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A GENERALIZATION OF SOME OF ORE'S THEOREMS

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Dedicated to Professor Ioan Purdea at his 60th anniversary

Abstract. The paper completes the results from [2] with new properties of finite π-solvable primitive groups, where π is an arbitrary set of primes. Thus we obtain a generalization for π-solvable groups of some of ORE's theorems from [5] given for solvable groups and being of special interest in the formation theory.

1. Preliminaries

All groups considered in the paper are finite. We shall denote by π an arbitrary set of primes and by $\pi\prime$ the complement to π in the set of all primes.

Definition 1.1. a) Let G be a group, M and N two norma; subgroups of G such that $N \subseteq M$. The factor M/N is called a ***chief factor*** of G if M/N is a minimal normal subgroup of G/N.

b) A group G is said to be π-*solvable* if every chief factor of G is either a solvable π-group or a $\pi\prime$-group. Particularly, for π the set of all primes we obtain the notion of solvable group.

Definition 1.2. a) Let G be a group and W a subgroup of G. We define

$$core_G W = \cap\{W^g / g \in G\},$$

where $W^g = g^{-1}Wg$.

b) W is a stabilizer of G if W is a maximal subgroup of G and $core_G W = 1$.

c) A group G is *primitive* if there is a stabilizer W of G.

The following results will be used to prove the main theorems of this paper.

Theorem 1.3. ([1]) *Any solvable minimal normal subgroup of a finite group is abelian.*

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Theorem 1.4 (Schur-Zassenhaus) ([3], p.16) *Let G be a finite group and H a normal abelian subgroup of G such that $|G : H|$ and $|H|$ are relatively prime. Then:*

(a) H has a complement K in G, i.e. $HK = G$ and $H \cap K = 1$;

(b) all complements of H in G are conjugate under H.

Theorem 1.5. ([4], p.18) *If G is a group and M, M_1 are two normal subgroups of G such that $M \cap M_1 = 1$, then M and M_1 commute elementwise, i.e. $mm_1 = m_1m$ for any $m \in M$ and $m_1 \in M_1$.*

Theorem 1.6. (Dedekind identity) ([4], p.8) *If G is a group and A, B, C are subgroups of G such that $A \subseteq C \subseteq AB$, then*

$$C = (AB) \cap C = A(B \cap C).$$

Theorem 1.7. ([2]) *Let G be a primitive group and W a stabilizer of G. Then:*

(i) for any normal subgroup $K \neq 1$ of G we have $KW = G$;

(ii) for any minimal normal subgroup M of G we have $MW = G$;

(iii) there is not a normal subgroup $K \neq 1$ of G such that $K \subseteq W$.

2. Frattini argument for π-solvable groups

In [4], p.35, 7.8. the following well-known theorem called the "Frattini argument" is given: Let G be a group, N a normal subgroup of G and P a Sylow p-subgroup of N. Then $G = NN_G(P)$.

Our later considerations need a new form of the Frattini argument which we give below.

We remind that a subgroup H of a group G is called a *Hall π-subgroup* of G if $|H|$ is a π-number and $|G : H|$ is a $\pi\prime$-number. We also remind the Hall-Čunihin theorem:

Theorem 2.1. (Hall-Čunihin, [4], p.660) *If G is a π-solvable group, then:*

(a) G has Hall π-subgroups and Hall $\pi\prime$-subgroups;

(b) all Hall π-subgroups of G are conjugate in G; all Hall $\pi\prime$-subgroups of G are conjugate in G.

