_{0j} A^j.$$

A basic matrix B is said to be *primal feasible* if $\alpha_{0j} \geq 0$, $\forall j \in \mathcal{B}$ and *dual feasible* if $\alpha_{i0} \leq 0$, $\forall i \in \overline{\mathcal{B}}$, respectively. If B is primal feasible and dual feasible then B is called *optimal* or *optimal basis*. Simplex algorithm 2.0, is based on the extended properties of the optimal basis. About an optimal basis B, we say that it is $0-$*max dual feasible* if $\max\limits_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0$ and $0-$*min primal feasible* if $\min\limits_{j \in \mathcal{B}} \alpha_{0j} = 0$, respectively.
As regards the three situations of the algorithm, when there is a unique solution, a bounded set of solutions (but not a singleton) or the unbounded set of solutions, we reformulate the following result in concordance to Proposition 1.3, Proposition 1.4, and Proposition 1.5.

Theorem 1.6. *Suppose that $x^0 = x^B$ is a solution for (LP) obtained in* Step 1 *of the algorithm and that B is the optimal basis. The following implications apply:*

1. *If B is NOT $0-$max dual feasible then the solution generated by B is unique.*
2. *If B is $0-$max dual feasible, $0-$min primal feasible, and $\alpha_{\bar{i}j} \leq 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j \in \mathcal{B}$, then the set of solutions is unbounded.*
3. *If B is $0-$max dual feasible, $0-$min primal feasible, and $\alpha_{\bar{i}j} > 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j \in \mathcal{B}_0$, then the solution generated by B is unique.*
4. *If B is $0-$max dual feasible, $0-$min primal feasible, and exist $\bar{i} \in \overline{\mathcal{B}}_0$, $k \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{\bar{i}k} > 0$, then the solution generated by B is not unique.*

5. *If B is* $0-$*max dual feasible and it is NOT* $0-$*min primal feasible, then the solution generated is not unique.*

In general, the computer algebra systems such as Octave [3], WolframAlpha [4], use the interior point algorithm implemented (the function *glpk* in Octave has the the parameter *param* that allows to use two-phase primal/dual simplex). None of these return more than one solution at one input. When the instruction/command has the parameter for the initial starting point (ex. Matlab), by changing it may be successful for returning another solution if it exists.

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