Theorem 2.2. (The Frattini argument for π-solvable groups) *Let G be a π-solvable group, N a normal subgroup of G and P a Hall π-subgroup (or a Hall $\pi\prime$-subgroup) of N. Then $G = NN_G(P)$.*

Proof. Clearly $NN_G(P) \subseteq G$. Let now $g \in G$. Then $P^g \subseteq N^g = N$, hence P^g is also a Hall π-subgroup (or a Hall $\pi\prime$-subgroup) of N. But N, as a subgroup of the π-solvable group G, is a π-solvable group too. Thus, applying 2.1, P and P^g are conjugate in N. It follows that $P^g = P^n$, where $n \in \mathrm{N}$. This implies $gn^{-1} \in N_G(P)$. Then

$$g = (gn^{-1})n \in N_G(P)N = NN_G(P).$$

This proves that $G \subseteq NN_G(P)$, hence $G = NN_G(P)$. □

3. A generalization of some of ORE's theorems

Given in [5] for solvable groups, the so-called ORE's theorems are of special interest in the formation theory. Here we establish a generalization for π-solvable groups of some of ORE's theorems, where π is an arbitrary set of primes. Particularly, for π the set of all primes, we obtain ORE's theorems.

In [2] we proved the following results similar to some of ORE's:

Theorem 3.1. *Let G be a primitive π-solvable group. If G has a minimal normal subgroup which is a solvable π-group, then G has one and only one minimal normal subgroup.*

Corollary 3.2. *If G is a primitive π-solvable group, then G has at most one minimal normal subgroup which is a solvable π-group.*

Corollary 3.3. *If a primitive π-solvable group G has a minimal normal subgroup which is a solvable π-group, then G has no minimal normal subgroups which are $\pi\prime$-groups.*

Theorem 3.4. *If G is a primitive π-solvable group and N is a minimal normal subgroup of G which is a solvable π-group, then $C_G(N) = N$.*

The first result of this paper examines the converse of 3.4:

Theorem 3.5. *Let G be a π-solvable group such that:*

(i) there is a minimal subgroup M of G which is a solvable π-group and $C_G(M) = M$;

(ii) there is a minimal normal subgroup L/M of G/M such that L/M is a $\pi\prime$-group. Then G is primitive.

Proof. Suppose $M = G$. Then $G/M = 1$, hence $L/M = 1$ giving a contradiction. Thus $M \neq G$. Further, by 1.3 M is abelian.

By (ii) $|L/M|$ is a π'-number and by (I) $|M|$ is a π-number. It follows that $(|L/M|, |M|) = 1$. Applying now theorem 1.4, we conclude that M has a complement L_0 in L, i.e. $ML_0 = L$ and $M \cap L_0 = 1$.

Put $W = N_G(L_0)$. We shall prove that W is a stabilizer of G, i.e. W is a maximal subgroup of G and $core_G W = 1$.

Indeed, $W \neq G$, for otherwise $N_G(L_0) = G$ and hence $L_0 \lhd G$. So M and L_0 are two normal subgroups of G such that $M \cap L_0 = 1$. By 1.5 M and L_0 commute elementwise. Hence $L_0 \subseteq C_G(M) = M$. Thus $L = ML_0 = M$ and $L/M = 1$ contradicting (ii).

We note that $MW = G$ and $M \cap W = 1$. Indeed, applying 2.2 to the π-solvable group G, $L \lhd G$ and L_0 a Hall π'-subgroup of L (since $L_0 \simeq L_0/1 = L_0/M \cap L_0 \simeq ML_0/M = L/M$ is a π'-group and $|L : L_0| = |ML_0 : L_0| = |M : M \cap L_0| = |M|$ is a π-number), we obtain:

$$G = LN_G(L_0) = ML_0N_G(L_0) = MN_G(L_0) = MW.$$

To prove that $M \cap W = 1$, let us first show that $M \cap W \lhd G$. Let $g \in G = MW$, $g = m_1 w$, with $m_1 \in M$, $w \in W$ and let $m \in M \cap W$. Then

$$g^{-1}mg = (m_1w)^{-1}m(m_1w) = v^{-1}(m_1^{-1}mm_1)w,$$

where $m_1^{-1}mm_1 \in M \cap W$ since $M \cap W$ is normal in the abelian group M, and

$$w^{-1}(m_1^{-1}mm_1)w \in M \cap W$$

since $M \cap W$ is normal in W. Hence $g^{-1}mg \in M \cap W$. Now from $M \cap W \lhd G$, $M \cap W \subseteq M$ and M minimal normal subgroup of G it follows that $M \cap W = 1$ or $M \cap W = M$. The last condition is impossible because it implies that $M \subseteq W$ and hence the contradiction $G = MW = W$. So $M \cap W = 1$.

To prove that W is a maximal subgroup of G, we remind that $W \neq G$ and let us show that $W \leq W^* < G$ imply $W = W^*$. Suppose that $W < W^*$. Let $w^* \in W^* \setminus W \subset G = MW$. It follows that $w^* = mw$, with $m \in M$ and $w \in W$. Hence $m = w^*w^{-1} \in M \cap W^*$. But $G = MW \subseteq MW^* \subseteq G$ imply $G = MW^*$. Hence $M \cap W^* = 1$ (proof like the above $M \cap W = 1$). Thus $m = 1$ and $w^* = w \in W$, a contradiction. Then $W = W^*$.

Finally, we prove that $core_G W = 1$. Since $M \cap core_G W \lhd G$, $M \cap core_G W \subseteq M$, $M \cap core_G W \neq M$ (for otherwise $M \subseteq core_G W$ and so the contradiction $G = MW = W$)

and M being a minimal normal subgroup of G we have $M \cap core_G W = 1$. By 1.5 M and $core_G W$ commute elementwise. It follows that $core_G W \subset C_G(M) = M$ which implies $core_G W = M \cap core_G W = 1$. □

The following two theorems generalize some of ORE's theorems.

Theorem 3.6. *If G is a π-solvable group satisfying (i) and (ii) from 3.5, then ant two stabilizers W_1 and W_2 of G are conjugate in G.*

Proof. By 3.5 G is primitive. Like in the proof of theorem 3.5 we note that $M \neq G$ and M is abelian. By 3.1 M is the only minimal normal subgroup of G.

Let $W = N_G(L_0)$ be the stabilizer of G given in the proof of theorem 3.5. Hence $ML_0 = L$ and $M \cap L_0 = 1$. We also know that $MW = G$ and $M \cap W = 1$.

We shall prove that W and W_1 are conjugate in G, and that W and W_2 are conjugate in G. It follows that W_1 and W_2 are conjugate in G. It is enough to prove for W and W_1, the proof for W and W_2 being similar.

Put $L_1 = W_1 \cap L$. Let us show that $L_0 = W \cap L$. First we note that $L_0 \subseteq ML_0 = L$, $L_0 \subseteq N_G(L_0) = W$ hence $L_0 \subseteq W \cap L$. Conversely, if $x \in W \cap L = N_G(L_0) \cap L$ then $x \in N_G(L_0)$ and $x \in L = ML_0 = L_0M$ which imply $L_0^x = L_0$, where $x = l_0 m$, $l_0 \in L_0$, $m\hat{\imath}M$. So $(L_0^1)^m = L_0$ which means that $m \in N_G(L_0) = W$. Then $m \in M \cap W = 1$ hence $m = 1$ and $x = l_0 \in L_0$. This proves that $W \cap L \subseteq L_0$.

We know that L_0 is a complement of M in L. L_1 is also a complement of M in L. Indeed, $ML_1 = M(W_1 \cap L)$ and by 1.6 $M(W_1 \cap L) = (MW_1) \cap L$. So $ML_1 = (MW_1) \cap L$. But $MW_1 = G$ for otherwise we have $W_1 \subseteq MW_1 \subset G$ which implies $W_1 = MW_1$ since W_1 is maxim in G and so $M \subseteq W_1$, in contradiction with 1.7.(iii). Thus $ML_1 = G \cap L = L$. Further, $M \cap L_1 = 1$ since

$$M \cap L_1 = M \cap (W_1 \cap L) = (M \cap W_1) \cap L$$

and $M \cap W_1 = 1$ as we shall see below. First note that $M \cap W_1 \lhd G$. Indeed, if $x \in G = MW_1$, $x = m_1 w_1$ with $m_1 \in M$, $w_1 \in W_1$, and $m \in M \cap W_1$ then, using that M is abelian and that $M \cap W_1 \lhd W_1$, we have:

$$x^{-1}mx = (m_1w_1)^{-1}m(m_1w_1) = w_1^{-1}m_1^{-1}mm_1w_1 = w_1mm_1^{-1}m_1w_1 = w_1^{-1}mw_1 \in M \cap W_1.$$

Now from $M \cap W_1 \lhd G$, $M \cap W_1 \subseteq M$, $M \cap W_1 \neq M$ (for otherwise $M \subseteq W_1$, contradicting 1.7.(iii)) and M minimal norma subgroup of G we obtain $M \cap W_1 = 1$.

By 1.4.(b) L_0 and L_1 are conjugate under M, i.e. $L_0 = L_1^m$ for some $m \in M$. Further, $L_0 \subseteq W \cap W_1^m$ since $L_0 = N_G(L_0) \cap L_0 = W \cap L_0 \subseteq W$ and $L_1 0 = L_1^m = (W_1 \cap L)^m \subseteq W_1^m$. Moreover, from $L_0 \lhd N_G(L_0) = W$ and $L_0 = L_1^m = (W_1 \cap L)^m \lhd W_1^m$ it follows $L_0 \lhd WW_1^m$.

We shall prove that $W = W_1^m$, which means that W and W_1 are conjugate in G. Let us suppose that $W \neq W_1^m$. From $W \leq WW_1^m \leq G$ and $W \neq WW_1^m$ ($W = WW_1^m$ is impossible because it implies $W_1^m \subseteq W$ hence $W_1 \subseteq W^k \subset G$ and $W_1 = W^k$, where $k = m^{-1}$, since W_1 is maximal in G; but this leads to the contradiction $W = W_1^m$) since W is maximal in G it can be inferred that $WW_1^m = G$. Thus $L_0 \lhd WW_1^m = G$ so that $W = N_G(L_0) = G$, a contradiction. It follows that $W = W_1^m$. □

Theorem 3.7. *If G is a primitive π-solvable group, $V < G$ such that there is a minimal normal subgroup M of G which is a solvable π-group and $MV = G$, then V is a stabilizer of G.*

Proof. $M \cap V$ is a normal subgroup of G. Indeed, let $g \in G = MV = VM$, $g = vm$ for some $v \in V$, $m \in M$ and let $x \in M \cap V$. Since $M \cap V \lhd V$ and since by 1.3 M is abelian we have:

$$g^{-1}xg = (vm)^{-1}x(vm) = m^{-1}(v^{-1}xv)m = m^{-1}m(v^{-1}xv) = (v^{-1}xv \in M \cap V.$$

Now $M \cap V = 1$ since M is a minimal normal subgroup of G and since $M \cap V \lhd G$, $M \cap V \subseteq M$, $M \cap V \neq M$ (supposing $M \cap V = M$ it follows $M \subseteq V$ and so $G = MV = V$, in contradiction with $V < G$).

Let us verify that V is a stabilizer of G.

First, V is a maximal subgroup of G. Indeed, $V \neq G$ and we shall prove that $V \leq V^* < G$ imply $V = V^*$. Suppose $V < V^*$ and let $v^* \in V^* \setminus V \subset G = MV$. Then $v^* = mv$ for some $m\hat{\imath}M$, $v \in V$, Hence $m = v^*v^{-1} \in M \cap V^*$. We prove that $M \cap V^* = 1$. From $G = MV \subseteq MV^* \subseteq G$ it follows $MV^* = G$. Hence $M \cap V^* \lhd G$ (as the above proof for $M \cap V \lhd G$). Since M is a minimal normal subgroup of G and since $M \cap V^* \lhd G$, $M \cap V^* \subseteq M$, $M \cap V^* \neq M$ (supposing $M \cap V^* = M$ we have $M \subseteq V^*$, hence $G = MV^* = V^*$, a contradiction) it follows $M \cap V^* = 1$. Thus $m = 1$ and so $v^* = v \in V$, a contradiction.

Finally, $core_G V = 1$. Indeed, suppose $core_G V \neq 1$. By 3.1 M is the only minimal normal subgroup of G. Thus since $core_G V \lhd G$ we have $M \subseteq core_G V$. But $core_G V \subseteq V$ and so $M \subseteq V$. Then $G = MV = V$, a contradiction. □

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